Sistemas Dinámicos Hamiltonianos

Posgrado en ciencias matemáticas UNAM IIMAS

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Pull-back $f: M \to M_1$, α k-forma, $p \in M$, $X_i \in T_pM$

$$(f^*\alpha)_p (X_1, \dots, X_k) = \alpha_{f(p)} (df(X_1), \dots, df(X_k))$$

Nota: el p.b. es como composición.

Prop
El pull-back de un producto cuña, es el producto cuña de los pull-backs.

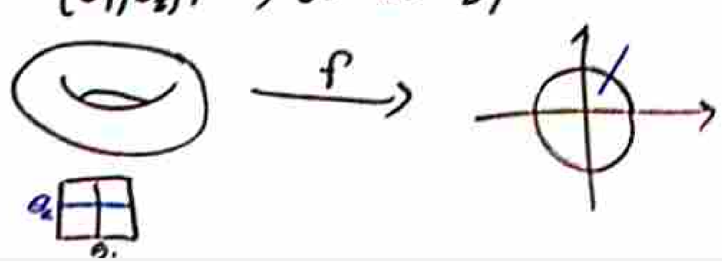
$$f^*(\alpha \wedge \beta) = f^*\alpha \wedge f^*\beta$$

Dem $\alpha_{f(p)}(df(X_1), \dots, df(X_k)) \wedge \beta_{f(p)}(df(X_1), \dots, df(X_l))$

$$= \alpha \wedge \beta_{f(p)} (df(X_1), \dots, df(X_k), df(X_1), \dots, df(X_l))$$

Ejemplo $f: \mathbb{T}^2 \to \mathbb{R}^2$

$$(\theta_1, \theta_2) \longmapsto (e^{\theta_1}\cos\theta_2, e^{\theta_1}\operatorname{sen}\theta_2) = (\phi, \psi)$$



$d\phi$ y $d\psi$ dos 1-formas en $T^*_{(\phi,\psi)}\mathbb{R}^2 \simeq \mathbb{R}^2$

$$\boxed{f^* d\phi = d(e^{\theta_1}\cos\theta_2) = e^{\theta_1}\cos\theta_2\, d\theta_1 - e^{\theta_1}\operatorname{sen}\theta_2\, d\theta_2}$$

$X_1 \in T_{(\theta_1,\theta_2)}\mathbb{T}^2 \qquad X_1 = v_1 \frac{\partial}{\partial\theta_1} + v_2 \frac{\partial}{\partial\theta_2}$

$$f^* d\phi (X_1) = e^{\theta_1}\cos\theta_2\, d\theta_1\left(v_1 \frac{\partial}{\partial\theta_1} + v_2 \frac{\partial}{\partial\theta_2}\right) - e^{\theta_1}\operatorname{sen}\theta_2\, d\theta_2\left(v_1\frac{\partial}{\partial\theta_1} + v_2\frac{\partial}{\partial\theta_2}\right)$$

$$\boxed{d\theta_1\left(\frac{\partial}{\partial\theta_1}\right) = 1, \quad d\theta_1\left(\frac{\partial}{\partial\theta_2}\right) = 0}$$

$$= e^{\theta_1}\cos\theta_2\, v_1 - e^{\theta_1}\operatorname{sen}\theta_2\, v_2$$

$$f^* d\psi = d(e^{\theta_1}\operatorname{sen}\theta_2) = e^{\theta_1}\operatorname{sen}\theta_2\, d\theta_1 + e^{\theta_1}\cos\theta_2\, d\theta_2$$

$$\cdot f^* d\phi \wedge f^* d\psi = (e^{\theta_1}\cos\theta_2\, d\theta_1 - e^{\theta_1}\operatorname{sen}\theta_2\, d\theta_2) \wedge (e^{\theta_1}\operatorname{sen}\theta_2\, d\theta_1 + e^{\theta_1}\cos\theta_2\, d\theta_2)$$

