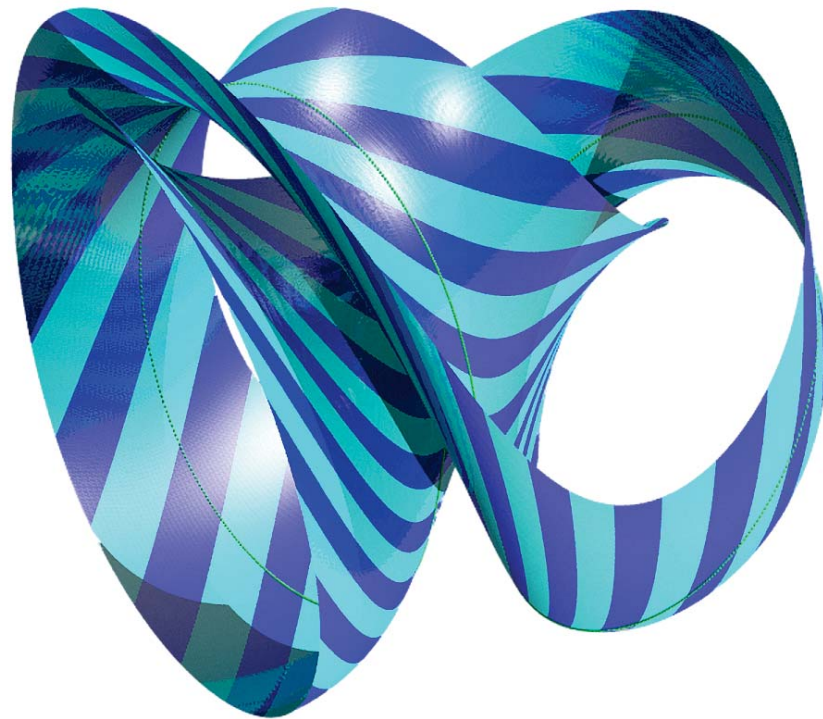


Computer assisted proofs in Nonlinear Dynamics



Jean-Philippe Lessard
McGill University

Many problems in differential equations can be reduced to questions about the zeros of some (possibly infinite dimensional) nonlinear function. Examples include equilibrium solutions, periodic solutions, their attached invariant manifolds, connecting orbits, invariant tori, etcetera. In many applications, especially when there are no small parameters, the resulting nonlinear equations are too difficult to solve by hand, and we resort to numerical simulations to better understand the problem.

Jay Mireles James
Florida Atlantic University

The main question addressed in this workshop is this: suppose we have computed a good numerical approximation -- can we establish the existence of a true solution nearby? Combining tools from functional analysis, complex analysis, numerical analysis, and interval computing, we see that for many of the problems mentioned above the answer is yes. Over the three days we will give a broad and example driven overview of the field of validated numerics.



An Introduction to Computer Assisted Proofs In Nonlinear Analysis

1 de agosto 4-7pm
2 de agosto 4-7pm
3 de agosto 4-6pm

Salón 200
Edificio Anexo
IIMAS - UNAM



Nonlinear Equations

- Let X, Y be Banach spaces, and $F: X \rightarrow Y$ a continuous (or better yet smooth) map.
- We are interested in solutions of the equation
$$F(x) = 0$$
- For example $x \in X$ could be an equilibrium or periodic solution of a differential equation, a chart or covering map for an invariant manifold, a connecting orbit, ...
- In many cases there are powerful numerical methods for finding approximate solutions.

