Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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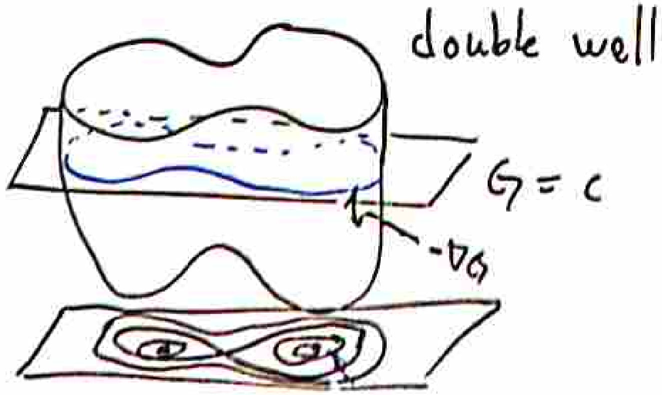
- We know that eigenvalues are real.
- We also know that $-\nabla G$ points in the steepest descent direction

$\Rightarrow$ **Prop**

A non-degenerate critical point of the gradient flow of G is always a saddle or a node of $\dot{x} = -\nabla G(x)$

Moreover, if the critical point is a strict local min (local max) of G, then we have a stable (or unstable) node.



To prove this prop rigorously we use the concept of a Lyapunov function.

Def Let x_0 be a fixed point of $\dot{x} = f(x)$, $x \in \mathbb{R}^n$.

A continuous function $V: \mathcal{U} \to \mathbb{R}$ where $\mathcal{U} \subset \mathbb{R}^n$ is open and contains x_0, is a Lyapunov function for $\dot{x} = f(x)$ at x_0 if

(i) $V(x_0) = 0$

(ii) $V(x) > 0$ if $x \in \mathcal{U} \setminus \{x_0\}$

(iii) $V \in C^1(\mathcal{U} \setminus \{x_0\})$ and if $x(t)$ is a sln in this set, then $\frac{d}{dt} V(x(t)) = \nabla V(x(t)) \cdot f(x(t)) \leq 0$ (stability)

Moreover,

(iv) If $\dot{V}(x(t)) < 0$ $\forall x \in \mathcal{U} \setminus \{x_0\}$ (asymptotic stability)

then V is a strict Lyapunov function.

Thm [Chicone 06, Thm 1.55]

If there is a Lyapunov function V at x_0; then x_0 is stable. If V is a strict Lyapunov function, then x_0 is asymptotically stable (The proof is in the book of Chicone).

Proof of the proposition

•) The fact that the fixed point is a saddle or a node follows from the fact that eigenvalues are real.

••) Suppose that the min (max) of G is strict, then we need to prove that the node is stable (unstable)

We define $V(x) = G(x) - G(x_0)$

from (iii) $\dot V(x(t)) = \nabla G(x(t)) \cdot (-\nabla G(x(t))) = -\|\nabla G(x)\|^2 < 0$

this implies (iv) $\Rightarrow$ min $\leftrightarrow$ stable node

max $\leftrightarrow$ unstable node //

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(i) $V(x_0) = 0$ ✓ $V(x_0) = G(x_0) - G(x_0)$

(ii) $V(x) > 0$ if $x \in \mathcal{U} \setminus \{x_0\}$ ✓ Since the min is strict, possibly l

