Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

Renato Calleja, 31 de enero de 2024

Ecuaciones Diferenciales Ordinarias.

Tareas (~6) 40% (físicamente o vía email).

2 exámenes parciales 60% (presenciales) (mediados de abril / finales de mayo ó junio)

Participación **Bono.**

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Banach Spaces

Stefan Banach

Banach spaces

Which properties of $\mathbb{R}^n$ can be translated into infinite dimensions?

Abstract linear vector spaces

Take X a vector space over $\mathbb{R}$ (or $\mathbb{C}$). This is a collection of elements $\{x, y, z, \dots\}$ such that for every $x, y \in X$

i) $x + y \in X$

ii) $x + y = y + x$

iii) There is an element $0 \in X$ s.t. $x + 0 = x$

iv) If $a, b \in \mathbb{R}$ (or $\mathbb{C}$) $ax \in X$ (scalar multipication)

$1 \cdot x = x$ $\quad (ab)x = a(bx) = b(ax)$ $\quad \forall x \in X$

$(a+b)x = ax + bx$

Normed linear space

If for every $x \in X$ there is a number $|x|$ that is called the norm of x that satisfies,

i) $|x| > 0$ for $x \neq 0$ and $|0| = 0$

ii) $|x + y| \leq |x| + |y|$ (triangle inequality)

iii) $|ax| = |a| \cdot |x|$ for every $a \in \mathbb{R}$ (or $\mathbb{C}$) and $x \in X$

Cauchy sequences

A sequence $\{x_n\}$ in a normed linear space $\mathfrak{X}$ converges to an $x \in \mathfrak{X}$ if

$$\lim_{n\to\infty} |x_n - x| = 0$$

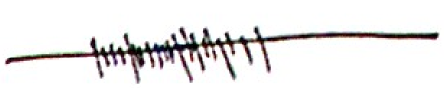
(We write $\lim_{n\to\infty} x_n = x$)

__Cauchy Sequences__

$\{x_n\}$ is a __Cauchy Sequence__ if $\forall \varepsilon > 0$ $\exists$ $N(\varepsilon) > 0$ such that $|x_n - x_m| < \varepsilon$ whenever $n, m \geq N(\varepsilon)$.

We will say that a space $\mathfrak{X}$ is __complete__ whenever all the Cauchy Sequences converge to an element $x \in \mathfrak{X}$

$\mathbb{Q} \subset \mathbb{R}$ Careful This is not a vec space over the reals.



- A complete normed linear space is called a __Banach Space__
- A ε-neighborhood of $x \in \mathfrak{X}$ is $\{y \in \mathfrak{X} : |y - x| < \varepsilon\}$

$S \subset \mathfrak{X}$ is open if $\forall x \in \mathfrak{X}$ $\exists$ an ε-neighborhood of x contained in S.

- $x \in \mathfrak{X}$ is a __limit point__ of S if every ε-neighborhood of a point x contains points of S.
- S is __closed__ if it contains all of its limit pts.

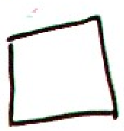
S

- The __closure__ of S is the union of S with all of its limit points.
- S is __dense__ in $\mathfrak{X}$ if its closure is S.

Compact sets

- $S \subset X$, $A \subset \mathbb{R}$, & $V_a, a \in A$ is a collection of open sets of X s.t. $\bigcup_{a \in A} V_a \supset S$, then V_a is an open cover of S.
- $S \subset X$ is compact if every V_a open cover contains a finite subcover.

$\mathbb{R}^n$

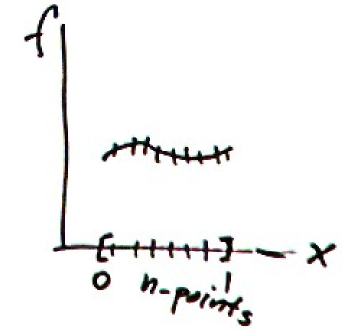


— equivalent to closed and bounded.

In Banach spaces the definition of compact is equivalent to the following,
$S \subset X$ (Banach) is compact if every sequence $\{x_n\}$ has a converging subsequence.

Example 1 $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a Banach space
With any norm $|x| = \sup_i |x_i|$, $\Sigma_i |x_i|$, $\sqrt{\Sigma_i |x_i|^2}$

Example 2
Let Ω be a compact subset of $\mathbb{R}^n$ (or $\mathbb{C}^n$) and let $C(\Omega, \mathbb{R}^n)$ be the linear space of continuous functions that take Ω into $\mathbb{R}^n$.