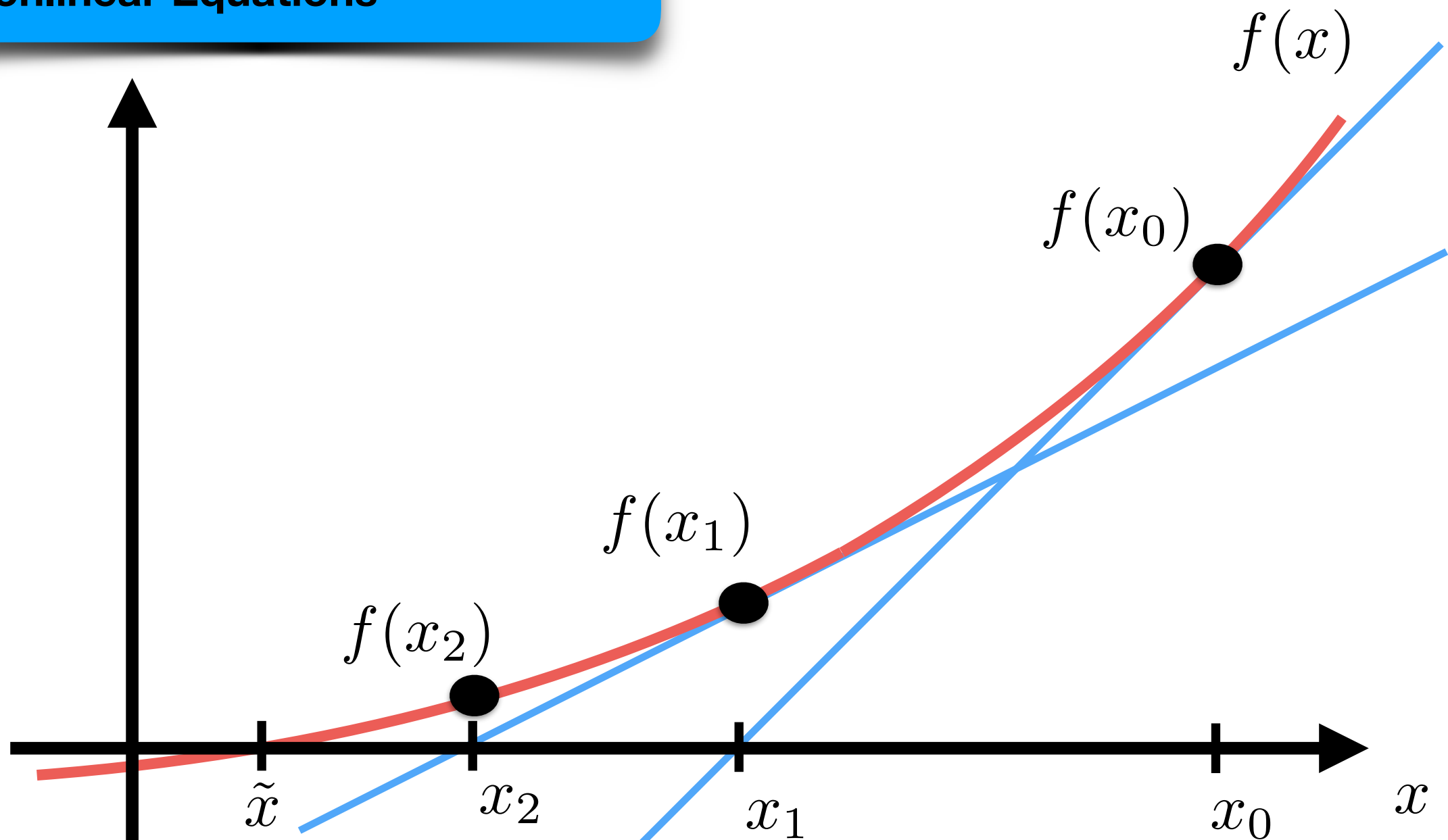
Nonlinear Equations

- **Main Question:** given a “good enough” approximate solution $\bar{x} \in X$ with $F(\bar{x}) \approx 0$, can we conclude that there exists a unique true solution “nearby”?
- The answer is often “yes”, depending on X , Y , F , and $\bar{x}$.
- Requires making precise terms “good enough” and “nearby”.

A theorem answering the main question is called an
a-posteriori existence theorem.

- Consider the situation in one dimension...

Nonlinear Equations



Newton's method

$$x_{n+1} = x_n + h_n$$

where

$$f'(x_n)h_n = -f(x_n)$$

Nonlinear Equations

Theorem (a-priori analysis of Newton): Suppose that f is C^2 . Assume that $f(\tilde{x}) = 0$, and that $f'(\tilde{x}) \neq 0$. Then there exists an $\epsilon > 0$ so that for all $a \in (\tilde{x} - \epsilon, \tilde{x} + \epsilon)$ the Newton sequence initialized with $x_0 = a$ has,

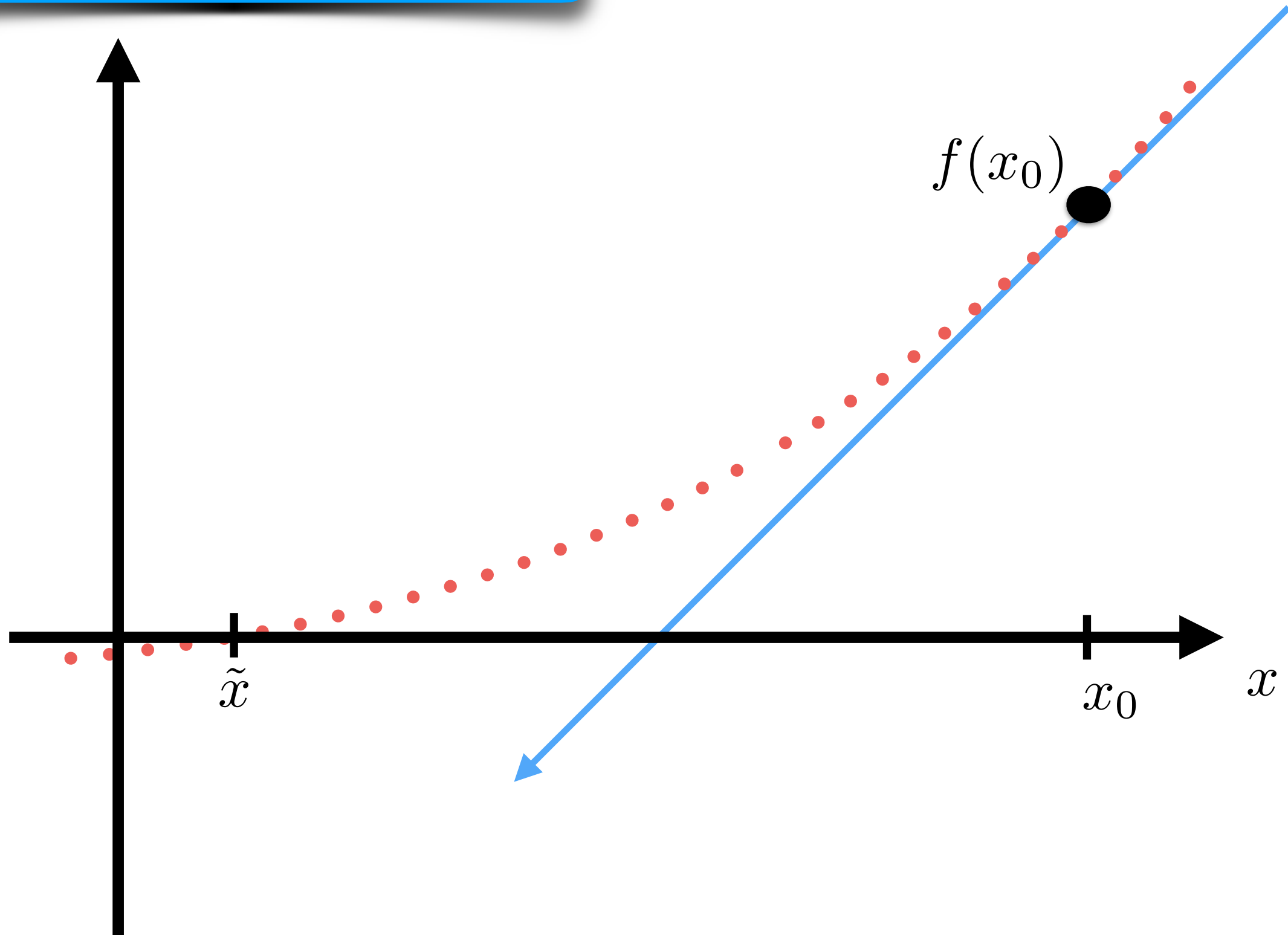
$$\lim_{n \rightarrow \infty} x_n = \tilde{x}.$$

Moreover,

$$|x_{n+1} - \tilde{x}| \leq C|x_n - \tilde{x}|^2$$

- If f is C^2 then Newton is appropriate – converges rapidly.
- What about an a-posteriori result?
Should be possible...

