Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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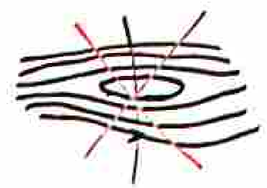
Thm Γ per orb of $\dot{u} = f(u)$, P - Poincaré map defined in Σ, $p \in \Sigma \cap \Gamma$. If eig $(DP(p))$ are inside the unit circle, Γ is asymptotically stable.

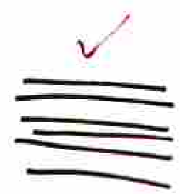
Pf

$\|DP(0)\| < \lambda < 1$, $\Sigma_0 \subset \Sigma \cap W$

$T(\sigma) \leq 2T^*$, $\|DP(\sigma)\| < \lambda < 1$, $\forall \sigma \in \Sigma_0$

We defined $\mathcal{U} = \{u(t;\sigma) : \sigma \in \Sigma_0, t > 0\}$





Take a point in $\mathcal{U}$

$\xi = u(\bar{t}, \bar{\sigma}) \in \mathcal{U}$, and consider

$H : \mathbb{R}^+ \times \Sigma_0 \longrightarrow \mathbb{R}^n$

$H(t,\sigma) = u(t,\sigma)$

And we will show that $DH(t,\sigma)$ is invertible.

Then the IFT gives us open neighborhoods $\tilde{V}$ of $(\bar{t},\bar{\sigma})$ and $\tilde{\mathcal{U}}$ of ξ.

In fact H is a diffeomorphism. There is a basis $\{f(\xi), s_1, \dots, s_{n-1}\}$ of $\mathbb{R}^n$ $(*)$

$$\underset{\text{in the basis }(*)}{DH} = \begin{pmatrix} \partial_t u_1 & \partial_1 u_1 & \cdots & \partial_n u_n \\ \vdots & \vdots & & \\ & & & \partial_n u_n \\ \partial_t u_n & \partial_1 u_n & \cdots & \end{pmatrix} \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & f(\xi) & \tilde{s}_1 \cdots & \tilde{s}_{n-1} \\ 0 & & & \end{pmatrix}$$

$$= \begin{pmatrix} \partial_t u_1 & \langle \nabla_\sigma u_1, f(\xi)\rangle & \langle \nabla_\sigma u_1, \tilde{s}_1\rangle & \cdots & \langle \nabla_\sigma u_1, \tilde{s}_{n-1}\rangle \\ \vdots & \vdots & \vdots & & \vdots \\ \partial_t u_n & \langle \nabla_\sigma u_n, f(\xi)\rangle & \langle \nabla_\sigma u_n, \tilde{s}_1\rangle & & \langle \nabla_\sigma u_n, \tilde{s}_{n-1}\rangle \end{pmatrix}$$

$n \times (n+1)$ matrix.

. The gradient $\nabla_\sigma u_j$ is orthogonal to Σ.

So $\langle \nabla_\sigma u_j, f(\xi)\rangle = 0$ and $\langle \nabla_\sigma u_j, \tilde{s}_\ell\rangle$ has rank $n-1$.

and $\displaystyle \frac{d}{dt} u(t+\bar{t}, \bar{\sigma})\Big|_{t=0} = \frac{d}{dt} u(t, \underbrace{u(\bar{t},\bar{\sigma})}_{=\xi})\Big|_{t=0} = \frac{d}{dt} u(t,\xi)\Big|_{t=0}$

$= f(\xi)$

So the matrix has rank n and H is a diffeomorphism.

Thm Γ per orb of $\dot{u} = f(u)$, P - Poincaré map defined in Σ, $p \in \Sigma \cap \Gamma$. If eig $(DP(p))$ are inside the unit circle, Γ is asymptotically stable.

