Sistemas Dinámicos Hamiltonianos

Posgrado en ciencias matemáticas UNAM

IIMAS

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Variedad inestable del mapeo estándar

$$\theta_{n+1} = \theta_n + p_n + \frac{\varepsilon}{2\pi}\,\mathrm{sen}(2\pi\theta_n)$$

$$p_{n+1} = p_n + \frac{\varepsilon}{2\pi}\,\mathrm{sen}(2\pi\theta_n)$$

$$F\begin{pmatrix}\theta\\ p\end{pmatrix} = \begin{pmatrix}\theta + p + \frac{\varepsilon}{2\pi}\,\mathrm{sen}(2\pi\theta)\\ p + \frac{\varepsilon}{2\pi}\,\mathrm{sen}(2\pi\theta)\end{pmatrix}, \quad F: \mathbb{T}\times\mathbb{R} \to \mathbb{T}\times\mathbb{R}$$

$$F\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \qquad z^* = \begin{pmatrix}0\\0\end{pmatrix}$$

$$DF\begin{pmatrix}\theta\\ p\end{pmatrix} = \begin{pmatrix}1 + \varepsilon\cos(2\pi\theta) & 1\\ \varepsilon\cos(2\pi\theta) & 1\end{pmatrix}$$

$$DF(z^*) = \begin{pmatrix}1+\varepsilon & 1\\ \varepsilon & 1\end{pmatrix} \qquad \begin{array}{l}\det DF = 1\\ \mathrm{tr}\,DF = 2+\varepsilon\end{array}$$

Vaps.

$$\lambda^2 - \mathrm{tr}\,DF\,\lambda + \det DF = 0$$

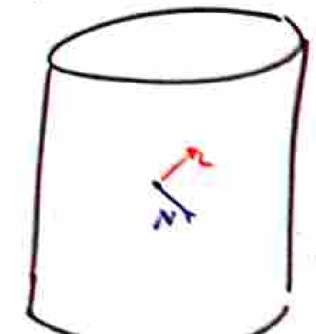
$$\Rightarrow \lambda^2 - (2+\varepsilon)\lambda + 1 = 0$$

$$\lambda = \frac{2+\varepsilon \pm\sqrt{(2+\varepsilon)^2 - 4}}{2}$$

$$= 1 + \frac{\varepsilon}{2} \pm \sqrt{\left(1+\frac{\varepsilon}{2}\right)^2 - 1} = \lambda_+, \lambda_-$$

$$\lambda_- < 1 < \lambda_+, \quad \lambda_+ \cdot \lambda_- = 1$$

$$v_{+,-} = \begin{pmatrix}1\\ -\varepsilon/2 \pm \sqrt{(1+\varepsilon/2)^2 - 1}\end{pmatrix}$$



$$\Lambda_L = \lambda_+, \quad \Lambda_N = \lambda_-$$

$$P = (L \,\vdots\, N) = \begin{pmatrix}1 & 1\\ -\varepsilon/2 + \sqrt{(1+\varepsilon/2)^2 - 1} & -\varepsilon/2 - \sqrt{(1+\varepsilon/2)^2 - 1}\end{pmatrix}$$

$$DF(z^*)P = P\Lambda \Rightarrow \begin{pmatrix}1+\varepsilon & 1\\ \varepsilon & 1\end{pmatrix}P = P\begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}$$

Variedad inestable del mapeo estándar

$$F\begin{pmatrix}\theta\\ p\end{pmatrix}=\begin{pmatrix}\theta + p + \frac{\varepsilon}{2\pi}\,\mathrm{sen}(2\pi\theta)\\ p + \frac{\varepsilon}{2\pi}\,\mathrm{sen}(2\pi\theta)\end{pmatrix}$$

nalíticas:

$$\begin{cases} F: \mathbb{T}\times\mathbb{R}\to\mathbb{T}\times\mathbb{R} & z^* = \begin{pmatrix}0\\0\end{pmatrix}\\ f: \mathbb{R}\to\mathbb{R} & f(s)=\lambda_+ s\\ W: \mathbb{R}\to\mathbb{T}\times\mathbb{R} & \lambda_+ = 1+\frac{\varepsilon}{2}+\sqrt{(1+\varepsilon/2)^2-1}\end{cases}$$

$$F(W(s)) = W(f(s)) \Rightarrow F(W(s)) = W(\lambda_+ s) \quad (\text{Invs})$$

$$\begin{pmatrix}1+\varepsilon & 1\\ \varepsilon & 1\end{pmatrix} L = L\lambda_+ \qquad V^L \in T_{\binom{0}{0}} W$$