Nonlinear Equations



Should be possible...

Nonlinear Equations

A little background:

- To generalize this argument to higher (infinite) dimensions we need the contraction mapping theorem.

Def: Let X be a metric space and $f: X \rightarrow X$ be a map. Suppose there is a $0 \leq \kappa < 1$ so that

$$d(f(x), f(y)) < \kappa d(x, y) \quad x, y \in X$$

We call f a contraction on X .

Thm: suppose that X is a complete metric space and that $f: X \rightarrow X$ is a contraction mapping. Then f has a unique fixed point in X .

Nonlinear Equations

A little background:

- If X is a Banach space and $B \subset X$ is closed, then B is a complete metric space with the metric space.
- In a Banach space the contraction condition is

$$\|f(x) - f(y)\|_X < \kappa \|x - y\|_X$$

for some $0 \leq \kappa < 1$.

Nonlinear Equations

A little background:

- Recall that $F: X \rightarrow Y$ with X, Y Banach spaces, is *Fréchet differentiable* at $x_0 \in X$, if there exists a bounded linear operator $A: X \rightarrow Y$ having that

$$\lim_{h \rightarrow 0} \frac{\|F(x_0 + h) - F(x_0) - Ah\|_Y}{\|h\|_X} = 0.$$

Such an operator, if it exists, is unique. We write

$$DF(x_0) := A$$

and call this the derivative of F at x_0 .

Nonlinear Equations

A little background:

- This associates with $x \in X$ an element $DF(x)$ of the Banach space $B(X, Y)$. We say that $F \in C^1(X, Y)$ if this correspondence is continuous.
- Thinking of $DF(x): X \rightarrow B(X, Y)$ as a map between Banach spaces lets us define second derivatives the same way.
- One can prove a *mean value inequality*:

Suppose that $F \in C^1(X, Y)$ and that $M \geq 0$ has

$$\sup_{w \in B} \|DF(w)\|_{B(X, Y)} \leq M \quad B \subset X \quad (B \text{ convex})$$

Then $\|F(x_1) - F(x_2)\|_Y \leq M \|x_1 - x_2\|_X \quad x_1, x_2 \in B.$

Nonlinear Equations

A little background:

- One can prove a *mean value inequality*:

Suppose that $F \in C^1(X, Y)$ and that $M \geq 0$ has

$$\sup_{w \in B} \|DF(w)\|_{B(X, Y)} \leq M \quad B \subset X \quad (B \text{ convex})$$

Then $\|F(x_1) - F(x_2)\|_Y \leq M \|x_1 - x_2\|_X \quad x_1, x_2 \in B.$

- Suppose that $B \subset X$ is closed and convex.

If

$$\sup_{w \in B} \|DF(w)\|_{B(X, Y)} < 1$$

then F is a contraction on the complete metric space B .

Nonlinear Equations

A-posteriori Analysis of Nonlinear Equations

Nonlinear Equations

The setup:

■ Let X, Y be Banach spaces and $F \in C^1(X, Y)$.

■ We are interested in an $\bar{x} \in X$ having

$$\|F(\bar{x})\|_Y \ll 1. \quad (\text{an approximate solution})$$

■ In general we have no reason to expect that F is a contraction near $\bar{x}$.

■ In this context Newton's method is

$$x_{n+1} = x_n + h_n$$

where h_n is a solution of the linear equation

$$DF(x_n)h_n = -F(x_n)$$

Nonlinear Equations

The setup:

- This suggests we look for fixed points of the operator

$$T(x) = x - DF(x)^{-1}F(x) \quad (\text{Newton operator})$$

near $\bar{x}$.

- But this problem is often too hard...

$DF(x)$ varies in a nonlinear way

so $DF(x)^{-1}$ is a nightmare...

- **So cheat:** replace $DF(x)^{-1}$ with $DF(\bar{x})^{-1}$. Get:

$$T(x) = x - DF(\bar{x})^{-1}F(x) \quad (\text{Newton-like operator})$$

Nonlinear Equations

The setup:

■ Still too difficult... $DF(\bar{x})$ may be hard to invert.

■ Cheat again:

replace $DF(\bar{x})$ with any $A^\dagger \in B(X, Y)$ you like.

replace $DF(\bar{x})^{-1}$ with any $A \in B(Y, X)$ you like.

Define $T(x) = x - AF(x)$

Well... would actually be nice if:

$$A^\dagger \approx DF(\bar{x})$$

$$A \approx \text{inv}(A^\dagger)$$

If A is one to one
then fixed points of T
correspond to zeros
of F .

Nonlinear Equations

Theorem (a-posteriori Newton-Kantorovich with bells and whistles)

Suppose that $F: X \rightarrow Y$ is continuously differentiable and that

$\bar{x} \in X$, $A^\dagger \in B(X, Y)$, and $A \in B(Y, X)$, with A one-to-one.

