Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

Renato Calleja, 25 de abril de 2024

Thm

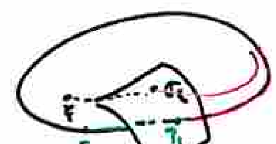
Γ asympt. stable, then $\exists\, V$ nbhd of Γ s.t.
$\forall \xi \in V$ there is $\eta \in \Gamma$ s.t.

$$\lim_{t\to\infty} |u(t;\xi) - u(t;\eta)| = 0$$

Pf

Claim $\sigma_\xi \in \Sigma$, there is $q_\xi \in \Gamma$ s.t.

$$\lim_{t\to\infty} |u(t,\sigma_\xi) - u(t,q_\xi)| = 0$$



Notice that if this works, then it should work $\forall \xi \in V$,
For instance, we know that $\exists\, t_\xi \in [0, 2T(p)]$ st. $u(t_\xi,\xi) = \sigma_\xi$ $\quad \xi = u(-t_\xi, \sigma_\xi)$
so we define $r = u(-t_\xi, q_\xi)$

$$|u(t,\xi) - u(t,r)| = |u(t-t_\xi, \sigma_\xi) - u(t-t_\xi, q_\xi)| \to 0$$

If the claim is true, the thm follows.

(Claim)
Take $\sigma \in \Sigma$ and define

$$S_n := n\,T(p) - \sum_{j=0}^{n-1} T(P^j(\sigma))$$

$$T(p) = T(P^n(\sigma)) + \underbrace{(n+1)T(p) - \sum_{j=0}^{n} T(P^j(\sigma))}_{= S_{n+1}} \underbrace{- n\,T(p) + \sum_{j=0}^{n-1} T(P^j(\sigma))}_{= -S_n}$$

$$= T(P^n(\sigma)) + S_{n+1} - S_n \Rightarrow S_{n+1} - S_n = T(p) - T(P^n(\sigma))$$

$$\Rightarrow |S_{n+1} - S_n| = |T(p) - T(P^n(\sigma))| \le 2\|DT(p)\|\,|p - P^n(\sigma)| \qquad p = P^n(p)$$

$$\le 2c\,\|DT(p)\|\,\lambda^n\,|p-\sigma|, \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \sum_{n=0}^{\infty} |S_{n+1} - S_n| \le 2c\,\|DT(p)\|\,|p-\sigma| \sum_{n=0}^{\infty} \lambda^n = \frac{1}{1-\lambda} < \infty$$

$\to$ Then the series is absolutely convergent.

So $|S_{n+1} - S_n| \xrightarrow[n\to\infty]{} 0$

Also $|S_n| \le |S_1| + \sum_{j=1}^{n} |S_{j+1} - S_j| \le |S_1| + 2c\,\|DT(p)\|\,|p-\sigma|\,\frac{1}{1-\lambda}$

$$\lim_{n\to\infty} S_n = S$$

Thm

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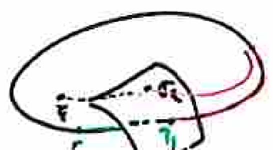

Notice that if this works, then it should work $\forall \xi \in V$,
For instance, we know that $\exists\, t_\xi \in [0, 2T(p)]$ st. $u(t_\xi, \xi) = \sigma_\xi$
$\xi = u(-t_\xi, \sigma_\xi)$
so we define $r = u(-t_\xi, \eta_\xi)$
$$|u(t,\xi) - u(t,r)| = |u(t-t_\xi, \sigma_\xi) - u(t-t_\xi, \eta_\xi)| \to 0$$
If the claim is true, the thm follows.

Pf (Claim)
Take $\sigma \in \Sigma$ and define
$$S_n := n\,T(p) - \sum_{j=0}^{n-1} T(\rho^j(\sigma))$$
$$T(p) = T(\rho^n(\sigma)) + \underbrace{(n+1)T(p) - \sum_{j=0}^{n} T(\rho^j(\sigma))}_{=S_{n+1}} \underbrace{- n\,T(p) + \sum_{j=0}^{n-1} T(\rho^j(\sigma))}_{=-S_n}$$
$$= T(\rho^n(\sigma)) + S_{n+1} - S_n \Rightarrow S_{n+1} - S_n = T(p) - T(\rho^n(\sigma))$$
$$\Rightarrow |S_{n+1} - S_n| = |T(p) - T(\rho^n(\sigma))| \le 2\|DT(p)\|\,|p - \rho^n(\sigma)|$$
$p = \rho^n(p)$
$$\le 2c\,\|DT(p)\|\,\lambda^n\,|p - \sigma|, \quad \forall n \in \mathbb{N}.$$
$$\Rightarrow \sum_{n=0}^{\infty} |S_{n+1} - S_n| \le 2c\,\|DT(p)\|\,|p-\sigma| \sum_{n=0} \lambda^n = \frac{1}{1-\lambda} < \infty$$
Then the series is absolutely convergent.
So $|S_{n+1} - S_n| \xrightarrow[n\to\infty]{} 0$
Also $|S_n| \le |S_1| + \sum_{j=1}^{n} |S_{j+1} - S_j| \le |S_1| + 2c\,\|DT(p)\|\,|p-\sigma| \frac{1}{1-\lambda}$
$$\lim_{n\to\infty} S_n = S$$

Thm

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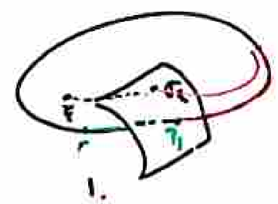

Now let $\varepsilon > 0$

By compactness of $u: [-K,K] \times \Sigma \to \mathbb{R}^n$
then u is unif. continuous, so there is $\delta > 0$ s.t.
if $|t_1 - t_2| + |\sigma_1 - \sigma_2| < \delta$ then

$$|u(t_1,\sigma_1) - u(t_2,\sigma_2)| < \varepsilon$$

Pf (Claim)

For every $t \geq 0$ there is an n integer and a number $s(t)$ s.t.,

$$t = nT(p) + s(t), \quad 0 \leq s(t) \leq 2T(p)$$

$$s(t) = t - nT(p)$$

For $q = u(s;p)$,

$$|u(t;\sigma) - u(t;q)| = |u(s(t), u(nT(p),\sigma)) - u(s(t), \overbrace{u(nT(p),q)}^{q})|$$

$$= |u(s(t), u(nT(p),\sigma)) - u(s(t),q)|$$

$$\leq |u(nT(p),\sigma) - q|\, e^{LT(p)} \xrightarrow[n\to\infty]{} 0$$

This proves the claim and by the observation it also proves the theorem //

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Using the concept of the asymptotic phase we can define an "isochrone" in a neighborhood of the limit cycle.

Def (Isochrone)

The isochrone is the geometrical site of all the points that have the same asymptotic phase.

$$\Gamma_\eta = \{\xi \in V : u(t,\xi) \text{ has the asymp. phase} = \eta\}$$

(In fact this set is a manifold, and the collection of all these manifolds for $\eta \in \Gamma$ foliate the basin of attraction of Γ.)