Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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Tarea 6 # 1(c) $(a) \to (b)$

$p^* = +A^{-1}b \to p^* = -A^{-1}b$

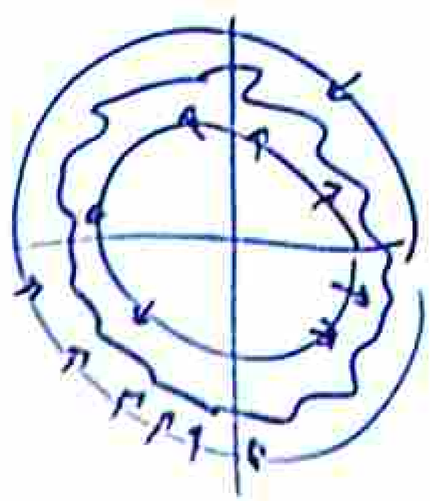
<u>Teo</u> (Poincaré - Bendixson)

Ω compacto, conexo, no-vacío y el ω-límite de un punto $\xi \in \mathbb{R}^n$.

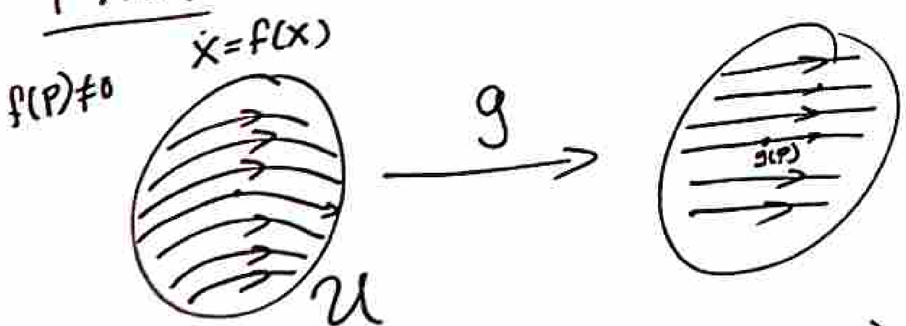
Si Ω no tiene un punto fijo $\Rightarrow$ Ω es orb. per.

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2$$

$$\dot{r} = f(r,\theta)$$
$$\dot{\theta} = g(r,\theta)$$



Thm



$$\dot y = Dg(g^{-1}(y))f(g^{-1}(y))$$

$$(\dot y_1, \dot y_2, \dots, \dot y_n) = (1, 0, \dots 0)$$

Pf The idea is to rectify at the point and then use the IFT.

Let $\{e_1, \dots, e_n\}$ be the standard base of $\mathbb{R}^n$.

We'll define a matrix B whose first column is $f(p)$, then we complete the matrix with the basis $\{e_2, \dots, e_n\}$

$$B = (f(p) \,|\, e_2 \,|\, \dots \,|\, e_n) \quad (B^{-1})$$

$$Be_1 = f(p) \Rightarrow e_1 = B^{-1}f(p)$$

Define

$$H_1 : \mathbb{R}^n \to \mathbb{R}^n \qquad \text{So } H_1(p) = 0$$

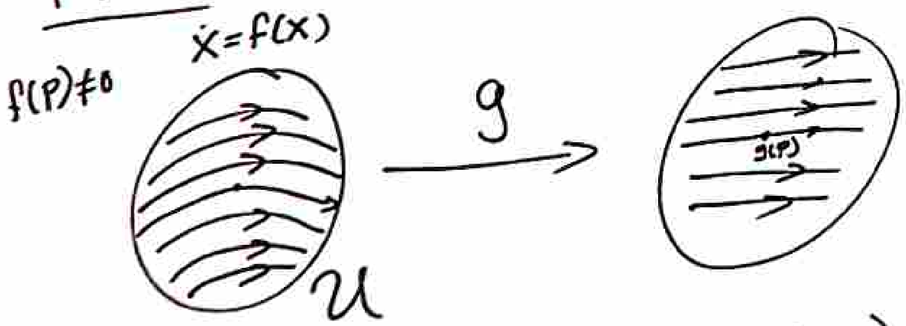
$$H_1(x) = B^{-1}(x-p) \quad , \quad DH_1(p)\,f(p) = e_1$$

$$z := H_1(x) \quad (x = H_1^{-1}(z))$$

$$\dot z = DH_1(x)\dot x = B^{-1}f(x) = B^{-1}f(H_1^{-1}(z)) := f_1(z)$$

Notice that $f_1(0) = B^{-1}f(p) = e_1$ (The conclusion is valid for 1 point.)

