Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

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Variedades tranversales

$P, Q \subset \mathbb{R}^n$: subvariedades. $\dim M = n$. $\dim P = k_1$.
$\dim Q = k_2$.

$x \in P \cap Q$. $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k_1}$.

$g : V \subset \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k_2}$.

$h : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k_1+n-k_2}$.

$x \longmapsto (f(x), g(x))$ $\quad T_xP + T_xQ = \mathbb{R}^n$

$\underline{dh_x} = (df_x, dg_x)$.

tiene rango máx.

$k_1 + k_2 \geq n \Rightarrow \underline{n-k_1+n-k_2} \leq n$

Por teo. de la dim

$n = \dim \operatorname{Ker} dh_x + \dim \operatorname{Im} dh_x$.

$\operatorname{Ker} dh_x = \operatorname{Ker} df_x \cap \operatorname{Ker} dg_x$

$T_xP \cap T_xQ$.

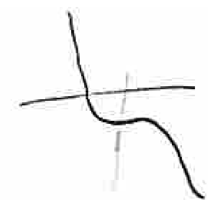
$\dim(T_xP \cap T_xQ) = \dim T_xP + \dim T_xQ - \dim(T_xP + T_xQ)$

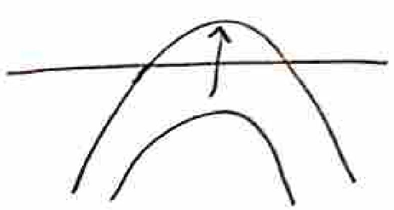
$= k_1 + k_2 - n$.

$\dim \operatorname{Im} dh_x = n + n - k_1 - k_2 = n - k_1 + n - k_2$.

Normal forms

$f_{SN}(u,\alpha) = \alpha - u^2$ $\quad (\alpha - u^3)$





Transversal intersection.

$f_1(u,\alpha) = \underbrace{\alpha_1 + \alpha_2 + \alpha_3}_{\beta} - u^2$

Codimension of a bifurcation

It is the smallest dimension in parameter space that can produce the bifurcation

$f_{AH}(r,\theta,\alpha) = (r(\alpha \pm r^2), 1)$

Normal form

Is an equation (usually simple) that describes a particular bifurcation and that is top. equivallent to another equation having the same bifurcation.

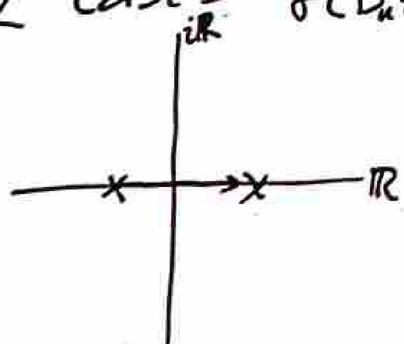
Ex

$\dot{u} = \alpha \pm u^2$ are conjugate to equations that have the S-N bifurcation.

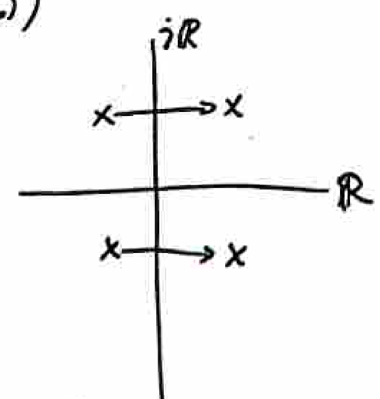
Normal forms (Saddle-node bifurcation)

A local bifurcation can only occur when a fixed point is not hyperbolic.