Let $Y_0, Z_0, Z_1 > 0$ be positive constants, and $Z_2: [0, \infty) \rightarrow [0, \infty)$ be a positive function, all satisfying the following conditions:

- $\|AF(\bar{x})\|_X \leq Y_0$, (small defect/a-posteriori error)
- $\|\text{Id} - AA^\dagger\|_{B(X)} \leq Z_0$, (approximate inverse)
- $\|A(A^\dagger - DF(\bar{x}))\|_{B(X)} \leq Z_1$, (approximate derivative)
- $\sup_{x \in \overline{B_r(\bar{x})}} \|A(DF(\bar{x}) - DF(x))\|_{B(X)} \leq Z_2(r)r$ (local control)

Define the function $p(r) = Z_2(r)r^2 - (1 - Z_0 - Z_1)r + Y_0$. If there is an $r > 0$, so that $p(r) < 0$, then there exists a unique $\tilde{x} \in B_r(\bar{x})$ so that

$$F(\tilde{x}) = 0.$$

Nonlinear Equations

Remarks:

- In many applications it can be arranged that $Z_2(r)$ and hence $p(r)$ are polynomial.
So this is sometimes called the method of *radii-polynomials*.
Reduces infinite dimensional problem to 1D nonlinear.
- In applications we look for an interval $I = [r_-, r_+]$ so that $r \in I$ implies $p(r) < 0$.
(Note that $p(0) = Y_0 > 0$ and $p(r) \rightarrow \infty$ and $r \rightarrow \infty$).
- In this case r_- gives the sharpest error bounds, and (by uniqueness) r_+ gives a lower bound on isolation.

Nonlinear Equations

Remarks:

- The theorem and its proof require small modifications of the classic Newton-Kantorovich Theorem. The proof follows very closely the proof of the implicit function theorem. The role of $A, A^\dagger$ in the theorem is to emphasize the appearance of numerical approximations/truncations of linear operators in computer assisted proofs.
- Similar theorems are found throughout the literature on computer assisted methods of proof in nonlinear analysis.
(A bibliography appears at the end of the slides)
- This approach to computer assisted proof is in the functional analytic tradition going back to the work of Lanford, Eckmann, Wittwer, and Koch on fixed points of renormalization operators and the first proof of the Feigenbaum conjectures

Nonlinear Equations

Proof: Assume that $r > 0$ has $p(r) < 0$.

Define the operator $T: \overline{B_r(\bar{x})} \subset X \rightarrow X$, by

$$T(x) = x - AF(x), \quad x \in \overline{B_r(\bar{x})}.$$

We will show that:

- T maps into $B_r(\bar{x})$.
- T is a contraction on $\overline{B_r(\bar{x})}$.

By the Banach fixed point theorem we will conclude that T has a unique fixed point $\tilde{x} \in \overline{B_r(\bar{x})}$.

Since A is one-to-one this gives that $\tilde{x}$ is the unique zero of F in $\overline{B_r(\bar{x})}$.

Nonlinear Equations

Proof: First, note that T is continuously differentiable on $\overline{B_r(\bar{x})}$ as F is. Moreover $DT(x) = \text{Id} - ADF(x)$ for all $x \in \overline{B_r(\bar{x})} \subset X$.

We want to bound the norm

$$\begin{aligned} \|DT(x)\|_{B(x)} &= \|\text{Id} - ADF(x)\|_{B(X)} && \text{(add and subtract } AA^\dagger, ADF(\bar{x})\text{)} \\ &\leq \|\text{Id} - AA^\dagger\|_{B(X)} + \|A(A^\dagger - DF(\bar{x}))\|_{B(x)} + \|A(DF(\bar{x}) - DF(x))\|_{B(X)} \\ &\leq Z_0 + Z_1 + Z_2(r)r. \end{aligned}$$

But $p(r) = Z_2(r)r^2 - (1 - Z_1 - Z_0)r + Y_0 < 0$,

Implies $Z_2(r)r^2 + (Z_1 + Z_0)r + Y_0 < r$,

or $Z_2(r)r + (Z_1 + Z_0) + \frac{Y_0}{r} < 1$,

But $Y_0, r > 0$, so

$$\|DT(x)\|_{B(X)} \leq Z_2(r)r + Z_1 + Z_0 < 1, \quad x \in \overline{B_r(\bar{x})}.$$

Nonlinear Equations

Proof: Now let $x \in \overline{B_r(\bar{x})}$. Consider

$$\begin{aligned}
 \|\bar{x} - T(x)\|_X &\leq \|\bar{x} - T(\bar{x})\|_X + \|T(x) - T(\bar{x})\|_X && \text{(add and subtract } T(\bar{x})) \\
 &\leq \|AF(\bar{x})\|_X + \sup_{w \in \overline{B_r(\bar{x})}} \|DT(w)\|_{B(X)} \|x - \bar{x}\|_X \\
 &\leq Y_0 + (Z_2(r)r + Z_1 + Z_0) \|x - \bar{x}\|_X \\
 &\leq Y_0 + (Z_2(r)r + Z_1 + Z_0) r \\
 &\leq Z_2(r)r^2 + (Z_1 + Z_0) r + Y_0 \\
 &< r
 \end{aligned}$$

as $p(r) < 0$.

Then $T\left(\overline{B_r(\bar{x})}\right) \subset B_r(\bar{x}) \subset \overline{B_r(\bar{x})}$.

And since X is a Banach space $\overline{B_r(\bar{x})}$ is a complete metric space.

Nonlinear Equations

Proof: Choose $x, y \in \overline{B_r(\bar{x})}$ and consider

$$\begin{aligned} \|T(y) - T(x)\|_X &\leq \sup_{w \in \overline{B_r(\bar{x})}} \|DT(w)\|_{B(X)} \|x - y\|_X \\ &\leq (Z_2(r)r + Z_1 + Z_0) \|x - y\|_X \end{aligned}$$

with $Z_2(r)r + Z_1 + Z_0 < 1$.