Thm



$$\dot{y} = Dg(g^{-1}(y))f(g^{-1}(y))$$

$$(\dot{y}_1, \dot{y}_2, \dots, \dot{y}_n) = (1, 0, \dots 0)$$

<u>Pf</u> The idea is to rectify at the point and then use the IFT.

We have that $\dot{z} = B^{-1} f(H_1^{-1}(z)) := f_1(z)$

Now we define $H_2 : \mathbb{R}^n \to \mathbb{R}^n$, $H_2(t, z_2, z_3, \dots, z_n) = \varphi(t, (0, z_2, \dots, z_n))$

→ where φ is the flow of $\dot{z} = f_1(z)$.

Note that $H_2(0) = 0$

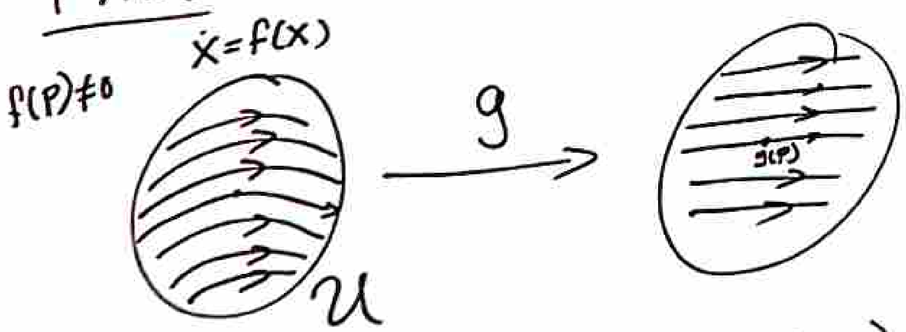
$$DH_2(0)e_1 = \frac{d}{dt} H_2(\overbrace{t, 0, 0, 0, \dots, 0}^{te_1})\Big|_{t=0} = \frac{d}{dt}\varphi(t;0)\Big|_{t=0} = f_1(0) = e_1$$

Now we need to look at $j = 2, 3, \dots, n$

$$DH_2(0)e_j = \frac{d}{dt} H_2(\underbrace{te_j}_{(0,\dots,\underset{j\text{th}}{t},0,\dots,0)})\Big|_{t=0} = \frac{d}{dt}(te_j)\Big|_{t=0} = e_j$$

$DH_2(0) = Id_{n\times n}$. By the (inverse function thm) we know that $\exists\, U$ & V neighborhoods of 0 such that $H_2 : U \to V$ is a diffeomorphi

Thm



$$\dot{y} = Dg(g^{-1}(y))f(g^{-1}(y))$$

$$(\dot{y}_1, \dot{y}_2, \dots, \dot{y}_n) = (1, 0, \dots 0)$$

Pf The idea is to rectify at the point and then use the IFT.

Define $y = H_2^{-1}(z)$, then $(z = H_2(y))$

$$\dot{z} = DH_2(y)\dot{y} \quad \text{or} \quad \dot{y} = DH_2^{-1}(y)\dot{z} = DH_2^{-1}(y) f_1(H_2(y))$$

$$DH_2^{-1}(y) f_1(H_2(y)) := f_2(y) \Leftrightarrow DH_2(y) f_2(y) = f_1(H_2(y))$$

So given $y \in \mathcal{U}$, we can write $y = (s, y_2, \dots, y_n)$

and $\gamma(t) = (s+t, y_2, \dots, y_n)$ $\quad \varphi(t, \varphi(s, 0, y_2, \dots, y_n))$

By definition of H_2:

$$DH_2(y)e_1 = \frac{d}{dt} H_2(\gamma(t))\Big|_{t=0} = \frac{d}{dt}\varphi(s+t, (0, y_2, \dots, y_n))$$

