Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

Renato Calleja, 22 de mayo de 2024

Bifurcation theory

saddle-node $\}$ generic
Andronov-Hopf

pitchfork $\}$ symmetric
trans-critical

Towards topological equivalence.

$$\dot{x} = f(x,\alpha) \quad , \quad \dot{y} = g(y,\beta) \qquad (*)$$

$$x, y \in \mathbb{R}^n \quad , \quad \alpha, \beta \in \mathbb{R}^m$$

with flows

$$\varphi(t; x, \alpha) \ \& \ \psi(t; y, \beta)$$

Def

The two systems $(*)$ are topologically equivalent

if there is a homeomorphism $\beta = p(\alpha)$

of the parameter spaces and a parameter

(continuously) dependent homeomorphism $h_\alpha : \mathbb{R}^n \to \mathbb{R}^n$

of the phase spaces that map orbits

of one system into orbits of the second one

and it preserves the direction of time.

$$h_\alpha(\varphi(t; x, \alpha)) = \psi(t; h_\alpha(x), p(\alpha))$$

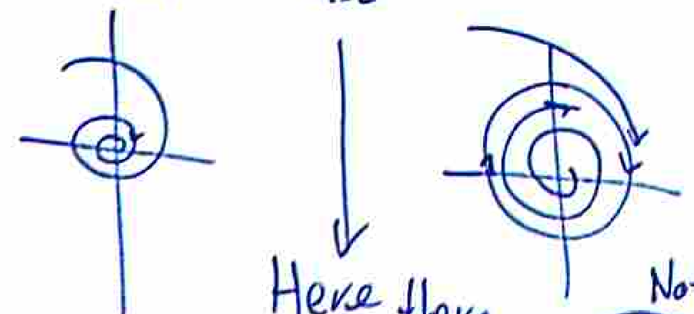
Bifurcation theory

Def

A system $\dot u = f(u,\alpha)$ undergoes a bifurcation at $\alpha=\alpha_0$ if for every $\epsilon>0$ $\exists$ $\alpha_1, \alpha_2 \in B_\epsilon(\alpha_0)$, such that, $\dot u = f(u,\alpha_1)$ & $\dot u = f(u,\alpha_2)$ are not topologically equivalent.

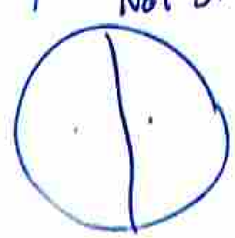
Andronov - Hopf

$\alpha<0$ $\quad$ $\alpha=0$ $\quad$ $\alpha>0$



Here there is a bifn.

Not struct. stable.



Def

We say that a vector field $f:\mathcal{U}\subset\mathbb{R}^n\to\mathbb{R}^n$ is <u>structurally stable</u> (relative to the C^1 topology) if there exists a an open set $\mathcal{V}_{rel\,C^1}$ which is a neighborhood of f s.t. $\forall g\in\mathcal{V}_{rel\,C^1}$ f & g are topologically equivalent.

Note

$$B_\varepsilon(f)=\left\{ g\in C^1(\mathcal{U}) \;\middle|\; \sup_{x\in\mathcal{U}}|f(x)-g(x)|<\varepsilon,\ \sup_{x\in\mathcal{U}}\|Df(x)-Dg(x)\|<\varepsilon \right\}$$

