Ecuaciones Diferenciales Ordinarias

Posgrado en ciencias matemáticas UNAM

IIMAS

Renato Calleja, 21 de febrero de 2024

Caso lineal de coeficientes constantes

Last class

- Liouville's formula.

$\det \Phi(t) = \det \Phi(t_0)\, e^{\int_{t_0}^{t} tr\, A(s)ds}$

- Homework 2 on the webpage
Due March 6^{th} (2 weeks)

Linear differential equation with constant coefficients

$\dot{x} = Ax$, $x(t_0)=x_0$, $A \in M(\mathbb{R})^{n\times n}$ (nxn matrix)

If A has a Jordan Canonical Form.

there is a semi-group e^{tA}

and $X(t) = e^{(t-t_0)A} x_0$

First, we focus on the system of the form

$\dot{X} = AX$, A nxn constant matrix.

Prop

Let A be real nxn matrix and $\dot{x} = Ax$.

i) The function $e^{\lambda t} v$ is a real solution if and only if (iff) v is a solution of $Av = \lambda v$ with λ its eigenvalue.

ii) If v is a complex eigenvector $v = u + iw$, $u, w \in \mathbb{R}^n$, with eigenvalue $\lambda = \alpha + \beta i$, $\beta \neq 0$, then

$e^{\alpha t}[\cos(\beta t) u - \sin(\beta t) w]$,

$e^{\alpha t}[\sin(\beta t) u + \cos(\beta t) w]$

i) $X(t) = e^{\lambda t} v \Rightarrow \dot{x}(t) = \lambda e^{\lambda t} v = e^{\lambda t} \lambda v = e^{\lambda t} A v = AX$.

ii) $\tilde{X}(t) = e^{(\alpha + i\beta)t}(u + iw) = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)][u + iw]$

de Moivre's formula

$= e^{\alpha t}(\cos(\beta t) u - \sin(\beta t) w) + i e^{\alpha t}[\sin(\beta t) u + \cos(\beta t) w]$

$\tilde{X}(t) = X_r(t) + i X_i(t)$

$\frac{d\tilde{X}(t)}{dt} = \frac{d X_r(t)}{dt} + i \frac{d X_i(t)}{dt} = A\tilde{X}(t) = AX_r(t) + iAX_i(t)$

$\frac{dX_r(t)}{dt} = AX_r(t)$, $\frac{dX_i(t)}{dt} = AX_i(t)$

Caso sencillo

If A has a complete basis of eigenvectors $\{v_1, \dots, v_n\}$, $Av_\ell = \lambda_\ell v_\ell$, then the solutions $e^{\lambda_\ell t} v_\ell$ are linearly independent.

then

$$\Phi(t) = \left[e^{\lambda_1 t} v_1 \,|\, e^{\lambda_2 t} v_2 \,|\, \dots \,|\, e^{\lambda_n t} v_n \right]$$

is a fundamental solution matrix.

$$\det(A - \lambda I) = 0 \qquad (*)$$

- If $(*)$ has n different real or complex solutions then ✓
- If a λ has multiplicity then use Jordan Canonical Form. Later...

First we talk about e^{A}, the exponential matrix.

Prop

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$$e^{\alpha t}[\cos(\beta t) u - \sin(\beta t) w],$$

$$e^{\alpha t}[\sin(\beta t) u + \cos(\beta t) w]$$

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i) $x(t) = e^{\lambda t} v \Rightarrow \dot{x}(t) = \lambda e^{\lambda t} v = e^{\lambda t} \lambda v = e^{\lambda t} A v = A x.$

ii) $\tilde{x}(t) = e^{(\alpha + i\beta)t}(u + iw) = e^{\alpha t}[\cos(\beta t) + i \sin(\beta t)][u + iw]$

de Moivre's formula

$$= e^{\alpha t}[\cos(\beta t) u - \sin(\beta t) w] + i e^{\alpha t}[\sin(\beta t) u + \cos(\beta t) w]$$

$\tilde{x}(t) = x_r(t) + i x_i(t)$

$$\frac{d\tilde{x}(t)}{dt} = \frac{d}{dt} x_r(t) + i \frac{d}{dt} x_i(t) = A\tilde{x}(t) = A x_r(t) + i A x_i(t)$$

$$\frac{dx_r(t)}{dt} = A x_r(t), \quad \frac{dx_i(t)}{dt} = A x_i(t)$$

Matriz exponencial

Absolutely convergent series

When the sums of the absolute values of the summands are finite

a series $\sum_{n=0}^{\infty} a_n$ is a.c. if $\sum_{n=0}^{\infty} |a_n| = L$

for some real number.