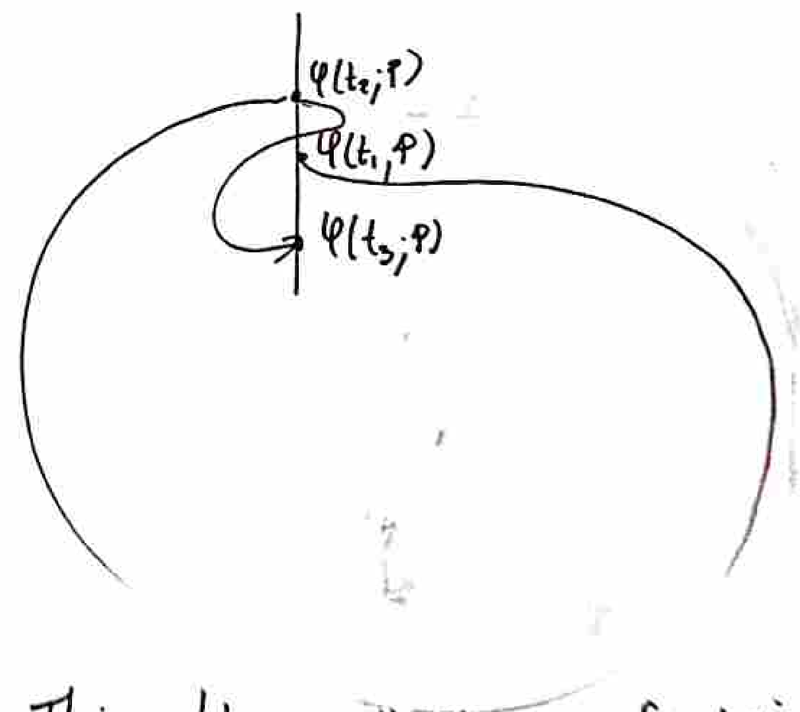
$$= \frac{d}{dt}\varphi(t, H_2(y))\Big|_{t=0} = f_1(H_2(y)) = DH_2(y) f_2(y)$$

$\Rightarrow e_1 = f_2(y)$, $\forall y \in \mathcal{V}$, then we've proven it for $\mathcal{V}$. //

Lemma

If Σ is a Poincaré section for the flow φ and $p \in \mathbb{R}^2$ then the orbit that passes through p intersects Σ in a monotone sequence. In other words, if $\varphi(t_1;p)$, $\varphi(t_2,p), \varphi(t_3,p) \in \Sigma$ with $t_1<t_2<t_3$ then either $\varphi(t_2;p)$ is between $\varphi(t_1;p)$ & $\varphi(t_2;p)$ or the three points are equal.

Pf Let's assume that the three points are different and $\varphi(t_1,p)$ is inbetween $\varphi(t_2,p)$ & $\varphi(t_3,p)$



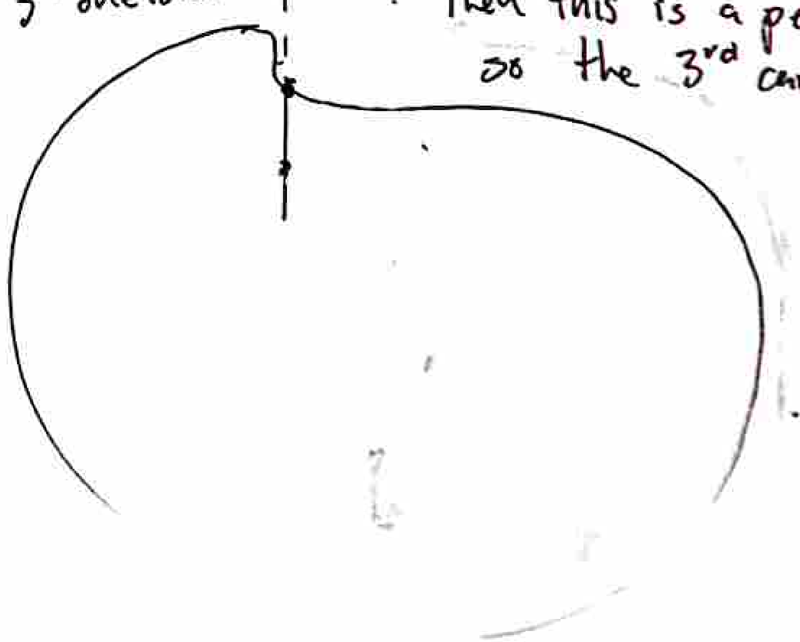
This would mean that there is a fixed point on Σ! contradiction.

