Computer-assisted proofs for radially symmetric solutions of PDEs

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Abstract

We obtain radially symmetric solutions of some nonlinear (geometric) partial differential equations via a rigorous computer-assisted method. We introduce all main ideas through examples, accessible to non-experts. The proofs are obtained by solving for the coefficients of the Taylor series of the solutions in a Banach space of geometrically decaying sequences. The tool that allows us to advance from numerical simulations to mathematical proofs is the Banach contraction theorem.

Keywords

Radially symmetric solutions · PDEs · Computer-assisted proofs
Taylor series · Contraction mapping theorem

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1 Introduction

In the last decade powerful techniques to bridge the divide, i.e., to turn numerical calculations into mathematically rigorous statements, have developed rapidly (see e.g. [8, 14, 26, 29, 32, 34]). The foundations for the field were laid in [10, 11, 15]. These methods have received renewed attention due to contemporary computational power as well as recent algorithmic advances. With a clever hybrid approach one can off-load the verification of intricate computationally intensive

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estimates to the computer to prove existence to infinite dimensional continuum problems near numerical approximations. This permits using classical ideas (such as fixed point methods) to generate results not accessible by any other means. Thus, going beyond simulations, we can get our hands on solutions of nonlinear (geometric) partial differential equations (PDEs) using the computer and still argue about them with all the rigour of a classical pencil and paper proof.

In this paper we present examples of this technique which we believe are valuable because we solve nontrivial nonlinear PDEs, while the technicalities are minimal (and the associated coding is easily manageable). In particular, we consider two instances of the nonlinear elliptic problem

$$\Delta u + f(u) = 0. \tag{1}$$

In the first type we consider $u = u(x)$ to be scalar with the independent variable x lying in the 2-sphere. In the second type, the unknown $u = u(x)$ is a vector in $\mathbb{R}^2$ and x lies in the unit disc in $\mathbb{R}^2$ (with Dirichlet boundary conditions). The specific choices for the nonlinearities f will be made explicit in Sections 3 and 4.

When restricting attention to solutions which are invariant under rotation (radially symmetric solutions), these PDEs reduce to ordinary differential equations (ODEs). The boundary value problems for these nonlinear ODEs are amenable to an approach based on power series. Numerically one can obtain approximations for finitely many of the Taylor coefficients. To prove that such a finitely truncated series corresponds to a solution of the differential equation, we will apply a Newton-Kantorovich type argument, see Section 2. This not only proves the existence of a solution, but also yields a rigorous bound on the truncation and approximation errors.

Finally, one ingredient that is needed in the analysis in order to control rounding errors, is computation using interval arithmetic. We refer to [19, 22, 27] for an introduction to interval arithmetic. The results in this paper make use of the package *Intlab* [21] for *Matlab*. All matlab scripts are available at [4].

2 A Newton-Kantorovich type theorem

When trying to solve numerically a finite dimensional nonlinear zero-finding for $F : \mathbb{R}^n \to \mathbb{R}^n$, the classical approach is to apply Newton's algorithm, where one iterates the map

$$T(x) = x - DF(x)^{-1}F(x),$$

until the residue gets small. Often this numerical outcome x^{num} is either assumed "good enough" (roughly, in applied mathematics) or "not rigorous" (roughly, in pure mathematics). Our goal is to merge these perspectives by providing a rigorous quantitative statement on how good this numerical outcome is, i.e., how close it is to a solution of $F(x) = 0$. The crucial observation is that Newton's method works so well because the map T is a contraction with a very small contraction constant on a neighborhood of a zero x^{sol} of F, provided that the Jacobian $DF(x^{\text{sol}})$ is invertible, i.e., x^{sol} is a non-degenerate zero of F (the generic situation). This allows us to *prove* that the numerical approximation x^{num} is close to the true solution x^{sol}.

Rather than going into detail about how to obtain such a computer-assisted proof, we first move to the infinite dimensional setting. After detailing in Theorem 2.1 below the setup in general Banach spaces, we return in Remark 2.2 and Example 2.3 to the (relatively straightforward) application to finite dimensional problems.

Consider now $F : X \to X'$ a smooth map between two Banach spaces. One of the main strategies of mathematical analysis is to turn the zero finding problem $F(x) = 0$ into a fixed point problem. The Newton operator itself is usually impractical because the inverse of the

derivative of an infinite dimensional map is hard to work with. Instead, one may *choose* an injective linear map $A \in L(X', X)$ and study the fixed point problem

$$x = T(x) \stackrel{\text{def}}{=} x - AF(x).$$

The main problem is how to choose A such that T is a contraction on some neighborhood of the (unknown) fixed point that we are looking for. In the context of a computer-assisted proof, the approach is the following. First, by your favorite method from scientific computing, choose a finite dimensional 'projection' F^{num} of F, solve it numerically to find x^{num}, and reinterpret x^{num} as an element of the infinite dimensional space, which we denoted by $\bar{x}$ (hence $F^{\text{num}}(x^{\text{num}}) \approx 0$ and $F(\bar{x}) \approx 0$). We expect the solution to be close to $\bar{x}$, hence we would like to choose A an approximate inverse of $DF(\bar{x})$.