Then T is a strict contraction on the complete metric space $\overline{B_r(\bar{x})}$ as desired. Moreover, since T maps into $B_r(\bar{x})$ so the unique fixed point $\tilde{x} \in \overline{B_r(\bar{x})}$ given by the Contraction Mapping Theorem has $\tilde{x} \in B_r(\bar{x})$. (Not on the boundary). Again, since A is one-to-one $T(\tilde{x}) = \tilde{x}$ implies that $F(\tilde{x}) = 0$.



Nonlinear Equations

Remarks:

- Suppose that F is only defined on an open set $U \subset X$.
- Suppose also that F is C^2 on $\overline{B_{r_*}(\bar{x})} \subset U$, and that

$$\sup_{x \in \overline{B_{r_*}(\bar{x})}} \|D^2 F(x)\|_{\text{bi-linear}(X \times X, Y)} \leq M.$$

- Then we have the bound

$$\sup_{x \in \overline{B_r(\bar{x})}} \|A(DF(\bar{x}) - DF(x))\|_{B(X)} \leq \|A\|_{B(Y, X)} M r$$

for all $0 < r \leq r_*$.

- Define $Z_2(r) = \|A\|_{B(Y, X)} M$ for $0 < r \leq r_*$,
and $p(r) = Z_2(r)r^2 - (1 - Z_1 - Z_0) + Y_0$.

A nearly identical proof shows that if $0 < r \leq r_*$, and $p(r) < 0$, then there is a unique zero of F in $\overline{B_r(\bar{x})}$.

Computer Assisted Validation

- In practice we need:
 - A numerical approximate solution $\bar{x} \in X$.
A numerical approximation $A^\dagger$ of $DF(\bar{x})$.
A numerical approximation A of $\text{inv}(A^\dagger)$.
(Also some understanding of the asymptotics...)
 - The ability to check some bounds like

$$\|AF(\bar{x})\|_X \leq Y_0$$

This is where *interval arithmetic* comes in...

Computer Assisted Validation

Remarks:

- Ideas go back to the book *Interval Analysis* by Ramon E. Moore in 1966
- The literature on interval arithmetic is vast.
The Journal *Reliable Computing* is devoted to this topic.
- Many of the examples we discuss this week are implemented using IntLab running under MatLab:

S.M. Rump: INTLAB - INTerval LABoratory. In Tibor Csendes, editor, Developments in Reliable Computing, pages 77-104. Kluwer Academic Publishers, Dordrecht, 1999.

Computer Assisted Validation

- **Interval arithmetic:** software for managing round-off.

Example: $1/3$ does not have floating point representation.

But computer can tell you – Thm:

$$\frac{1}{3} \in [0.3333333333333333, 0.3333333333333334]$$

Thm:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \in [1.2833333333333333, 1.2833333333333334]$$

Cor:

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \leq 1.2833333333333334$$

Computer Assisted Validation

- **Interval arithmetic:** software for managing round-off.

$$\cos([-0.1, 0.1]) \subset [0.99500416527802, 1.0000000000000000]$$

$$\frac{1}{\sqrt{0.5 - [0.49, 0.495]}} \subset [9.9999999999999999, 14.14213562373095]$$

The MatLab library IntLab is a great option.

There are also C^{++} and Julia libraries...

Computer Assisted Validation

Example: Prove that $f(x) = \cos(x)$ has a fixed point.

Newton's method started from $x_0 = 0$ converges after 5 steps to $\bar{x} = 0.739085133215161$

Let's apply the radii-polynomial method to $F(x) = \cos(x) - x$.

Then $DF(x) = -\sin(x) - 1$ and $D^2F(x) = -\cos(x)$

Let $A^\dagger = -1.67361202918321$

(by evaluating: $-\sin(\bar{x}) - 1$ in double precision).

Let $A = -0.59751004567530$

(by evaluating: $1/A^\dagger$ in double precision).

Let $r_* = 0.1$ and $M = |\cos([\bar{x} - r_*, \bar{x} + r_*])|$ (this is an interval)

Computer Assisted Validation

Example: Prove that $f(x) = \cos(x)$ has a fixed point.

Let's apply the radii-polynomial method to $F(x) = \cos(x) - x$.

$$\bar{x} = 0.739085133215161$$

$$r_* = 0.1$$

$$\text{Let } A^\dagger = -1.67361202918321$$

$$M = [0.66814379567638, 0.80264177641192]$$

$$\text{Let } A = -0.59751004567530$$

Check that

$$|AF(\bar{x})| \in [0, 1.32673882030072 \times 10^{-16}] = Y_0$$

$$|1 - AA^\dagger| \in [0, 1.111 \times 10^{-16}] = Z_0$$

$$|A(A^\dagger - DF(\bar{x}))| \in [0, 1.2 \times 10^{-16}] = Z_1$$

$$|A|M \in [0.3993, 0.4796] = Z_2(r) \quad r \leq 1.0$$

One checks using the quadratic formula that

$$p(r) = Z_2 r^2 - (1 - Z_1 - Z_0)r + Y_0 < 0$$

$$\text{for } r \in [1.327 \times 10^{-16}, 0.1]$$

Then $\cos(x)$ has a unique fixed point x_* with $x_* \in [\bar{x} - 1.327 \times 10^{-16}, \bar{x} + 1.327 \times 10^{-16}]$

Computer Assisted Validation

Example: Prove that $f(x) = \cos(x)$ has a fixed point.

Recall from numerical analysis that $f(x) = \cos(x)$ is a contraction near its fixed point...
What if we try to directly apply the contraction mapping theorem?

$$r = 1.6 \times \varepsilon_{\text{machine}} \qquad \varepsilon_{\text{machine}} = 2^{-52} \approx 2.22044 \times 10^{-16}$$

$$B = [\bar{x} - r, \bar{x} + r]$$

Then $B = [0.73908513321516, 0.73908513321517]$
and we check that $\cos(B) = [0.73908513321516, 0.73908513321517]$
and that $|\sin(B)| = [0.67361202918321, 0.67361202918322]$

Again, by the contraction mapping theorem we have a unique fixed point in B .