$$= \cancel{e^{2\theta_1}\cos\theta_2 \operatorname{sen}\theta_2\, d\theta_1 \wedge d\theta_1} \to 0 + e^{2\theta_1}\cos^2\theta_2\, d\theta_1 \wedge d\theta_2$$

$$+ e^{2\theta_1}\operatorname{sen}^2\theta_2 \underbrace{d\theta_2 \wedge d\theta_1}_{d\theta_1 \wedge d\theta_2} - \cancel{e^{2\theta_1}\operatorname{sen}\theta_2\cos\theta_2\, d\theta_2 \wedge d\theta_2} \to 0$$

$$= e^{2\theta_1}\, d\theta_1 \wedge d\theta_2$$

$$f^*(d\phi \wedge d\psi) = d(e^{\theta_1}\cos\theta_2) \wedge d(e^{\theta_1}\operatorname{sen}\theta_2)$$

$$= (e^{\theta_1}\cos\theta_2\, d\theta_1 - e^{\theta_1}\operatorname{sen}\theta_2\, d\theta_2) \wedge (e^{\theta_1}\operatorname{sen}\theta_2\, d\theta_1 + e^{\theta_1}\cos\theta_2\, d\theta_2)$$

$$= e^{2\theta_1}\, d\theta_1 \wedge d\theta_2$$

Pull-back $f: \mathcal{M} \to \mathcal{M}_1$, α k-forma, $p \in \mathcal{M}$, $X_i \in T_p\mathcal{M}$

$$(f^*\alpha)_p (X_1, \dots, X_k) = \alpha_{f(p)} (df(X_1), \dots, df(X_k))$$

Nota: el p.b. es como composición.

Otro vector $X_2 = v_{21} \frac{\partial}{\partial \theta_1} + v_{22} \frac{\partial}{\partial \theta_2}$

$$f^*(d\phi \wedge d\psi)(X_1, X_2) = e^{2\theta_1} d\theta_1 \wedge d\theta_2 (X_1, X_2)$$

$$= e^{2\theta_1} (d\theta_1(X_1) d\theta_2(X_2) - d\theta_1(X_2) d\theta_2(X_1))$$

$$= e^{2\theta_1} (v_{11} \cdot v_{22} - v_{21} v_{12}) = e^{2\theta_1} \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix}$$

Pull-back

$f: \mathcal{M} \to \mathcal{M}$, α k-forma, $p \in \mathcal{M}$, $X_i \in T_p\mathcal{M}$

$$(f^*\alpha)_p (X_1, \dots, X_k) = \alpha_{f(p)} (df(X_1), \dots, df(X_k))$$

Nota: el p.b. es como composición.

Otro vector $X_2 = v_{21}\frac{\partial}{\partial \theta_1} + v_{22}\frac{\partial}{\partial \theta_2}$

$$f^*(d\phi \wedge d\psi)(X_1, X_2) = e^{2\theta_1} d\theta_1 \wedge d\theta_2 (X_1, X_2)$$

$$= e^{2\theta_1}(d\theta_1(X_1)\, d\theta_2(X_2) - d\theta_1(X_2)\, d\theta_2(X_1))$$

$$= e^{2\theta_1}(v_{11} \cdot v_{22} - v_{21} v_{12}) = e^{2\theta_1} \begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix}$$

Producto interior

Sea α una k-forma en una variedad $\mathcal{M}$ y X es un vectorial.

El producto interno $i_X \alpha$ (también conocido como contracción) se define por

$$(i_X\alpha)(v_2, \dots, v_k) = \alpha_p (X(p), v_2, \dots, v_k)$$

lleva una k-forma a una (k-1)-forma.

Ejemplo

$\mathcal{M} = \mathbb{T}^2$

$d\theta_1$ y $d\theta_2$ son 2 1-formas.