How to determine when A is an appropriately accurate approximate inverse? And on which neighborhood of $\bar{x}$ do we get that T is a contraction? The following theorem is one way to make this precise. It uses, as an intermediate tool, and approximation $A^\dagger \in L(X, X')$ of $DF(\bar{x})$.

Theorem 2.1. [Radii polynomial approach in Banach spaces] *Let X and X' be Banach spaces. Denote the norm on X by $\|\cdot\|_X$. Consider bounded linear operators $A^\dagger \in L(X, X')$ and $A \in L(X', X)$. Assume $F\colon X \to X'$ is C^1, that A is injective and that*

$$AF\colon X \to X. \tag{2}$$

Consider an approximate solution $\bar{x}$ of $F(x) = 0$ (usually obtained using Newton's method on a finite dimensional projection). Let Y_0, Z_0, Z_1, and Z_2 be positive constants satisfying

$$\|AF(\bar{x})\|_X \leq Y_0 \tag{3}$$

$$\|I - AA^\dagger\|_{B(X)} \leq Z_0 \tag{4}$$

$$\|A[DF(\bar{x}) - A^\dagger]\|_{B(X)} \leq Z_1, \tag{5}$$

$$\|A[DF(c) - DF(\bar{x})]\|_{B(X)} \leq Z_2 r, \quad \text{for all } \|c - \bar{x}\|_X \leq r, \tag{6}$$

where $\|\cdot\|_{B(X)}$ is the induced operator norm for bounded linear operators from X to itself. Define the radii polynomial

$$p(r) \stackrel{\text{def}}{=} Z_2 r^2 - (1 - Z_1 - Z_0)r + Y_0. \tag{7}$$

If there exists $r_0 > 0$ such that

$$p(r_0) < 0,$$

then there exists a unique $\tilde{x} \in B_{r_0}(\bar{x}) \stackrel{\text{def}}{=} \{x \in X \mid \|x - \bar{x}\|_X \leq r_0\}$ satisfying $F(\tilde{x}) = 0$.

Proof. Using the mean value theorem, it is not hard to show that T maps $B_{r_0}(\bar{x})$ into itself, and that T is a contraction on that ball with contraction constant $\kappa \leq Z_0 + Z_1 + Z_2 r_0 < 1$. The result then follows directly from the Banach contraction theorem. We refer to [13] for additional details. □

This theorem fits in a long tradition of quantitative Newton-Kantorovich type theorems. The bounds are parametrized in terms of r, so that an appropriate r does not need to be guessed in advance but rather can be determined as the final step of the proof process. Moreover, it allows us to obtain balls for an interval of values for the radius. Small values of r_0 give us the tightest control on the distance between the solution and the numerical approximation, while large values of r_0 provide us with the best information about the isolation of the solution.

Before we show, in Sections 3 and 4 how to apply Theorem 2.1 in practice in infinite dimensional settings, we first come back to finite dimensional problems.

Remark 2.2. *In finite dimensions Theorem 2.1 can be implemented very easily and generally, if one has an interval arithmetic package, and if one has explicit formulas for D_jF_i and $D^2_{jk}F_i$ for $1 \le i,j,k \le n$. We use the ∞-norm $|x| = \max_{1\le i\le n}|x_i|$. Let $x^{\rm num}$ be the numerical approximation of a solution (e.g. found using Newton iterations). Let A be a numerical computed (hence approximate) inverse of the numerically computed $DF(x^{\rm num})$. Now compute* with interval arithmetic *$A^\dagger = DF(x^{\rm num})$, which implies we may set $Z_1 = 0$. Then evaluate, again with interval arithmetic, the following three computable expressions:*

1. *the residue*
$$Y_0 = \sup\Big(\max_{1\le i\le n} |AF(x^{\rm num})|_i\Big),$$
where absolute values are taken component-wise, and sup *denotes the supremum of the interval obtained.*

2. *the matrix norm*
$$Z_0 = \sup\left(\max_{1\le i\le n}\sum_{1\le j\le n} |(I_n - AA^\dagger)_{ij}|\right).$$
By checking that $Z_0 < 1$, one verifies that the hypothesis in Theorem 2.1 that A is injective.

3. *the second derivative estimate (which provides a bound* (6) *via the mean value theorem)*
$$Z_2 = \sup\left(\max_{1\le i\le n}\sum_{1\le k,m\le n}\Big|\sum_{1\le j\le n} A_{ij}D^2_{km}F_j\big(x^{\rm num} + [-r_*, r_*]\big)\Big|\right),$$
where $x^{\rm num} + [-r_*, r_*]$ is the vector of intervals with components $[x^{\rm num}_k - r_*, x^{\rm num}_k + r_*]$. Here we choose a loose a priori upper bound r_* on the value of r, and we bound the second derivative uniformly on this ball of radius r_*.