Proposition

If $A \in \mathcal{L}(\mathbb{R}^n)$ (or $\mathcal{L}(E)$, E a Banach space)

then the series $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ is absolutely convergent.

Proof

Partial sums $S_N = \sum_{k=0}^{N} \frac{1}{k!} A^k \in \mathcal{L}(\mathbb{R}^n)$

$$\|S_N\| \leq 1 + \|A\| + \tfrac{1}{2}\|A^2\| + \dots + \tfrac{1}{N!}\|A^N\|$$

$$\leq 1 + \|A\| + \tfrac{1}{2}\|A\|^2 + \dots + \tfrac{1}{N!}\|A\|^N < \sum_{k=0}^{\infty} \tfrac{1}{k!}\|A\|^k = e^{\|A\|}$$

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HW2 You will prove that $\mathcal{L}(\mathbb{R}^n)$ is a Banach Algebra $\|AB\| \leq \|A\|\,\|B\|$

Observation

The partial sum is Cauchy Sequence

$$\|S_m - S_n\| = \left\|\sum_{k=n+1}^{m} \frac{1}{k!} A^k\right\| \leq \sum_{k=n+1}^{m} \frac{1}{k!}\|A^k\|$$

$$\leq \frac{\|A\|^{n+1}}{(n+1)!} \sum_{j=0}^{\infty} \frac{1}{j!}\|A\|^j < \varepsilon$$

All this can be extended to $\mathcal{L}(E)$.

$\Rightarrow$ The partial sums converge to $e^A \in \mathcal{L}(E)$.

We define

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

This operator has very similar properties to the ordinary exponential (whenever we have commuting matrices).

Propiedades de la matriz exponencial

Proposition

Suppose $A, B \in \mathcal{L}(E)$

i) If $A \in \mathcal{L}(E)$, then $e^A \in \mathcal{L}(E)$ (After you prove that $\mathcal{L}(\mathbb{R}^n)$ is a Banach algebra)

ii) If B is non-singular, then $Be^A B^{-1} = e^{BAB^{-1}}$

iii) If $AB = BA$, then $e^{A+B} = e^A e^B$.

iv) $e^{-A} = (e^A)^{-1}$ In particular the range of the exponential map $A \longmapsto e^A$ is the general linear group $GL(E)$, the invertible elements of $\mathcal{L}(E)$.

v) $\frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA} A$. In particular e^{tA} is the principal fundamental matrix solution $\dot{X} = AX$, at $t=0$.

$\Phi(t) = e^{tA}$, $\dot{\Phi}(t) = A\Phi(t)$, $\Phi(0) = I_d$.

vi) $\|e^A\| \leq e^{\|A\|}$.

Observation

The partial sum is Cauchy Sequence

$$\| S_m - S_n \| = \| \sum_{k=n+1}^{m} \frac{1}{k!} A^k \| \leq \sum_{k=n+1}^{m} \frac{1}{k!} \|A^k\|$$

$$\leq \frac{\|A\|^{n+1}}{(n+1)!} \sum_{j=0}^{\infty} \frac{1}{j!} \|A\|^j < \varepsilon$$

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vi) $\|e^A\| \le e^{\|A\|}$.

Proof

i) ✓, ii) $S_N(A) = \sum_{k=0}^{N} \frac{1}{k!} A^k$

$$B S_N(A) B^{-1} = \sum_{k=0}^{N} \frac{1}{k!} B(\underbrace{A \cdot A \cdot \ldots \cdot A}_{k\text{-times}})B^{-1}$$

$$= \sum_{k=0}^{N} \frac{1}{k!} \underbrace{BAB^{-1}BAB^{-1} \ldots BAB^{-1}}_{k\text{-times}}$$