Pf

$\|DP(0)\| < \lambda < 1$, $\Sigma_0 \subset \Sigma \cap W$

$T(\sigma) \le 2T^*$, $\|DP(\sigma)\| < \lambda < 1$, $\forall \sigma \in \Sigma_0$

Now, we need to see that $\mathcal{U} \subset V$. ($\xi = u(\bar{t}, \bar{\sigma})$)

We notice that if $\xi = u(\bar{t}, \bar{\sigma}) \in \mathcal{U}$ then there exist n, s,

$$\bar{t} = \sum_{j=0}^{n-1} T(P^j(\sigma)) + s, \quad 0 \le s < T(P^n(\sigma)) < 2T^*$$

Then $u(\bar{t}, \bar{\sigma}) = u\left(s + \sum_{j=0}^{n-1} T(P^j(\sigma)), \sigma\right) = u(s, P^n(\bar{\sigma}))$

Finally, to see that solutions with init. cond inside of $\mathcal{U}$ converge to the Γ, take $\xi \in \mathcal{U}$

$\xi = u(\bar{t}, \bar{\sigma})$

Now consider the distance

$$\mathrm{dist}(u(t, \xi), \Gamma) = \mathrm{dist}(u(t, u(s, P^n(\bar{\sigma}))), \Gamma)$$

$$= \mathrm{dist}(u(t+s, P^n(\bar{\sigma})), \Gamma) \longrightarrow 0$$

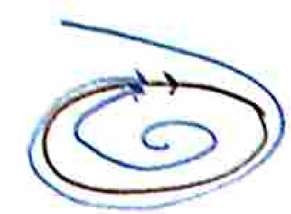
Now, we have to ~~take~~ n and t to ∞.

- Observe that n and t both go to ∞ together.

So Γ is asympt. stable. //

Def (Limit cycle)

An asymptotically stable periodic orbit is called a limit cycle.



Asymptotic period and phase

Coupled oscillators.

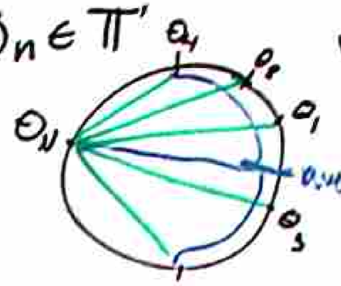
There is a question about coherence.

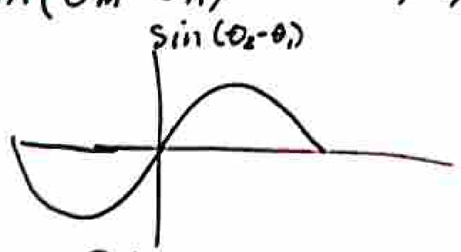
- Related to synchronization.

Kuramoto model

$$\dot{\theta}_n = \omega_n + \frac{k}{N}\sum_{m=0}^{N} \sin(\theta_m - \theta_n) \qquad n=1,\dots,N$$

$\theta_n \in \mathbb{T}^1$





Steven Strogatz
(Cornell Univ.)

Mean Field. Every particle interacts with a mean field as opposed to interacting with each particle.

When there is a Limit cycle we have asymptotic periods and phases.

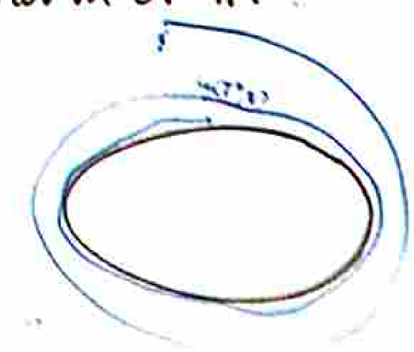
Thm (Prop 2.131, Chicone '06) [Asymptotic period]

Let Γ be a limit cycle with period T^*.

Then there exists a neighborhood V of Γ s.t. if $\xi \in V$ then

$$\lim_{t\to\infty} | u(t+T^*;\xi) - u(t;\xi) | = 0$$

$|\cdot|$ is any norm of $\mathbb{R}^n$



Asymptotic period and phase

Pf

Let $\varepsilon > 0$, by a previous lemma $\exists\, W$ neighborhood of Γ and $\delta < \varepsilon/2$ s.t. for $\xi, \eta \in W$, $|\xi - \eta| < \delta$

$\Rightarrow |u(t;\xi) - u(t;\eta)| < \varepsilon/2$, $t \in [0, 2T^*]$.

In particular $|u(T^*, \xi) - u(T^*, \eta)| < \varepsilon/2$

By hypothesis there $\exists\, \tau > 0$ suff. large s.t.