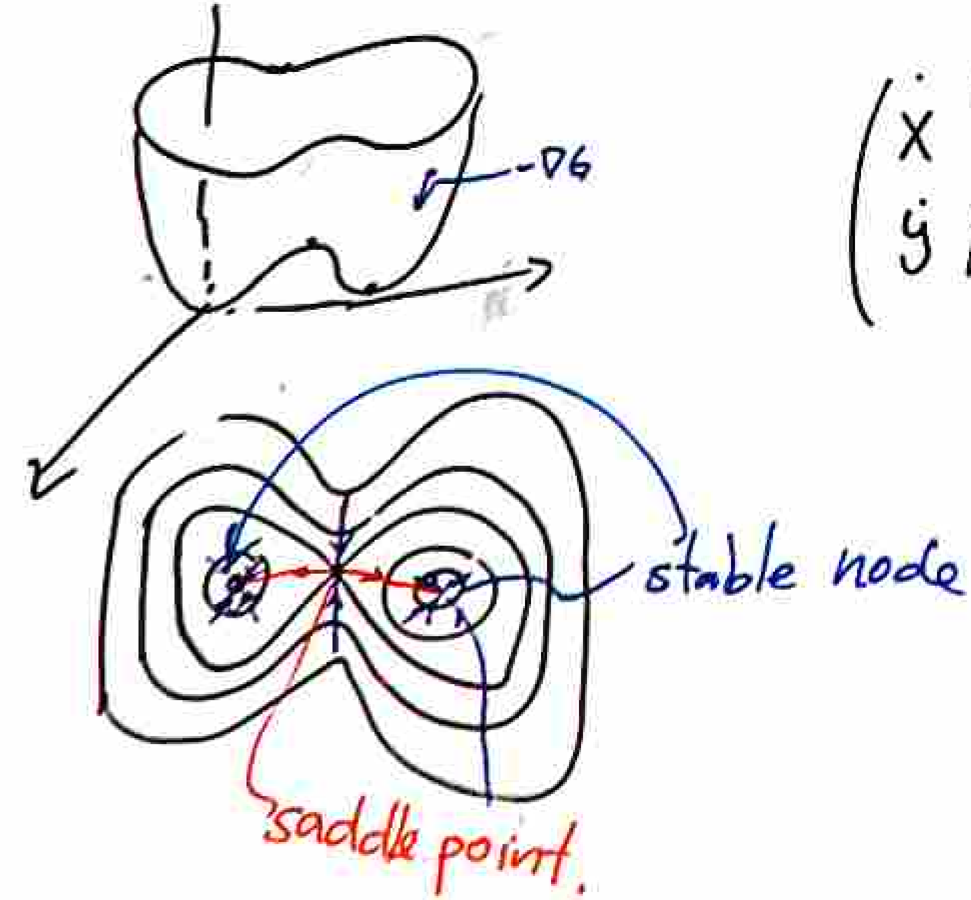
(iii) $V \in C^1(\mathcal{U} \setminus \{x_0\})$ and if $x(t)$ is a sln in this set, then $\frac{d}{dt} V(x(t)) = \nabla V(x(t)) \cdot f(x(t)) \leq 0$ (stability) ✓ $\dot V(x(t)) = -\|\nabla G(x)\|^2 < 0$

Moreover,

(iv) If $\dot V(x(t)) < 0$ $\forall x \in \mathcal{U} \setminus \{x_0\}$ (asymptotic stability) then V is a strict Lyapunov function. ✓

Example

$$G(x,y) = x^2(x-1)^2 + y^2$$



$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = -\nabla G(x,y)$$

Systems on the plane & Poincaré-Bendixson theory

We have a pretty complete description of the asymptotic dynamics on the plane.

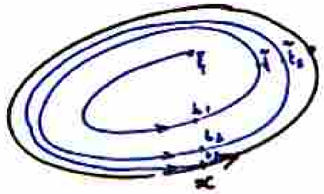
Def Let $\varphi(t, \xi)$ in $\mathbb{R}^n$ be the flow of a system.

A point $x \in \mathbb{R}^n$ is an ω-limit point of $\xi \in \mathbb{R}^n$ if there exists a sequence of times $t_1 \le t_2 \le t_3 \le \dots$ s.t. $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \varphi(t_n; \xi) = x$.

The ω-limit set is the collection of all the ω-limit points an is denoted $\omega(\xi)$.

The corresponding set for $t_n \to -\infty$ is called the α-limit set.

A stable limit cycle



$$\varphi(t_n, \xi) \xrightarrow{t_n \to \infty} x \Rightarrow x \in \omega(\xi)$$

In fact it is clear that the stable limit cycle is $\omega(\xi)$.

About α & ω-limit sets.

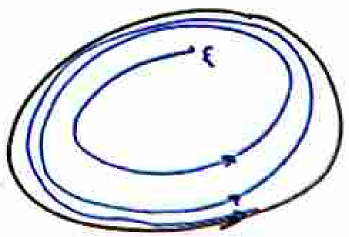
Prop [Chione 06, Prop 1.167]
The ω-limit set of a point is closed and invariant.

Pf
First we note that the empty set is closed and invariant.
If it is not empty take $x \in \omega(\xi)$
$\Rightarrow \exists \{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ s.t. $t_n \xrightarrow[n\to\infty]{} \infty$ $\varphi(t_n, \xi) \xrightarrow[n\to\infty]{} x$
Let's start flowing from a point x. Take $T \in \mathbb{R}$
$\varphi(T, x)$. Consider the sequence $\{T+t_n\}_{n=1}^{\infty}$
$T+t_n \xrightarrow[n\to\infty]{} \infty$. Now, notice that
$\lim\limits_{n\to\infty} \varphi(T+t_n; \xi) = \lim\limits_{n\to\infty} \varphi(T; \varphi(t_n; \xi)) = \varphi(T; x)$
So $\varphi(T; x) \in \omega(\xi)$ and $\omega(\xi)$ is invariant.

To see that $\omega(\xi)$ is closed we note that

$$\omega(\xi) = \bigcap_{\tau \geq 0} cl\,\{\varphi(t;\xi) : t \geq \tau\}$$

[The intersection of a collection of closed sets. Which is necessarily closed.



About α & ω-limit sets.