$X = -\frac{\partial}{\partial \theta_2}$ y $\alpha = d\theta_1 \wedge d\theta_2$, $X_1 = v_{11}\frac{\partial}{\partial \theta_1} + v_{12}\frac{\partial}{\partial \theta_2}$

$X_2 = v_{21}\frac{\partial}{\partial \theta_1} + v_{22}\frac{\partial}{\partial \theta_2}$

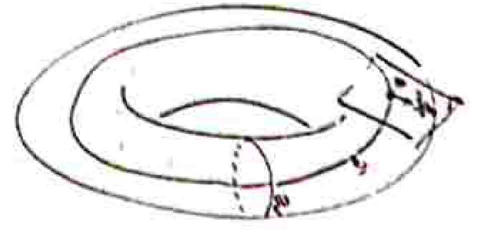
$$i_X \alpha = i_{-\frac{\partial}{\partial \theta_2}} d\theta_1 \wedge d\theta_2$$

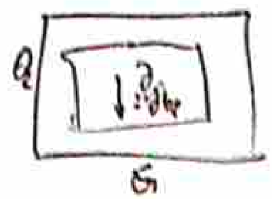
$$i_X\alpha(X_1) = d\theta_1 \wedge d\theta_2\left(-\frac{\partial}{\partial \theta_2}, v_1\frac{\partial}{\partial \theta_1} + v_2\frac{\partial}{\partial \theta_2}\right)$$

$$= d\theta_1\left(-\frac{\partial}{\partial \theta_2}\right) d\theta_2\left(v_1\frac{\partial}{\partial \theta_1} + v_2\frac{\partial}{\partial \theta_2}\right) - d\theta_1\left(v_1\frac{\partial}{\partial \theta_1} + v_2\frac{\partial}{\partial \theta_2}\right) d\theta_2\left(-\frac{\partial}{\partial \theta_2}\right)$$

$$= +v_{11}$$

$$(i_{X_1}\alpha)(X_1) = v_{21}v_{12} - v_{11}v_{22} = -\begin{vmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{vmatrix}$$





Proposición

Sea α una k-forma y β una l-forma en una variedad $\mathcal{M}$. Entonces

$$i_X(\alpha \wedge \beta) = (i_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (i_X \beta)$$

Dem Tarea

Prop

La derivada exterior conmuta con el pull-back

$$d(f^* \alpha) = f^* d\alpha$$

donde α es una k-forma en $\mathcal{N}$ y $f: \mathcal{M} \to \mathcal{N}$ mapeo suave.

Def

Decimos que una k-forma α es exacta si existe una $(k-1)$-forma β tal que $\alpha = d\beta$.

Ejemplo

$\mathcal{M} = \mathbb{T} \times \mathbb{R}$ $\quad (\theta, P)$

Si $\beta = P\,d\theta$ $\quad d\beta = dP \wedge d\theta$

y $\alpha = dP \wedge d\theta$

entonces $\alpha = d\beta$, α es exacta.

$f: \mathbb{T} \times \mathbb{R} \longrightarrow \mathbb{T} \times \mathbb{R}$ (mapeo estándar)

$(\theta, P) \longmapsto (\theta + P + \operatorname{sen}(\theta) \pmod{2\pi}, P + \operatorname{sen}(\theta))$

Tarea f es canónico. $Df^T J Df = J$

$\det(Df^T)\det(J)\det(Df) = \det(J)$

$$f^*\beta = (P + \operatorname{sen}(\theta))\, d(\theta + P + \operatorname{sen}(\theta))$$
$$= (P + \sin(\theta))[(1 + \cos(\theta))d\theta + dP]$$
$$d(f^*\beta) = [dP + \cos\theta\, d\theta] \wedge [(1 + \cos(\theta))d\theta + dP]$$
$$= dP \wedge d\theta + \cos(\theta)\, \cancel{dP \wedge d\theta + \cos\theta\, d\theta \wedge dP} \to 0$$
$$= dP \wedge d\theta$$
$$f^* d\beta = f^*\alpha = d(P + \operatorname{sen}\theta) \wedge d(\theta + P + \operatorname{sen}(\theta))$$
$$= [dP + \cos(\theta)d\theta] \wedge [(1 + \cos(\theta))d\theta + dP]$$
$$= dP \wedge d\theta = \alpha \qquad f^*\alpha = \alpha$$

f preserva la forma simpléctica.