2 Cases $\sigma(D_u f(u_0, \alpha_0))$



S-N



A-H

Consider

$$\dot u = f(u, \alpha)$$ and assume that

$$f(0,0) = 0 \quad \& \quad Df(0,0) \text{ is not hyperbolic.}$$

Let's prove the following

<u>Lemma</u> [Kuznetsov '98, Lemma 3.1, pg 82]

The system $\dot y = \alpha - y^2 + \psi(y, \alpha) =: F(y, \alpha)$ is topologically equivalent at the origin to

$$\dot u = \alpha - u^2$$

where $\psi(y, \alpha) = O(y^3)$

Normal forms (Saddle-node bifurcation)

$\dot{y} = \alpha - y^2 + \psi(y,\alpha)$ (1) top. equiv. $\dot{u} = \alpha - u^2$ (2)

Pf

Let's consider the set of fixed points of (1)

$\mathcal{M} = \{(y,\alpha) : \alpha - y^2 + \psi(y,\alpha) = F(y,\alpha) = 0\}$

$(0,0) \in \mathcal{M}$ & notice that $D_\alpha F(0,0) = 1$

so by the IFT we can write

$\mathcal{M} = \{\alpha = g(y)\}$ where $g(y)$ is smooth

and defined for $|y|$ small.

Moreover, if we write $\psi(y,\alpha) = C_1(\alpha)y^3 + C_2(\alpha)y^4 + h.o.t.$

So when $\alpha = 0$ we have that

$g(y) = y^2 + O(y^3)$

So for $|y|$ small, there are two fixed points $y_1(\alpha)$, $y_2(\alpha)$ that are close to $\pm\sqrt{\alpha} = x_{1,2}(\alpha)$.

Define

$$h_\alpha(x) = \begin{cases} x, & \text{if } \alpha \le 0 \\ a(\alpha) + b(\alpha)x, & \text{if } \alpha > 0 \end{cases}$$

where a and b are determined by the condition

$h_\alpha(x_j(\alpha)) = y_j(\alpha)$.

It is easy to see that h_α is a homeomorphism and that it conjugates the equations. //

Normal forms (Saddle-node bifurcation)

• $\dot{y} = \alpha - y^2 + \psi(y,\alpha)$ (1) top. equiv. $\dot{u} = \alpha - u^2$ (2)

Now $\dot{x} = f(x,\alpha)$, $x\in\mathbb{R}$, $\alpha\in\mathbb{R}$.

$$f(x,\alpha) = c_0(\alpha) + c_1(\alpha)x + c_2(\alpha)x^2 + O(x^3)$$

We need that f has a fixed point at $(0,0)$

$$f(0,0)=0 \Rightarrow c_0(0)=0$$

Step 1

We use the translation $\xi = x+\delta$, $\delta(\alpha)$

$$\Rightarrow \dot{\xi} = [c_0(\alpha) - c_1(\alpha)\delta + O(\delta^2)] + \xi^2[c_2(\alpha) + O(\delta)] + O(\xi^3)$$

Step 2

$$\mu = \mu(\alpha) = c_0'(0)\alpha + O(\alpha^2)$$
$$= c_0'(0)\alpha + \alpha^2\phi(\alpha)$$

And impose that $\mu'(0) = c_0'(0) = D_\alpha f(0,0) \neq 0$

$$\dot{\xi} = \mu + a(\mu)\xi^2 + O(\xi^3), \quad a(\mu) = c_2(0) + O(\alpha(\mu$$

Step 3

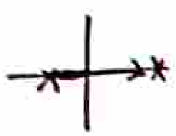
Define $\eta = |a(\mu)|$ and $\beta = |a(\mu)|\mu$.

then

$$\dot{\eta} = \beta + (\operatorname{sign}[a(0)])\eta^2 + O(\eta^3)$$

Thm (Normal form of the saddlenode bifn)

$\dot{x} = f(x,\alpha)$, $x\in\mathbb{R}$, $\alpha\in\mathbb{R}$, f smooth.



Suppose that $f(0,0)=0$, $D_x f(0,0)=0$

•) $D_x^2 f(0,0)\neq 0$, ••) $D_\alpha f(0,0)\neq 0$

Then the system is top. equiv. to $\dot{\eta} = \beta \pm \eta^2$.