$$= \sum_{k=0}^{N} \frac{1}{k!} (BAB^{-1})^k = S_N(BAB^{-1}) \to e^{BAB^{-1}}$$

iii) Cauchy Products

$$\left(\sum_{i=0}^{\infty} a_i\right) \cdot \left(\sum_{j=0}^{\infty} b_j\right) = \sum_{k=0}^{\infty} C_k = \sum_{k=0}^{\infty} \overbrace{\left(\sum_{\ell=0}^{k} a_{k-\ell}\, b_\ell\right)}^{C_k}$$

$$e^{A+B} = \sum_{k=0}^{\infty} \frac{1}{k!}(A+B)^k = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\ell=0}^{k} \frac{k!}{(k-\ell)!\,\ell!} A^{k-\ell} B^\ell$$

$$= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} \left(\frac{A^{k-\ell}}{(k-\ell)!} \cdot \frac{B^\ell}{\ell!}\right)$$

$(A+B)^k = \sum_{\ell=0}^{k} \binom{k}{\ell} A^{k-\ell} B^\ell$

↳ only true when $AB = BA$

$$= \left(\sum_{i=0}^{\infty} \frac{A^i}{i!}\right) \cdot \left(\sum_{j=0}^{\infty} \frac{B^j}{j!}\right)$$

$$= e^A \cdot e^B$$

There are examples where this property fails.

Propiedades de la matriz exponencial

<u>Proposition</u>

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$\Phi(t) = e^{tA}$, $\dot{\Phi}(t) = A\Phi(t)$, $\Phi(0) = I_d$.

vi) $\|e^A\| \leq e^{\|A\|}$.

<u>Proof</u>

It only remains to prove v)

$$\lim_{s\to 0} \frac{1}{s}\left(e^{(t+s)A} - e^{tA}\right) = \lim_{s\to\infty} \frac{1}{s}\left(e^{sA} - I\right)e^{tA}$$

$$\boxed{tA \cdot sA = sA \cdot tA}$$

$$= \lim_{s\to 0} \Big(A + \underbrace{R(sA)}_{= \sum_{k=2}^{\infty} \frac{s^{k-1}A^k}{k!}}\Big)e^{tA}$$

if $|s|<1$

$$\|R(sA)\| \leq \sum_{k=2}^{\infty} \frac{|s|^{k-1}\|A\|^k}{k!} = |s| \sum_{k=2}^{\infty} \frac{|s|^{k-2}}{k!}\|A\|^k \leq |s| e^{\|A\|}$$

$$= Ae^{tA} \quad \checkmark$$

Calcular la matrix exponencial

Proposition

Suppose $A, B \in \mathcal{L}(E)$

i) If $A \in \mathcal{L}(E)$, then $e^A \in \mathcal{L}(E)$ (After you prove that $\mathcal{L}(\mathbb{R}^n)$ is a Banach algebra.)

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$\Phi(t) = e^{tA}$, $\dot{\Phi}(t) = A\Phi(t)$, $\Phi(0) = I_d$.

vi) $\|e^A\| \le e^{\|A\|}$.

If we hav a complete basis of eigenvectors $\{v_1, \dots, v_n\}$ Let the matrix $V = [v_1 | v_2 | \dots | v_n]$

$\exists V^{-1}$ the V satisfies $AV = \Lambda V$, $V^{-1}AV = \Lambda$

where $\Lambda = \operatorname{diag}\{\lambda_1, \dots, \lambda_n\}$,

$$V^{-1} e^A V = e^{V^{-1}AV} = e^{\Lambda} = e^{\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}} = \begin{pmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix}$$

$$e^A = V \begin{pmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix} V^{-1}$$

Forma canónica de Jordan

Theorem (Perko §1.8) Jordan Canonical Form

A - $n \times n$ matrix - has real eigenvalues λ_j, $j = 1, \ldots, k$ and complex eigenvalues $\lambda_j = a_j + i b_j$, $j = k+1, \ldots, m$ where $2(m-k-1)+k = n$. Then exists a basis $\{v_1, \ldots, v_k, u_{k+1}, w_{k+1}, \ldots, u_m, w_m\}$ of $\mathbb{R}^n$ such that v_j's and $(a_j + i w_j)$ are generalized eigenvectors of A associated to the eigenvalues. Furthermore, the invertible matrix $S = [v_1 | \ldots | v_k | u_{k+1} | w_{k+1} | \ldots | u_m | w_m]$ is such that $S^{-1} A S = \mathrm{diag}\{J_1, \ldots, J_r\}$ where J_ℓ is an elementary Jordan block, of the form

$$J_\ell = \begin{pmatrix} \lambda_j & 1 & 0 & & \\ 0 & \lambda_j & 1 & & \\ & & \ddots & & \\ & & & \lambda_j & 1 \\ & & & 0 & \lambda_j \end{pmatrix} \quad \text{for real } \lambda_j$$

$$J_\ell = \begin{pmatrix} D & I_2 & 0 & \\ 0 & D & I_2 & \\ & & \ddots & \\ & & 0 & D \end{pmatrix}, \; D = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}, \; I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for complex } \lambda_j.$$