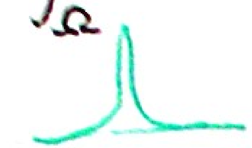
$$\begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix} \in \mathbb{R}^n$$

This property does not carry through with any norm.

We are going to prove that $C(\Omega, \mathbb{R}^n)$ is a Banach space with a particular choice of norm.

Norm 1
$$\|\phi\| = \max_{x \in \Omega} |\phi(x)|$$

Norm 2
$$\|\phi\|_1 = \int_\Omega |\phi(x)| dx$$



Arzela-Ascoli

A sequence of functions $\{\phi_n\}$ $n=1,2,3,\ldots$ in $C(\Omega,\mathbb{R}^n)$ <u>converges uniformly</u> in Ω if there exists a function ϕ such that $\forall\, \varepsilon>0$ $\exists\, N(\varepsilon)$ (depends only on ε) s.t.

$$|\phi_n(x)-\phi(x)|<\varepsilon \quad \forall n\geq N(\varepsilon),\ \forall x\in\Omega$$

Such a sequence is called <u>uniformly bounded</u> if there exists $M>0$ that

$$|\phi_n(x)|<M \quad \forall x\in\Omega$$

and $\forall\, n=1,2,\ldots$

And $\{\phi_n\}$ is <u>equicontinuous</u> if given $\varepsilon>0$ there exists a $\delta>0$ s.t.

$$|\phi_n(x)-\phi_n(y)|<\varepsilon,\quad n=1,2,\ldots$$

whenever $|x-y|<\delta$, $\forall x,y\in\Omega$.

A function $f\in C(\Omega,\mathbb{R}^n)$ is Lipschitz in Ω if there exists a constant k such that,

$$|f(x)-f(y)|\leq k|x-y|,\quad \forall x,y\in\Omega.$$

A sequence $\{\phi_n\}$ that all have the same Lipschitz constant is equicontinuous.

<u>Lema</u> (Arzela-Ascoli)

Any sequence of functions in $C(\Omega,\mathbb{R}^n)$ that is equicontinuous and uniformly bounded has a convergent subsequence.

<u>Lema</u>

If a sequence in $C(\Omega,\mathbb{R}^n)$ converges uniformly in Ω then the limiting function is in $C(\Omega,\mathbb{R}^n)$

$C(\Omega, \mathbb{R}^n)$ is complete

Norm 1

$$\|\phi\| = \max_{x\in\Omega} |\phi(x)|,$$

Let's prove that $C(\Omega, \mathbb{R}^n)$ is a Banach space.

Proof

$\{\phi_n\}$ is a Cauchy sequence in $C(\Omega, \mathbb{R}^n)$

$\Rightarrow \forall \varepsilon > 0 \quad \exists N(\varepsilon)$ s.t.

$\|\phi_n - \phi_m\| < \varepsilon \quad$, if $n,m \geq N(\varepsilon)$

For every $x \in \Omega$, $n,m \geq N(\varepsilon)$

$|\phi_n(x) - \phi_m(x)| \leq \|\phi_n - \phi_m\| < \varepsilon$

$\phi_n(x) \longrightarrow \phi(x)$ since $\mathbb{R}^n$ is complete.

Let's prove that the convergence is uniform.

We have that

$|\phi_n(x) - \phi_m(x)| < \varepsilon$

take $\lim_{m\to\infty} \Rightarrow |\phi_n(x) - \phi(x)| < \varepsilon$

$\forall n \geq N(\varepsilon)$, $\forall x \in \Omega$.

We know that ϕ_n is bounded

$|\phi_n(x) - \phi(x)| < 1$

then since Ω is compact then

ϕ is continuous by the lema.

We need to prove that $\|\phi_n - \phi\| \to 0$

$\|\phi_n - \phi_m\| < \varepsilon$, $n,m \geq N(\varepsilon)$

We know that $\forall x \in \Omega$

$|\phi_n(x) - \phi(x)| = \lim_{m\to\infty} |\phi_n(x) - \phi_m(x)| < \varepsilon$

$\|\phi_n - \phi\| < \varepsilon \Rightarrow \lim_{n\to\infty} \|\phi_n - \phi\| = 0$

$C(\Omega, \mathbb{R}^n)$ is complete. !!