Taylor

$$W(s) = z^* + \sum_{k\geq 1} W_k(s) = \sum_{k\geq 1}\begin{pmatrix}W_k^\theta\\ W_k^p\end{pmatrix} s^k$$

$$f(s) = \lambda_+ s$$

$$DF(z^*)DW(0) = DW(0)\lambda_+$$

A orden k

$$DF(z^*)W_k(s) - W_k(\Lambda_L s) - L f_k(s) - E_k(s) = 0$$

$$\begin{pmatrix}1+\varepsilon & 1\\ \varepsilon & 1\end{pmatrix}\begin{pmatrix}W_k^\theta\\ W_k^p\end{pmatrix}s^k - \begin{pmatrix}W_k^\theta\\ W_k^p\end{pmatrix}\lambda_+^k s^k - f_{1k}\lambda_+ s = E_k(s)$$

$$E_k(s) = \left[F\left(\sum_{\ell=1}^{k-1} W_\ell s^\ell\right)\right]_k - \cancelto{0}{\left[\sum_{\ell=1}^{k-1} W_\ell \lambda_+^\ell s^\ell\right]_k}$$

$$= \left[\begin{pmatrix}\sum_{\ell=1}^{k-1} W_\ell^\theta s^\ell + \sum_{\ell=1}^{k-1} W_\ell^p s^\ell + \frac{\varepsilon}{2\pi}\,\mathrm{sen}\left(2\pi\sum_{\ell=1}^{k-1} W_\ell^\theta s^\ell\right)\\ \sum_{\ell=1}^{k-1} W_\ell^p s^\ell + \frac{\varepsilon}{2\pi}\,\mathrm{sen}\left(2\pi\sum_{\ell=1}^{k-1} W_\ell^\theta s^\ell\right)\end{pmatrix}\right]_k$$

$$= \left[\begin{pmatrix}\frac{\varepsilon}{2\pi}\,\mathrm{sen}\left(2\pi\sum_{\ell=1}^{k-1} W_\ell^\theta s^\ell\right)\\ \frac{\varepsilon}{2\pi}\,\mathrm{sen}\left(2\pi\sum_{\ell=1}^{k-1} W_\ell^\theta s^\ell\right)\end{pmatrix}\right]_k$$

Variedad inestable del mapeo estándar

Serie de seno y coseno

$$S(x) = \operatorname{sen}(W(x)) \Rightarrow S'(x) = \cos(W(x))\,W'(x) = C(x)W'(x)$$

$$C(x) = \cos(W(x)) \Rightarrow C'(x) = -\operatorname{sen}(W(x))\,W'(x) = -S(x)W'(x)$$

$$S(x) = \sum_{k=0}^{\infty} S_k x^k \Rightarrow x S'(x) = \sum_{k=0}^{\infty} k\, S_k x^k$$

$$C(x) = \sum_{k=0}^{\infty} C_k x^k \Rightarrow x C'(x) = \sum_{k=0}^{\infty} k\, C_k x^k$$

$$x S'(x) = C(x)\, x W'(x), \quad W(x) = \sum_{k=0}^{\infty} W_k x^k, \; xW'(x) = \sum_{k=0}^{\infty} k W_k x^k$$

$$\Rightarrow \sum_{k=0}^{\infty} k\, S_k x^k = \left(\sum_{k=0}^{\infty} C_k x^k\right)\left(\sum_{k=0}^{\infty} k W_k x^k\right)$$

$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} (k-j) W_{k-j}\, C_j\right) x^k$$

Producto de Cauchy.

Despejando

$$S_k = \frac{1}{k} \sum_{j=0}^{k-1} (k-j) W_{k-j}\, C_j$$

$$C_k = -\frac{1}{k} \sum_{j=0}^{k-1} (k-j) W_{k-j}\, S_j$$

A orden k

$$DF(z^*) W_k(s) - W_k(\Lambda_L s) - L f_k(s) - E_k(s) = 0$$

$$\begin{pmatrix} 1+\varepsilon & 1 \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} W_k^\theta \\ W_k^p \end{pmatrix} s^k - \begin{pmatrix} W_k^\theta \\ W_k^p \end{pmatrix} \lambda_+^k s^k - \delta_{1k} \lambda_+ s = E_k(s)$$

$$E_k(s) = \left[F\left(\sum_{\ell=1}^{k-1} W_\ell s^\ell \right) \right]_k - \cancel{\left[\sum_{\ell=1}^{k-1} W_\ell \lambda_+^\ell s^\ell \right]_k} \to 0$$