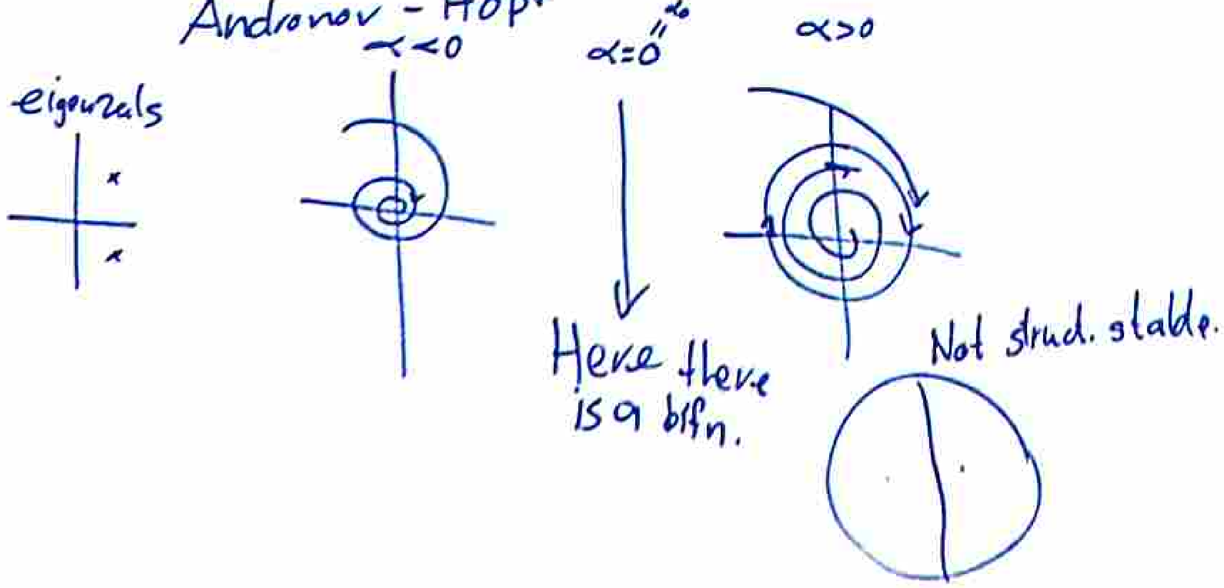
$$h(\varphi(t,x)) = \psi(t,h(x))$$

Bifurcation theory

Def

A system $\dot{u} = f(u, \alpha)$ undergoes a bifurcation at $\alpha = \alpha_0$ if for every $\epsilon > 0$ $\exists$ $\alpha_1, \alpha_2 \in B_\epsilon(\alpha_0)$, such that, $\dot{u} = f(u, \alpha_1)$ & $\dot{u} = f(u, \alpha_2)$ are not topologically equivalent.

Andronov - Hopf



About linear systems

Thm [Robinson '99, Thm 7.1]

Let's supose that the eigenvalues of A & B ($n \times n$ matrices) have real part different from zero and the sum of the dimensions of all the directions of all the eigenvalues with negative real part for A & B are equal (therefore the same should happen for the possitive real parts).

Then e^{tA} and e^{tB} are topologically equivalent.

This plus Grobman-Hartman

Bifurcation theory

We know that for local bifurcations we should look at the place where the real part of the eigenvalues changes sign.

For eigenvalues that have zero real part, we have a center manifold.

So a local bifurcation occurs when there is a center manifold.

Let $\dot{u} = f(u, \alpha)$ and (u_0, α_0) with u_0 is a fixed point for the system $\dot{u} = f(u, \alpha_0)$ so that $Df(u_0, \alpha_0)$ has an eigenvalue with real part equal to zero.

For the moment we assume that the rest of the eigenvalues have negative real part.

We can write the system as follows.

$$\left.\begin{aligned} \dot{\tilde{y}} &= \tilde{C}\tilde{y} + \tilde{F}(\tilde{y}, z, \alpha) \\ \dot{\alpha} &= 0 \\ \dot{z} &= Sz + G(\tilde{y}, z, \alpha) \end{aligned}\right\} \text{we know there is a center manifold here.}$$

Bifurcation theory

Example

$$\dot{x} = -x - y^2 - 2x^2$$
$$\dot{y} = \varepsilon y - xy$$

$u_0 = (0,0)$ fixed point

What is ε_0?

Let's look at the linearization around u_0.

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & \varepsilon\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} - \begin{pmatrix}y^2 + 2x^2\\ xy\end{pmatrix}$$

We look for ε_0 so that the system has a center space.

At $\varepsilon_0 = 0$ there is an eigenvalue that changes sign.