Lemma

If Σ is a Poincaré section for the flow φ and $p \in \mathbb{R}^2$ then the orbit that passes through p intersects Σ in a monotone sequence. In other words, if $\varphi(t_1; p)$, $\varphi(t_2, p), \varphi(t_3, p) \in \Sigma$ with $t_1 < t_2 < t_3$ then either $\varphi(t_2; p)$ is between $\varphi(t_1; p)$ & $\varphi(t_3; p)$ or the three points are equal.

Pf Let's assume that the three points are different and $\varphi(t_1, p)$ is inbetween $\varphi(t_2, p)$ & $\varphi(t_3, p)$

What happens if two of the points are equal and the 3rd one is not. Then this is a periodic orbit so the 3rd cannot be different



Lemma

If Σ is a section for the flow φ in $\mathbb{R}^2$ and $p \in \mathbb{R}^2$, then $\omega(p) \cap \Sigma$ contains at most one point.

Pf

Let's suppose that there are two points $x_1, x_2 \in \omega(p)$, $x_1 \neq x_2$

There exist sequences of times

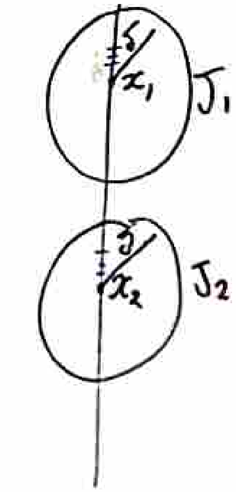
$$\{\varphi(t_i; p)\}_{i=1}^{\infty}, \ \{\varphi(s_i; p)\}_{i=1}^{\infty} \in \Sigma$$

so that $\varphi(t_i; p) \xrightarrow[i\to\infty]{} x_1$

$$\varphi(s_i; p) \xrightarrow[i\to\infty]{} x_2$$

Note that using the rectification lemma we can create the sequences that take the converging points to Σ.

Since $x_1 \neq x_2$ there is $\delta > 0$ so that



Then we know that $\exists N \in \mathbb{N}$ large enough so that

$$|\varphi(t_i, p) - x_1| < \delta$$

$i > N$.

and

$$|\varphi(s_i, p) - x_2| < \delta$$

$t_i \to \infty$

$s_i \to \infty$

$\varphi(t_{i_1}; p) \in J_1$

there is an s_{i_2} $\quad t_{i_1} < s_{i_2}$

$\varphi(s_{i_2}; p) \in J_2$

and a t_{i_2} s.t. $\quad s_{i_2} < t_{i_2}$

$\varphi(t_{i_2}; p) \in J_1$

This is not possible. //

Proof of the P-B thm

Suppose that $\omega(p)$ is non empty, compact, and does not contain any fixed points.

Take $q \in \omega(p)$.

First we will prove that the orbit that passes through q is closed.

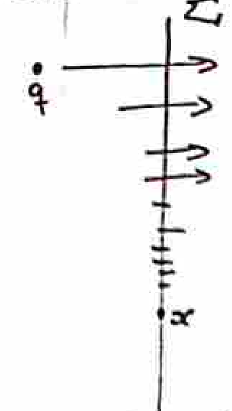
Since the ω-limit is a closed and invariant set $\omega(q) \subset \omega(p)$

We know that $\varphi(t_n; q) \to x$ and $\varphi(s_n; p) \to q$

then $\varphi(t_n + s_n; p) - x$
$= \varphi(t_n; \varphi(s_n; p)) - \varphi(t_n; q) + \varphi(t_n; q) - x \to 0$

This proves $\omega(q) \subset \omega(p)$

Take $x \in \omega(q)$ and since x is not a fixed point there is a section Σ, over x and by the previous lemma (rectification) there is a sequence of intersections of that converges to x.



All of these intersections belong to the $\omega(p)$.
This is a contradiction since we would have many points in $\omega(p)$ that belong to the Σ.

$\Rightarrow$ The intersection with Σ is one point

So q has to belong to a periodic orbit.

$q \in \Gamma$. Then we know that $\Gamma \subseteq \omega(p)$

We would need to prove that $\omega(p) \subseteq \Gamma$. I'll finish the argument next week.