$$= \left[\begin{pmatrix} \sum_{\ell=1}^{k-1} W_\ell^\theta s^\ell + \sum_{\ell=1}^{k-1} W_\ell^p s^\ell + \frac{\varepsilon}{2\pi} \operatorname{sen}\left(2\pi \sum_{\ell=1}^{k-1} W_\ell^\theta s^\ell\right) \\ \sum_{\ell=1}^{k-1} W_\ell^p s^\ell + \frac{\varepsilon}{2\pi} \operatorname{sen}\left(2\pi \sum_{\ell=1}^{k-1} W_\ell^\theta s^\ell\right) \end{pmatrix} \right]_k$$

$$= \left[\begin{pmatrix} \frac{\varepsilon}{2\pi} \operatorname{sen}\left(2\pi \sum_{\ell=1}^{k-1} W_\ell^\theta s^\ell\right) \\ \frac{\varepsilon}{2\pi} \operatorname{sen}\left(2\pi \sum_{\ell=1}^{k-1} W_\ell^\theta s^\ell\right) \end{pmatrix} \right]_k$$

$$= \begin{pmatrix} \varepsilon \frac{1}{k} \sum_{j=0}^{k-1} (k-j) W_{k-j}\, C_j \\ \varepsilon \frac{1}{k} \sum_{j=0}^{k-1} (k-j) W_{k-j}\, C_j \end{pmatrix}$$

Variedad inestable del mapeo estándar

$$\begin{pmatrix}1+\varepsilon & 1\\ \varepsilon & 1\end{pmatrix}\begin{pmatrix}W_k^\theta\\ W_k^p\end{pmatrix} - \begin{pmatrix}W_k^\theta\\ W_k^p\end{pmatrix}\lambda_+^k - \delta_{1,k}\lambda_+ = \varepsilon\frac{1}{k}\sum_{j=0}^{k-1}(k-j)W_{k-j}c_j\begin{pmatrix}1\\ 1\end{pmatrix}$$

Cambio de vars

$$P\begin{pmatrix}\xi_k^L\\ \xi_k^N\end{pmatrix} = \begin{pmatrix}W_k^\theta\\ W_k^p\end{pmatrix} \qquad DF(z^*)P = P\begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}$$

$$\boxed{\begin{pmatrix}\eta_k^L\\ \eta_k^N\end{pmatrix} = -P^{-1}E_k}$$

Ecuaciones normales.

$$\lambda_-\,\xi_k^N - \xi_k^N\lambda_+^k = \eta_k^N$$

$$\xi_k^N = \frac{\eta_k^N}{\lambda_- - \lambda_+^k} \qquad W_k = P\begin{pmatrix}\xi_k^L\\ \xi_k^N\end{pmatrix}$$

Ecuaciones tangentes

$$\lambda_+\xi_k^L - \xi_k^L\lambda_+^k - \delta_{1,k}\lambda_+ = \tilde\eta_k$$

W_k a todos los ordenes.

$\tilde\eta_k = \eta^L - T\xi_k^N = \eta^L$ porque pudimos diagonalizar.

$$\xi_k^L = \frac{\tilde\eta^L}{\lambda_+ - \lambda_+^k}$$

$W(s) =$

A orden k

$$DF(z^*)W_k(s) - W_k(\Lambda_L s) - L\,f_k(s) - E_k(s) = 0$$

$$\begin{pmatrix}1+\varepsilon & 1\\ \varepsilon & 1\end{pmatrix}\begin{pmatrix}W_k^\theta\\ W_k^p\end{pmatrix}s^k - \begin{pmatrix}W_k^\theta\\ W_k^p\end{pmatrix}\lambda_+^k s^k - \delta_{1,k}\lambda_+ s = E_k(s)$$

$$E_k(s) = \left[F\left(\sum_{l=1}^{k-1}W_l s^l\right)\right]_k - \cancel{\left[\sum_{l=1}^{k-1}W_l\lambda_+^l s^l\right]_k} \to 0$$

$$= \left[\begin{pmatrix}\sum_{l=1}^{k-1}W_l^\theta s^l + \sum_{l=1}^{k-1}W_l^p s^l + \frac{\varepsilon}{2\pi}\operatorname{sen}\left(2\pi\sum_{l=1}^{k-1}W_l^\theta s^l\right)\\ \sum_{l=1}^{k-1}W_l^p s^l + \frac{\varepsilon}{2\pi}\operatorname{sen}\left(2\pi\sum_{l=1}^{k-1}W_l^\theta s^l\right)\end{pmatrix}\right]_k$$