If $B = \mathrm{diag}(J_1, \ldots, J_r)$ then $B^{\cdot} = \mathrm{diag}\{J_1, \ldots, J_r\}$

$$e^{At} = S e^{t\,\mathrm{diag}\{J_1, \ldots, J_r\}} S^{-1} = S\,\mathrm{diag}(e^{tJ_1}, \ldots, e^{tJ_r}) S^{-1}$$

A Jordan block can be writen in

$J = \lambda I + N$, where $N^k = 0$ is nilpotent with $k = \dim J$.

$$e^{tJ} = e^{t\lambda I + tN} = e^{t\lambda I} e^{tN} = e^{t\lambda} \underbrace{\sum_{j=0}^{k-1} \frac{1}{j!} t^j N^j}_{\text{polynomial in } t}$$

For the complex.

$$D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a I_2 + \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$$

$$e^{tD} = e^{at} e^{\begin{pmatrix} 0 & -bt \\ bt & 0 \end{pmatrix}} = e^{at} \begin{pmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{pmatrix}$$

$$J = DI + \tilde{N}$$

$$e^{tJ} = e^{tD} \underbrace{\sum_{j=0}^{k-1} \frac{1}{j!} t^j \tilde{N}^j}_{\text{polynomial in } t}, \quad k = \dim J.$$

Froma canonica de Jordan

Theorem (Perko §1.8) Jordan Canonical Form

A ($n\times n$ matrix) has real eigenvalues λ_j, $j=1,\dots,k$ and complex eigenvalues $\lambda_j = a_j + i b_j$, $j=k+1,\dots,m$ where $2(m-k-1)+k=n$. There exists a basis $\{v_1,\dots,v_k,u_{k+1},w_{k+1},\dots,u_m,w_m\}$ of $\mathbb{R}^n$ such that v_j's and $(a_j + i w_j)$ are generalized eigenvectors of A associated to the eigenvalues.

Furthermore, the invertible matrix $S=[v_1|\dots|v_k|u_{k+1}|w_{k+1}|\dots|u_m|w_m]$ is such that $S^{-1}AS = \operatorname{diag}\{J_1,\dots,J_r\}$ where J_ℓ is an elementary Jordan block, of the form

$$J_\ell = \begin{pmatrix} \lambda_j & 1 & 0 & \\ 0 & \lambda_j & 1 & \\ & & \ddots & \\ & & \lambda_j & 1 \\ & & 0 & \lambda_j \end{pmatrix} \quad \text{for real } \lambda_j$$

$$J_\ell = \begin{pmatrix} D & I_2 & 0 & \\ 0 & D & I_2 & \\ & & \ddots & \\ & & 0 & D \end{pmatrix}, \quad D = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}, \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for complex } \lambda_j.$$

Prop (2.3+) (Chicone ...)

A $n\times n$ matrix, then e^{tA} is a matrix whose components are finite sums of terms of the form

$p(t)e^{at}\sin(bt)$ and $p(t)e^{at}\cos(bt)$

where $\lambda = a+ib$ is an eigenvalue of A, and $p(t)$ are polynomials of degree at most $n-1$.

This is about stability

$$e^{tJ} = e^{t\lambda I + tN} = e^{t\lambda I} e^{tN} = e^{t\lambda} \underbrace{\sum_{j=0}^{k-1} \frac{1}{j!} t^j N^j}_{\text{polynomial in } t}$$

For the complex.

$$D = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = aI_2 + \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix}$$

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$$J = DI + \tilde{N}$$

$$e^{tJ} = e^{tD} \underbrace{\sum_{j=0}^{k-1} \frac{1}{j!} t^j \tilde{N}^j}_{\text{polynomial in } t}, \quad k = \dim J.$$