$\text{dist}(u(t;\xi), \Gamma) < \delta$, $\forall t \geq \tau$

For every $t \geq \tau$ there is a $\eta \in \Gamma$ s.t.

$|u(t,\xi) - \eta| < \delta$

Then $|u(t+T^*, \xi) - u(t,\xi)| \leq |u(t+T^*, \xi) - u(T^*, \eta)|$

$+ |u(T^*, \eta) - u(t,\xi)| < \varepsilon/2 + \delta < \varepsilon$ //

When there is a Limit cycle we have asyptotic periods and phases!

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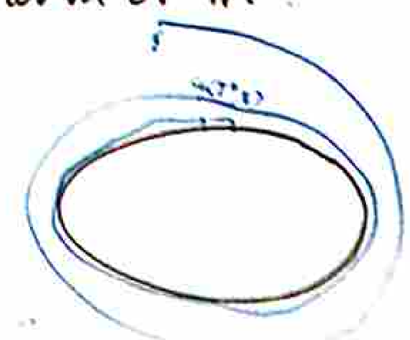


Asymptotic period and phase

If we require hyperbolicity of the limit cycle, then

Thm (Asymptotic phase)

If Γ is asymptotically stable then there exists a neighborhood V of Γ such that for every $\xi \in V$ there exists $\eta \in \Gamma$ s.t.

$$\lim_{t\to\infty} |u(t;\xi) - u(t;\eta)| = 0$$

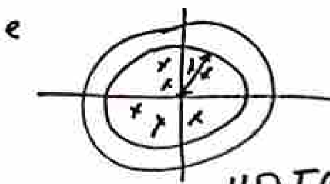
Pf Let Σ be a Poincaré section over $p \in \Gamma$, with P, and T.
By the hypothesis of the theorem and the regularity of T we can assume that $\forall \sigma \in \Sigma$,

$$\lim_{n\to\infty} P^n(\sigma) = p, \quad T(\sigma) < 2T(p) \quad \text{and}$$

$$\|DT(\sigma)\| < 2\|DT(p)\|$$

Hyperbolicity means that there is a $c > 0$, $\lambda \in (0,1)$ with $|p - P^n(\sigma)| \le c\lambda^n |p - \sigma|$ ($\Leftrightarrow |P^n(p) - P^n(\sigma)| \le c\lambda^n |p - \sigma|$)

since



Define $K := \dfrac{2c\|DT(p)\|}{1-\lambda} \cdot \sup_{\sigma \in \Sigma} |p - \sigma| + 2T(p)$

Now using the IFT we can prove that $\exists V$ ngbhd of Γ s.t. $\forall \xi \in V$ there is a time $t_\xi \ge 0$ with $\sigma_\xi := u(t_\xi, \xi) \in \Sigma$.

By the lemma we can choose V such that every solution with initial condition in V is defined for times $t \in [-K, K]$

By asymptotic stability we know that any solution defined in this way is going to be defined for every positive time.

Asymptotic period and phase

If we require hyperbolicity of the limit cycle, then

Thm (Asymptotic phase)

If Γ is asymptotically stable then there exists a neighborhood V of Γ such that for every $\xi \in V$ there exists $q \in \Gamma$ s.t.

$$\lim_{t\to\infty} |u(t;\xi) - u(t;q)| = 0$$

Pf

and if we restrict V to the set given by solutions that we can follow for a time K, then every solution in the set will be defined $\forall t \in [-K, K]$.

Next class we will prove the following claim

Claim Given any $\sigma_\xi \in \Sigma$, there is a $q_\xi \in \Gamma$ such that $\lim_{t\to\infty} |u(t,\sigma_\xi) - u(t;q_\xi)| = 0$

Hyperbolicity means that there is a $c>0$, $\lambda \in (0,1)$ with $|p - P^n(\sigma)| \le c\lambda^n |p-\sigma|$ ($\Leftrightarrow |P^n(p) - P^n(\sigma)| \le c\lambda^n |p-\sigma|$)

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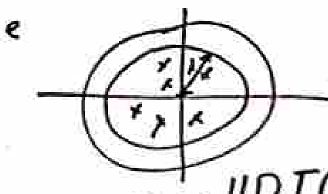

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