However the proof fails with $r = 1.5 \times \varepsilon_{\text{machine}}$.

With Newton-Krawczyk the proof goes through for $r = 1.1 \times \varepsilon_{\text{machine}}$.

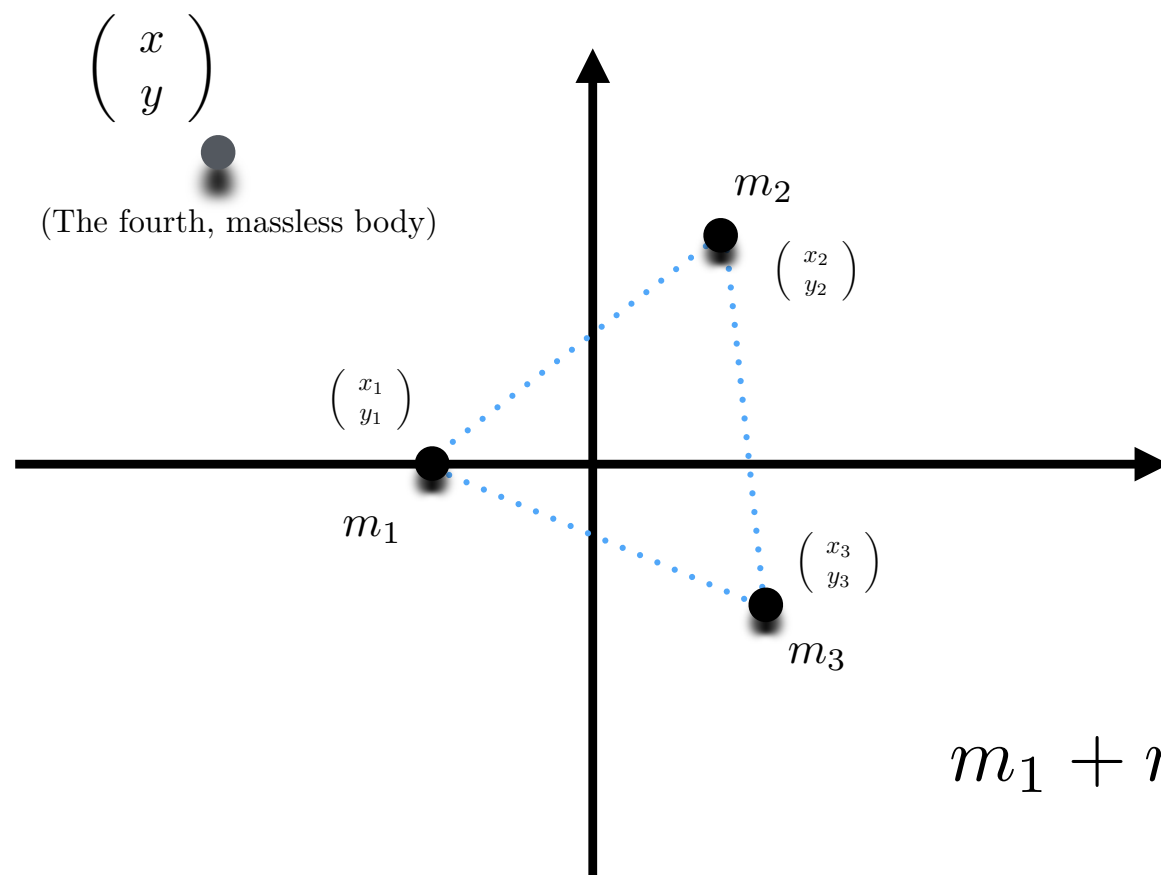
But not for $r = \varepsilon_{\text{machine}}$.

Computer Assisted Validation

- All “Interval Newton” methods require either guessing the ball to check, or iterating an interval operator.
- The radii-polynomial method requires neither of these, produces sharper bounds, and gives useful isolation information.
- But “Interval Newton” methods don’t have to analyze second derivative information... easy to implement.

Computer Assisted Validation

A restricted four body problem



$$r_j(x, y) = \sqrt{(x - x_j)^2 + (y - y_j)^2}$$

$$j = 1, 2, 3$$

$$m_1 + m_2 + m_3 = 1$$

$$x' = v$$

$$v' = 2w + x - m_1 \frac{x - x_1}{r_1(x, y)^3} - m_2 \frac{x - x_2}{r_2(x, y)^3} - m_3 \frac{x - x_3}{r_3(x, y)^3}$$

$$y' = w$$

$$w' = -2v + y - m_1 \frac{y - y_1}{r_1(x, y)^3} - m_2 \frac{y - y_2}{r_2(x, y)^3} - m_3 \frac{y - y_3}{r_3(x, y)^3}$$

Computer Assisted Validation

An equilibrium solution has

$$0 = v$$

$$0 = 2w + x - m_1 \frac{x - x_1}{r_1(x, y)^3} - m_2 \frac{x - x_2}{r_2(x, y)^3} - m_3 \frac{x - x_3}{r_3(x, y)^3}$$

$$0 = w$$

$$0 = -2v + y - m_1 \frac{y - y_1}{r_1(x, y)^3} - m_2 \frac{y - y_2}{r_2(x, y)^3} - m_3 \frac{y - y_3}{r_3(x, y)^3}$$

So, find $f(x, y) = \mathbf{0}$ where

$$f(x, y) = \begin{pmatrix} x - m_1 \frac{x - x_1}{r_1(x, y)^3} - m_2 \frac{x - x_2}{r_2(x, y)^3} - m_3 \frac{x - x_3}{r_3(x, y)^3} \\ y - m_1 \frac{y - y_1}{r_1(x, y)^3} - m_2 \frac{y - y_2}{r_2(x, y)^3} - m_3 \frac{y - y_3}{r_3(x, y)^3} \end{pmatrix}$$

$$m_1 + m_2 + m_3 = 1 \quad \text{and} \quad (x, y) \in \mathbb{R}^2 \setminus (x_1, y_1) \cup (x_2, y_2) \cup (x_3, y_3)$$