Normal forms

In general $\dot{u} = f(u, \alpha)$, $u \in \mathbb{R}^n$

$(0,0)$ is a fixed point

$D_x f(0,0)$ is not hyperbolic.

Using center manifold reduction we can transform

$\dot{x} = F(x, \alpha)$ $\rightsquigarrow$ Normal form.

$\dot{y} = Hy$

Lyapunov-Schmidt reduction

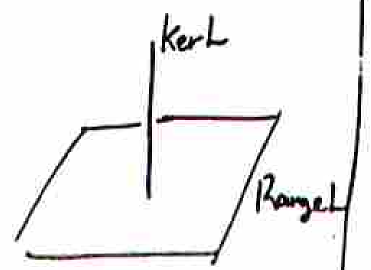
Suppose that we are interested in the fixed points of $\dot{u} = f(u,\alpha)$, $u \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$

Assume that $f(0,0)=0$ and $D_u f(0,0)$ is not hyperbolic.

$f(0,0)=0$, range $D_u f(0,0) = n-1$

$L := D_u f(0,0)$

$K := \ker L$ & $R := \text{range } L$



$\mathbb{R}^n = K \oplus K^\perp$, $\dim K = 1$

$\mathbb{R}^n = R^\perp \oplus R$, $\dim R^\perp = 1$

Let P be the proyection over R.

$P^2 = P$, $\ker P = R^\perp$, $PL = L$

We cannot apply the IFT because we have a kernel.

Since $u \in \mathbb{R}^n = K \oplus K^\perp$, write

$u = v + w$, $v \in K$, $w \in K^\perp$

If we want $f(u,\alpha)=0$ then

$Pf(u,\alpha)=0$

& $(I-P) f(u,\alpha)=0$

Define

$F : K \times K^\perp \times \mathbb{R} \longrightarrow R$

$(v, w, \alpha) \longmapsto Pf(v+w,\alpha)$

$D_w F(v,w,\alpha)\big|_0 = D_w Pf(v+w,\alpha)\big|_0 = PL = L$

this has range $n-1$ and we can apply the IFT.

$\Rightarrow w = w(v,\alpha)$ s.t. $Pf(v + w(v,\alpha),\alpha) = 0$

Lyapunov-Schmidt reduction

Suppose that we are interested in the fixed points of

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^n, \quad \alpha \in \mathbb{R}$$

We use $w(v, \alpha)$ to define

$$\phi : K \times \mathbb{R} \longrightarrow R^{\perp}$$
$$(v, \alpha) \longmapsto (I-P)\, f(v + w(v,\alpha), \alpha)$$

This is a 1 dimensional problem!

$$\phi(v, \alpha) = (I-P)\, f(v + w(v,\alpha), \alpha)$$

↳ Bifurcation equation.

For saddle-node

$$\dot{u} = f(u, \alpha), \quad u \in \mathbb{R}^1, \quad \alpha \in \mathbb{R}$$

A fixed point u_0 is a saddle-node at α_0 if $f(u_0, \alpha_0) = 0$, $D_u f(u_0, \alpha_0)$ has a zero eigenvalue with multiplicity 1.

(SNB1) $\exists$ $p_0 > 0$ and a curve $p \longmapsto \beta(p) \in \mathbb{R}^n \times$

s.t. $\beta(0) = (u_0, \alpha_0)$ and $f(\beta(p)) = 0$ $\forall$ $|p| < p_0$

(SNB2) The function β has a quadratic tangency with $\mathbb{R}^n \times \{\alpha_0\}$ at (u_0, α_0) ($\beta(p) = (\beta_1(p), \beta_2(p))$, $\beta_2(0) = \alpha_0$, $\beta_2'(0) = 0$, $\beta_2''(0)$

(SNB3) $p \neq 0$ then $D_u f(\beta(p))$ is hyperbolic.

We also know that 1 eigenvalue crosses through zero with non-zero velocity.