$$= \left[\begin{pmatrix}\frac{\varepsilon}{2\pi}\operatorname{sen}\left(2\pi\sum_{l=1}^{k-1}W_l^\theta s^l\right)\\ \frac{\varepsilon}{2\pi}\operatorname{sen}\left(2\pi\sum_{l=1}^{k-1}W_l^\theta s^l\right)\end{pmatrix}\right]_k$$

$$= \begin{pmatrix}\varepsilon\frac{1}{k}\sum_{j=0}^{k-1}(k-j)W_{k-j}c_j\\ \varepsilon\frac{1}{k}\sum_{j=0}^{k-1}(k-j)W_{k-j}c_j\end{pmatrix}$$

Variedad inestable del mapeo estándar

$$\begin{pmatrix}1+\varepsilon & 1\\ \varepsilon & 1\end{pmatrix}\begin{pmatrix}W_k^\theta\\ W_k^p\end{pmatrix} - \begin{pmatrix}W_k^\theta\\ W_k^p\end{pmatrix}\lambda_+^k - \delta_{1,k}\lambda_+ = \varepsilon\frac{1}{k}\sum_{j=0}^{k-1}(k-j)W_{k-j}c_j\begin{pmatrix}1\\ 1\end{pmatrix}$$

Cambio de vars

$$P\begin{pmatrix}\xi_k^L\\ \xi_k^N\end{pmatrix} = \begin{pmatrix}W_k^\theta\\ W_k^p\end{pmatrix}$$

En este caso

$$\begin{pmatrix}1+\varepsilon & 1\\ \varepsilon & 1\end{pmatrix}P = P\begin{pmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{pmatrix}$$

En general

$$DF(z^*)P = P\begin{pmatrix}\Lambda_L & T\\ 0 & \Lambda_N\end{pmatrix}$$

Ecuaciones normales.

$$\lambda_-\xi_k^N - \xi_k^N\lambda_+^k = .$$

$$\xi_k^N = \frac{\eta_k^N}{\lambda_- - \lambda_+^k}$$

$$W_k = P\begin{pmatrix}\xi_k^L\\ \xi_k^N\end{pmatrix}$$

W_k a todos los ordenes.

Ecuaciones tangentes

$$\lambda_+\xi_k^L - \xi_k^L\lambda_+^k - \delta_{1,k}\lambda_+ = \tilde{\eta}_k$$

$$\tilde{\eta}_k = \eta_k^L - T\xi_k^N = \eta^L$$ porque pudimos diagonalizar.

$$\xi_k^L = \frac{\tilde{\eta}_k}{\lambda_+ - \lambda_+^k}$$

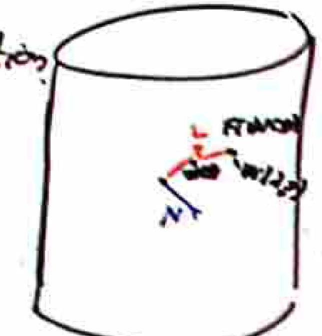
$$W(s) = \sum_{k=0}^{\infty} W_k s^k = \sum_{k=0}^{\infty} P\begin{pmatrix}\xi_k^L\\ \xi_k^N\end{pmatrix}s^k$$

$$= \sum_{k=0}^{\infty} P\begin{pmatrix}\dfrac{\eta_k^N}{\lambda_- - \lambda_+^k}\\ \dfrac{\tilde{\eta}_k}{\lambda_+ - \lambda_+^k}\end{pmatrix}s^k \rightsquigarrow (s/\lambda_+)^k$$

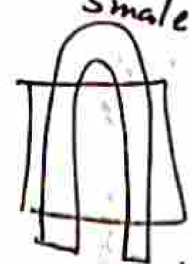
$$\frac{\eta_k^N}{\lambda_- - \lambda_+^k} \sim \lambda_+^{-k}$$

Parametrizaciones

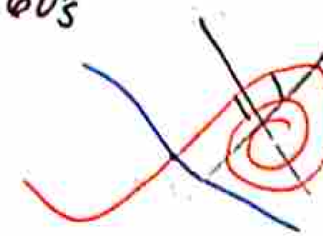
A. Haro
M. Canadell
J.M. Mondelo
J.Ll. Figueras
A. Luque.



Smale 60's



Comportamiento caótico.



"Maraña homoclínica"