$\rightarrow$ here there is a center manifold.

Center manifold $\underset{\text{stable}}{x} = W(\underset{\text{center}}{y}, \varepsilon) = A(\varepsilon) y^2 + B(\varepsilon) y^3 + \mathcal{O}(y^4)$

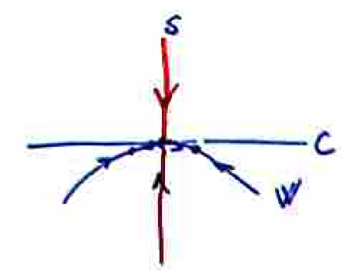
W is invariant $\quad \dot{x} = D_y W(y,\varepsilon)\dot{y}$, $D_y W(y,\varepsilon) = 2A(\varepsilon)y + 3B(\varepsilon)y^2 + \mathcal{O}(y^3)$

$$-x - y^2 - 2x^2 = D_y W(y,\varepsilon)(\varepsilon y - xy)$$

$$-A(\varepsilon)y^2 - B(\varepsilon)y^3 - y^2 - 2\,\mathcal{O}(y^4) = (2A(\varepsilon)y + 3B(\varepsilon)y^2 + \mathcal{O}(y^3))(\varepsilon y - A(\varepsilon)y^3 - B(\varepsilon)y^4) + \mathcal{O}(y^4)$$

We can write the system as follows.

$$\left.\begin{aligned}\dot{\tilde{y}} &= \tilde{C}\tilde{y} + \tilde{F}(\tilde{y}, z, \alpha)\\ \dot{\alpha} &= 0\\ \dot{z} &= Sz + G(\tilde{y}, z, \alpha)\end{aligned}\right\} \xrightarrow{\text{center manifold reduction}} \begin{aligned}\dot{\bar{y}} &= \tilde{C}\bar{y} + \bar{F}(\bar{y}, \bar{z}, \alpha)\\ \dot{\alpha} &= 0\\ \dot{\bar{z}} &= S\bar{z}\end{aligned}$$



At order y^2

$-A(\varepsilon) - 1 = 2\varepsilon A(\varepsilon)$

$\Rightarrow (2\varepsilon + 1)A(\varepsilon) = -1 \Rightarrow A(\varepsilon) = \frac{-1}{2\varepsilon + 1}$

At order y^3

I can come up with an expression for $B(\varepsilon)$

$\rightarrow W(y,\varepsilon) = -\frac{1}{2\varepsilon+1} y^2 + \mathcal{O}(y^3)$

The center manifold is $W(y,0) = -y^2 + \mathcal{O}(y^3)$

We can make a change of variables and re-write the system as,

$$\dot{\bar{x}} = -\bar{x}$$
$$\dot{\bar{y}} = \varepsilon\bar{y} + \frac{1}{2\varepsilon+1}\bar{y}^3 + \mathcal{O}(\bar{y}^4) \rightarrow \text{looks like } \dot{u} = \alpha u + u^3$$

There is a pitchfork bifurcation.

Bifurcation theory

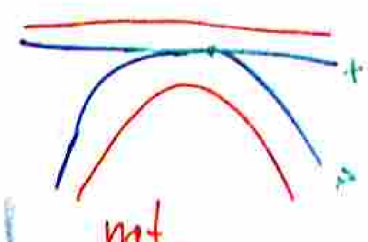
Normal forms

They are a tool to decide what kind of bifn we are dealing with.

NFs are related to the concept of tranversality.

Intuitively, an intersection is tranvesal if it cannot be "broken" by small perturbations

transversal



not.

Def

Let Y be a manifold and $X, Z \subset Y$ submanifolds.

We say that X & Z are tranverse at a point $x \in X \cap Z$ if

$$T_x X + T_x Z = T_x Y$$

A related concept is codimension

If X is a submanifold of Y, its codimension is $\operatorname{codim} X = \dim Y - \dim X$

Thm If $X, Z \subset Y$ are transversal submanifolds then

$$\operatorname{codim}(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z$$

$\mathbb{R}^3$