Sistemas Dinámicos Hamiltonianos

Posgrado en ciencias matemáticas UNAM IIMAS

Renato Calleja, 4 de marzo de 2025

Lema de Poincaré - Cartan

Una forma cerrada es localmente exacta.
$(d\alpha=0)$ (forma exacta $\alpha=d\beta$)

- si $d\alpha=0$ es cerrada, entonces existe una vecindad de cada punto tal que $\alpha=d\beta$.

α es una k-forma, β es una $k-1$ forma.

En coordenadas, podemos escribir esto,

$$(d\alpha)(v_0,\dots,v_k)=\sum_{i=0}^{k}(-1)^i v_i\left[\alpha(v_0,\dots,\hat{v}_i,\dots,v_k)\right]$$

quitamos el i-ésimo. j-ésimo

$$+\sum_{0\le i<j\le k}(-1)^{i+j}\alpha\left([v_i,v_j],v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_k\right)$$

la derivada exterior en coordenadas.

Dem

$$\beta(v_1,\dots,v_{k+1})=\int_0^1 t^{k-1}\alpha^{j,i_1,\dots,i_k}(tv_1,\dots,tv_k)v_i\,dt\,dv_{i_1}\wedge\dots\wedge dv_{i_k}$$

Al derivar esta expresión obtenemos la fórmula para α en coordenadas. //

La derivada de Lie

Se α una k-forma y X un campo vectorial con flujo φ_t. La derivada de Lie de α a lo largo de X está dada por

$$\underbrace{\mathcal{L}_X \alpha}_{\text{K-forma.}} = \lim_{t \to 0} \frac{\varphi_t^* \alpha - \alpha}{t} = \frac{d}{dt} \varphi_t^* \alpha \Big|_{t=0}$$

Teorema de la derivada de Lie

$$\frac{d}{dt} \varphi_t^* \alpha = \varphi_t^* \mathcal{L}_X \alpha$$

Propiedad del pull-back

$$(f \circ g)^* \alpha = g^* f^* \alpha$$

Dem

De la definición tenemos que.

$$\frac{d}{dt} \varphi_t^* \alpha = \frac{d}{dt} \varphi_{t+s}^* \alpha \Big|_{s=0} = \frac{d}{ds} \varphi_{t+s}^* \alpha \Big|_{s=0}$$

$$\boxed{\varphi_{t+s} = \varphi_t \circ \varphi_s} \qquad \varphi_{t+s}^* \alpha = \varphi_{s+t}^* \alpha = (\varphi_s \circ \varphi_t)^* \alpha$$
$$= \varphi_t^* \varphi_s^* \alpha$$

$$= \varphi_t^* \frac{d}{ds} \varphi_s^* \alpha \Big|_{s=0} = \varphi_t^* \mathcal{L}_X \alpha \quad //$$

La derivada de Lie

Una función es una 0-forma
(función de una variedad. M

$$\mathcal{L}_X f = \frac{d}{dt} \varphi_t^* f \Big|_{t=0} = \frac{d}{dt} f(\varphi_t) \Big|_{t=0} = X(f)$$

$$X = \sum_{i=1}^{n} X_i \frac{\partial}{\partial x_i}$$

y veamos que $\underbrace{X(f) = df(X)}$

derivadas direccionales de f
c.r.a. X.

$$X(f) = \sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i}$$

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

$$df(X) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i \left(\sum X_j \frac{\partial}{\partial x_j} \right) = \sum_{i=1}^{n} X_i \frac{\partial f}{\partial x_i}$$

La derivada de Lie para campos vectoriales

$$\mathcal{L}_X Y = \frac{d}{dt} (\varphi_t)^* Y \Big|_{t=0} = [X, Y]$$

La derivada de Lie para campos vectoriales

$$\mathcal{L}_X Y = \frac{d}{dt}(\varphi_t)^* Y \Big|_{t=0} = [X,Y]$$

$$\frac{d}{dt}(\varphi_t)^* Y \Big|_{t=0} (f) = \frac{d}{dt}\left(d(\varphi_{-t})\, Y \circ \varphi_t\right)\Big|_{t=0}(f)$$