Computer Assisted Validation

- If you choose $\|x\| := \max_{1 \leq j \leq n} |x_j|$

then the induced matrix norm is

$$\|A\|_M := \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}| \quad (\text{max rows - sum columns})$$

and the induced norm on bilinear forms is

$$\|B\|_Q := \max_{1 \leq i \leq n} \sum_{j=1}^n \sum_{k=1}^n |b_{ijk}|$$

Computer Assisted Validation

A restricted four body problem

- Numerical studies suggest that there are 8, 9, or 10.
(Simó – 77)
- Recent computer assisted proof by Leandro.
(See the Bibliography)

Computer Assisted Validation

A restricted four body problem

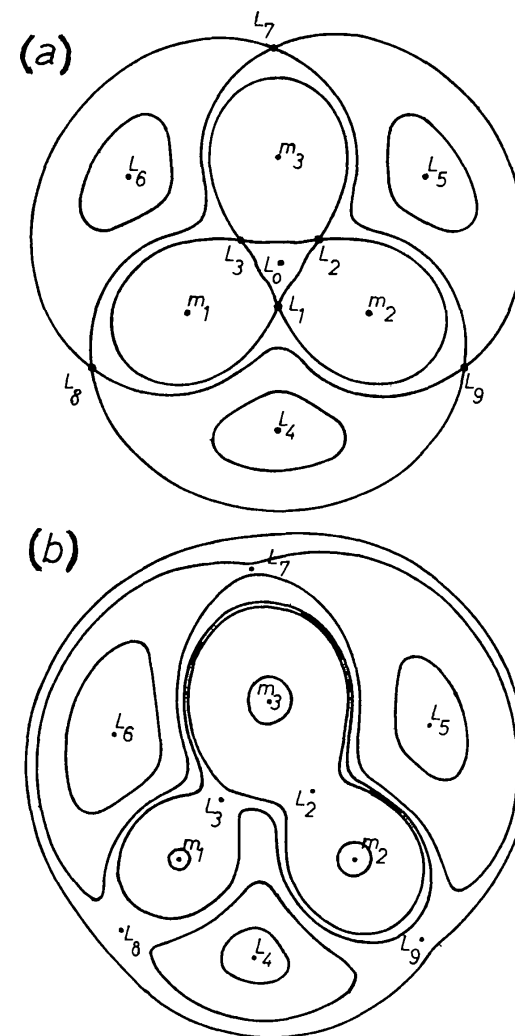
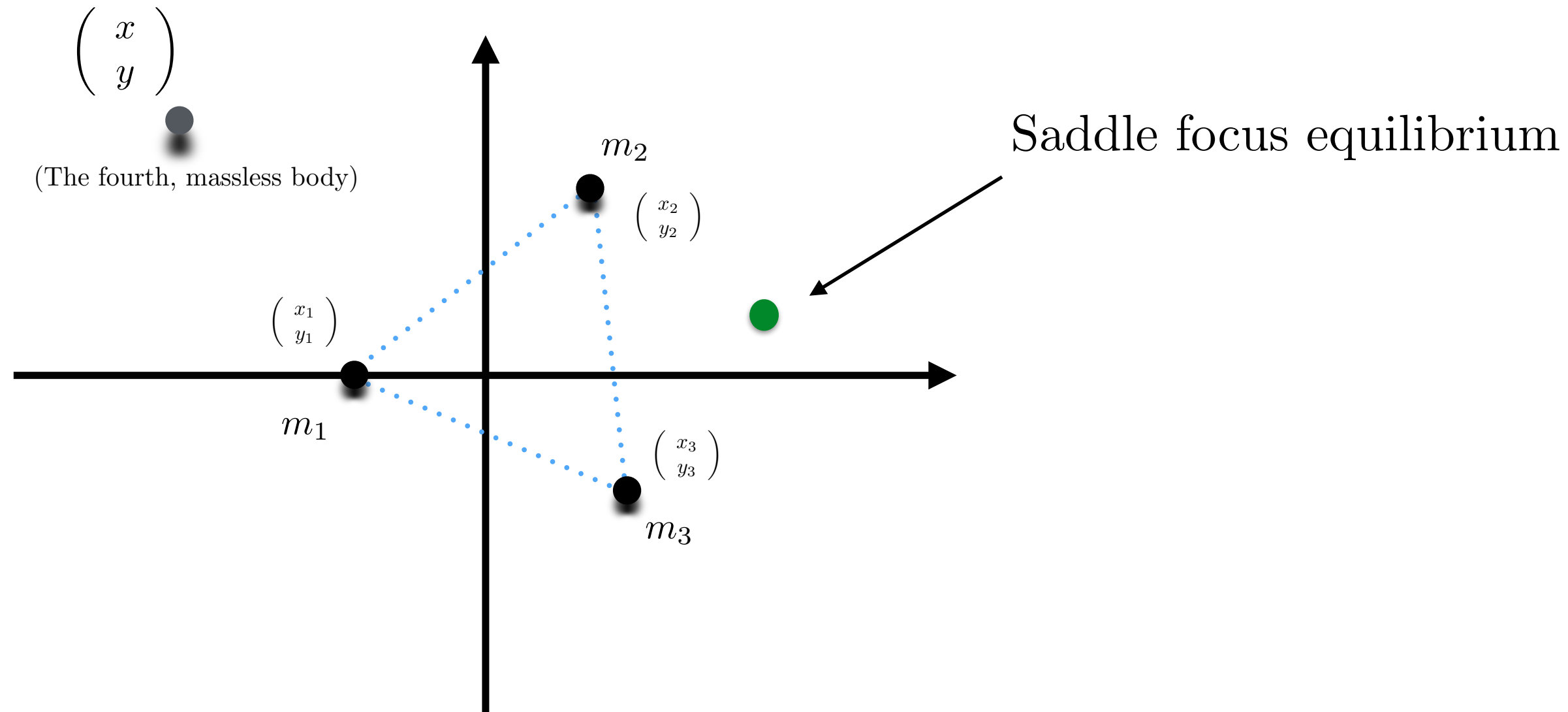


Fig. 1a-b. Relative equilibrium solutions for the 3 + 1 body, Lagrangian case, and equipotential curves.