Prop [Perko 96, pg 191]

Let $p \in \mathbb{R}^n$ and suppose that the orbit that passes through p is contained in a compact set of $\mathbb{R}^n$. Then $\omega(p)$ is non-empty, compact and connex. $\omega(p)$ is invariant.

Proof (Perko's book)

- The ω-limit set can be very complicated
  - → homoclinic and heteroclinic cycles, separatrix cycles, strange attractors.
- This sets are a lot more complicated in dim ≥ 3. In dimension 2 they are very well characterized.
  - → In dim 2 we have the Jordan curve theorem.

Thm [Chicone 06, Thm 1.175] (Jordan curve thm)

A (continuous) simple closed curve divides the plane into two connected components, one of them is bounded and the other one is not bounded. Both components have the closed curve as their boundary. (Pf: Basic result of algebraic topology).

→ A consequence of this theorem is Poincaré-Bendixson thm

About α & ω-limit sets.

Thm [Chicone 06, Thm 1.174]

If Ω is compact, non-empty, and is the ω-limit set of a flow in $\mathbb{R}^2$, and if Ω does not contain any fixed point, then Ω is a periodic orbit.

Def

A <u>connecting orbit</u> between 2 fixed points p and q is an orbit whose α-limit is q and whose ω-limit is p. (It is possible that $p=q$).

→ A heteroclinic orbit is a connecting orbit from p to q

→ A homoclinic orbit is a connecting orbit from p to itself.

Thm

Suppose that $S \subset \mathbb{R}^2$ is a positively invariant set with compact closure. Suppose aswell that there is a finite number of fixed points inside the closure of S. If $\xi \in S$, then $\omega(\xi)$ is:

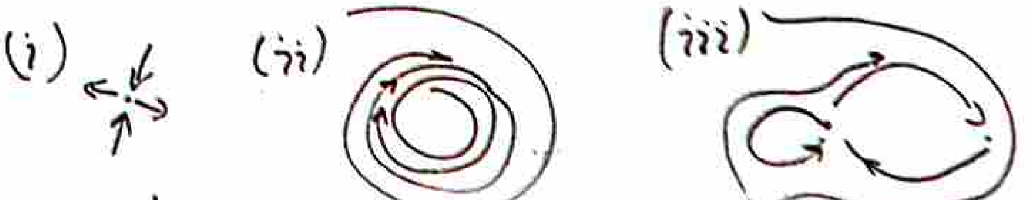
(i) A fixed point.

(ii) A periodic orbit.

(iii) The union of finite number of fixed points with a non-empty set or a countable collection of connecting orbits.

This is a complete characterization of the ω-limit sets of a p

(i) (ii) (iii)



Notice that there are no strange attractors.

About α & ω-limit sets.

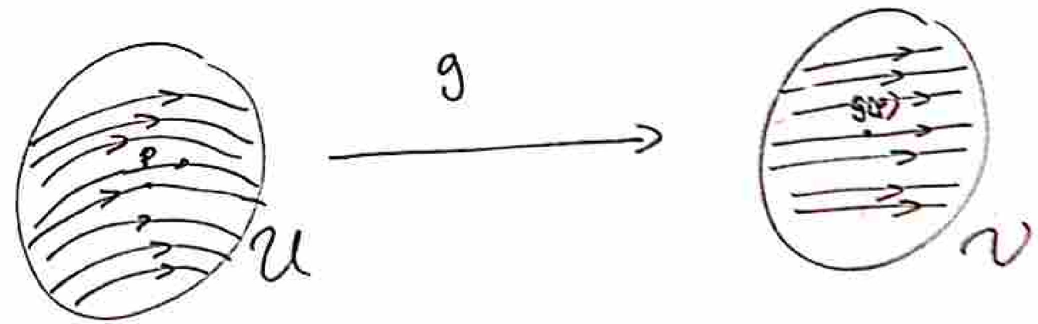
Def (Limit cycle)

A limit cycle is a periodic orbit that is the α or ω-limit of some point that is not on the periodic orbit.

We need some preliminary results to prove Poincaré-Bendixson.

Lemma (Rectification lemma)

Let $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$ and $f(p) \neq 0$ for $p \in \mathbb{R}^n$. Then there exist open sets $\mathcal{U}$ & $\mathcal{V} \subseteq \mathbb{R}^n$ with $p \in \mathcal{U}$ and a diffeomorphism $g : \mathcal{U} \to \mathcal{V}$ such that the diff eq under the new coordinates $y = g(x)$, i.e., $\dot{y} = Dg(g^{-1}(y))\, f(g^{-1}(y))$ is given by the trivial flow $(\dot{y}_1, \dots, \dot{y}_n) = (1, 0, 0, \dots, 0)$



g - rectifies the flow in a neighborhood $\mathcal{U}$ to another neighborhood $\mathcal{V}$.

This is going to be usefull for the proof of the Poincaré-Bendixson theorem because we are going to need to have an explicit expression of the flow.