Tarea

$$= \lim_{t\to 0} \frac{d\varphi_{-t} Y(f) - Y(f)}{t} = \lim_{t\to 0} d\varphi_{-t} \frac{Y(f) - d\varphi_t Y(f)}{t}$$

$$= \lim_{t\to 0} \frac{Y(f) - d\varphi_t Y(f)}{t} = \lim_{t\to 0} \frac{Y(f) - Y(f\circ\varphi_t)\circ\varphi_{-t}}{t}$$

Escribimos $\varphi_t(x) = \varphi(t,x)$ y hacemos Taylor a $f\circ\varphi_t$

$$f(\varphi(t,x)) = f(x) + t\,h(t,x)$$

$$h(0,x) = \frac{\partial}{\partial t} f(\varphi(t,x))\Big|_{t=0}$$

La derivada de Lie para campos vectoriales

Nos damos cuenta que,

$$X(f) = \frac{\partial}{\partial t} f(\varphi_t(x))\Big|_{t=0} = X(f)$$

$$\Rightarrow = \lim_{t\to 0} \frac{Y(f) - Y(f)\circ\varphi_{-t}}{t} - Y(h(t,x))\circ\varphi_t^{-1}$$

$$= \frac{d}{dt} Y(f)\circ\varphi_t(x) - Y(X(f))$$

$$= X(Y(f)) - Y(X(f)) = [X,Y](f)$$

$\forall$ función f. $\mathcal{L}_X Y = [X,Y]$ //

La derivada de Lie para campos vectoriales

$$\mathcal{L}_X Y = \frac{d}{dt}(\varphi_t)^* Y\Big|_{t=0} = [X,Y] = XY - YX$$

$$= (X_1+X_2)Y - Y(X_1+X_2)$$

Propiedades del corchete de Lie.

(a) $[X,Y]$ es bilineal

$$\mathcal{L}_{X_1+X_2} Y = \frac{d}{dt}(\varphi_1^t + \varphi_2^t)^* Y\Big|_{t=0}$$

$\varphi_1^t \quad \varphi_2^t$

(b) $[X,Y] = -[Y,X]$

(c) $[X, fY] = X(f)Y + f\cdot[X,Y]$

(d) Identidad de Jacobi

$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$

Ejemplo

$\mathbb{R}^n = \mathcal{M}$, $X = \frac{\partial}{\partial x_i}$, $Y = \frac{\partial}{\partial x_k}$

$$[X,Y](h) = \frac{\partial}{\partial x_i}\left(\frac{\partial h}{\partial x_k}\right) - \frac{\partial}{\partial x_k}\left(\frac{\partial h}{\partial x_i}\right)$$

$$= \frac{\partial^2 h}{\partial x_i \partial x_k} - \frac{\partial^2 h}{\partial x_k \partial x_i} = 0$$

Lema de Schwartz $\frac{\partial^2 h}{\partial x_i \partial x_k} = \frac{\partial^2 h}{\partial x_k \partial x_i}$

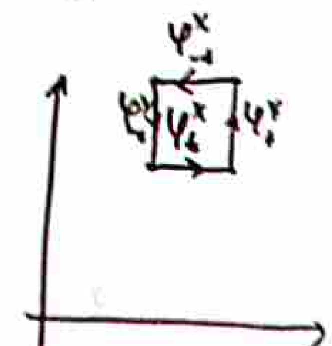
$\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\right] = 0 \iff X$ y Y conmutan.

En $\mathbb{R}^2 = (x,y)$ $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y}$.

$[X,Y] = 0$

$\varphi_t^X(x,y) = (x+t, y)$

$\varphi_t^Y(x,y) = (x, y+t)$



$\varphi^Y_{-t}\circ\varphi^X_{-t}\circ\varphi^Y_t\circ\varphi^X_t = id. \to$ tiene que ver con que $[X,Y]=0$