(a) Finite masses: $m_1 = m_2 = m_3 = 1/3$. (b) Finite masses: $m_1 = 0.2$, $m_2 = 0.3$, $m_3 = 0.5$.

Computer Assisted Validation



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Thm: (Henrard) suppose that $f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is a Hamiltonian vector field, with a saddle-focus equilibrium. If there exists transverse (relative to the energy section) homoclinic orbit then there is a cylinder of periodic orbits, parameterized by energy, accumulating to the homoclinic.

(Blue sky catastrophe)

Thm: (Devaney) suppose that $f: \mathbb{R}^4 \rightarrow \mathbb{R}^4$ is a Hamiltonian vector field, with a saddle-focus equilibrium. If there exists transverse (relative to the energy section) homoclinic orbit then f admits a chaotic horseshoe (near the homoclinic).

(See the Bibliography)

Computer Assisted Validation

The first step is to prove the existence of a saddle-focus.

Proposition: when $m_1 = 1/2$, $m_2 = 3/10$ and $m_3 = 1/5$ the CRFBP has a saddle-focus equilibrium with

$$\tilde{x} \in B_r(\bar{x})$$

where $r = 3 \times 10^{-15}$ and

$$\bar{x} = \begin{pmatrix} 0.927099246135636 \\ 0 \\ 0.217703423699760 \\ 0 \end{pmatrix}.$$

Computer Assisted Validation

Proof:

From Newton,

$$\bar{x} = 0.927099246135636, \quad \bar{y} = 0.217703423699760.$$

Evaluate $Df(\bar{x}, \bar{y})$ (no intervals) and get

$$A^\dagger := \begin{pmatrix} 2.074531863336581 & 0.163688766491296 \\ 0.163688766491296 & 1.448616847931906 \end{pmatrix}.$$

Compute an approximate inverse

$$A := \begin{pmatrix} 0.486372903543232 & -0.054958480394206 \\ -0.054958480394206 & 0.696523782188810 \end{pmatrix}.$$

Bound (using interval arithmetic)

$$\|A\|_M \leq 0.75148226258302$$

Computer Assisted Validation

Interval enclosure of the defect:

$$Af(\bar{x}, \bar{y}) \in 10^{-14} \begin{pmatrix} [-0.24035361087522, 0.18345710317770] \\ [-0.25639867955817, 0.29923189175017] \end{pmatrix}$$

Taking

$$Y_0 := 0.29923189175017 \times 10^{-14}.$$

we have

$$\|Af(\bar{x}, \bar{y})\| \leq Y_0$$

Again, compute the interval enclosure

$$\text{Id} - AA^\dagger \in$$

$$10^{-15} \begin{pmatrix} [-0.22204460492504, 0.11102230246252] & [0.00000000000000, 0.02775557561563] \\ [-0.01387778780782, 0.01387778780782] & [-0.22204460492504, 0.11102230246252] \end{pmatrix}$$

$$\text{so } \|\text{Id} - AA^\dagger\|_M \in [0, 0.24980018054067 \times 10^{-15}].$$

$$\text{Then take } Z_0 := 0.24980018054067 \times 10^{-15}.$$

Computer Assisted Validation

Next compute the enclosure

$$A(A^\dagger - Df(\bar{x}, \bar{y})) \in$$

$$10^{-13} \begin{pmatrix} [-0.05697595725541, 0.05451094681948] & [-0.08685491176201, 0.08979888842367] \\ [-0.12111986933565, 0.12523041771790] & [-0.08678718041998, 0.08338891260758] \end{pmatrix}$$

so

$$\|A(A^\dagger - Df(\bar{x}, \bar{y}))\|_M \in [0.0000000000000000, 0.20861933032547 \times 10^{-13}]$$

Then take

$$Z_1 := 0.20861933032547 \times 10^{-13}.$$

Computer Assisted Validation

Finally, we choose $r_* = 10^{-6}$

and check that the ball is in the domain.

Use interval arithmetic and known formulas to compute enclosure

$$\sup_{\mathbf{x} \in \overline{B_{r_*}(\bar{x}, \bar{y})}} \|D^2 F(\mathbf{x})\|_Q \in [131.144085137264, 140.058509630884]$$

and

$$\|A\|_M \sup_{\mathbf{x} \in \overline{B_{r_*}(\bar{x}, \bar{y})}} \|D^2 F(\mathbf{x})\|_Q \in [70.992409102564, 105.251485711422].$$

Then let $Z_2 = 105.251485711422$.

The polynomial is:

$$p(r) = Z_2 r^2 - (1 - Z_1 - Z_0)r + Y_0$$

Compute roots and see that $p(r) < 0$ when

$$r \in [2.992618149394393 \times 10^{-15}, 0.01]$$

Restricting to $r \leq r_*$ gives $r \in [3 \times 10^{-15}, 10^{-6}]$

Then there is a solution $(\tilde{x}, \tilde{y})$ with $\|(\bar{x} - \tilde{x}, \bar{y} - \tilde{y})\| < 3 \times 10^{-15}$

- More complicated invariant sets require more work.
Tomorrow I will discuss Taylor approximation of stable and unstable manifolds, as well as Taylor integrators for analytic vector fields.
- I'll also talk about how these tools are combined to produce atlases for large pieces of invariant manifolds. These computations will reveal many connecting orbits.
- Friday: validated numerics for (local) Taylor methods.
- But First: J.P. will illustrate the use of radii-polynomials for an infinite dimensional problem.

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