Since Y_0 and Z_0 are usually near machine precision, the quadratic formula then gives a very small r_0 for which $p(r_0) < 0$. After checking that $r_0 \le r_*$ we can then invoke Theorem 2.1 to prove that a (unique) zero $x^{\rm sol}$ of F lies within distance r_0 of $x^{\rm num}$.

Example 2.3. *We consider the circular restricted four body problem (CR4BP), where three bodies (with masses (m_1, m_2, m_3), normalized so that $m_1 + m_2 + m_3 = 1$ and $m_1 \ge m_2, m_3$) move in circular periodic orbits around their center of mass in a triangular configuration that is fixed in the co-rotating frame. A fourth* massless *satellite moves in the effective potential (in this co-rotating frame)*

$$\Omega(x, y; m_1, m_2, m_3) \stackrel{\rm def}{=} \frac{1}{2}(x^2 + y^2) + \sum_{i=1}^{3} \frac{m_i}{[(x - x_i)^2 + (y - y_i)^2]^{1/2}}.$$

Here (x, y) is the position of the satellite in the plane of the triangle, and the fixed positions (x_i, y_i) of the three bodies can be expressed in terms of their masses:

$$(x_1, y_1) = \big(-\tfrac{M}{2}, 0\big), \qquad (x_2, y_2) = \big(\tfrac{K_{2,3}}{M}, -\tfrac{\sqrt{3}m_3}{M}\big), \qquad (x_3, y_3) = \big(\tfrac{K_{3,2}}{M}, \tfrac{\sqrt{3}m_2}{M}\big),$$

where $K_{i,j} \stackrel{\rm def}{=} (m_1 - m_j)m_j + (2m_1 + m_j)m_i$ and $M \stackrel{\rm def}{=} 2(m_2^2 + m_2m_3 + m_3^2)^{1/2}$, see e.g. [5, 7].

The equilibria of the system are given by the critical points of the effective potential Ω:

$$F(x, y) \stackrel{\rm def}{=} \left(x - \sum_{i=1}^{3} \frac{m_i(x - x_i)}{[(x - x_i)^2 + (y - y_i)^2]^{3/2}}\,,\, y - \sum_{i=1}^{3} \frac{m_i(y - y_i)}{[(x - x_i)^2 + (y - y_i)^2]^{3/2}}\right) = 0\,. \quad (8)$$

It is known that the number of equilibrium points varies from 8 to 10 when the masses are varied (e.g. see [16] and [24]).

Using the general bounds introduced in Remark 2.2 for the finite dimensional case, we applied the radii polynomial approach to prove the existence of several solutions of (8)*, hence yielding rigorous bounds for relative equilibria of (CR4BP) Let us present a sample result. In case of equal masses* $m_1 = m_2 = m_3 = 1/3$*, the routine* `script_equilibria.m`*, available at [4], computes (using Newton's method)* $x^{\text{num}} = (-0.467592983336122\,,\, 0.809894804400869)$*, and yields the bounds* $Y_0 = 1.775 \times 10^{-15}$*,* $Z_0 = 1.23 \times 10^{-14}$ *and* $Z_2 = 12.6987$ *with the choice of* $r_* = 0.02$*. In this case, we obtain* $p(r_0) < 0$ *for any* $r_0 \in [1.78 \times 10^{-15}, 0.02]$*, with* p *the radii polynomial defined in* (7)*. In Figure 1 one can find several sample results, where each point has been rigorously validated using the computer program.*

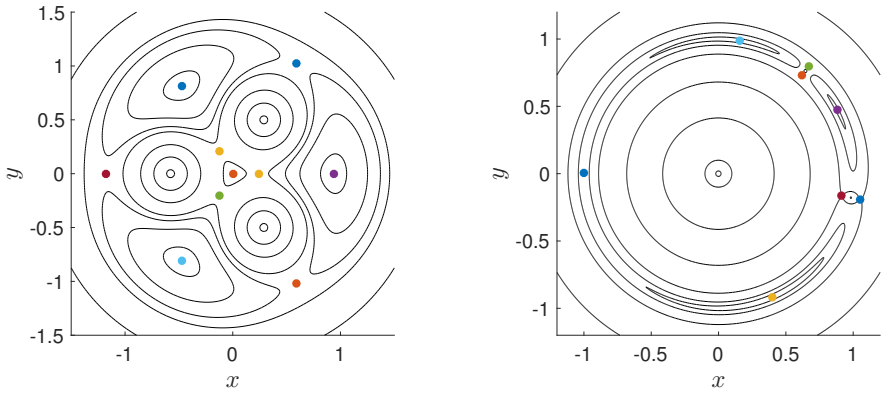

Figure 1: (Left) Ten relative equilibria of (CR4BP) with equal masses. (Right) Eight relative equilibria of (CR4BP) with masses $m_1 = 0.9987451087$, $m_2 = 0.0010170039$ and $m_3 = 0.0002378873$. In both plots, some level sets of the effective potential Ω are depicted.

We now turn our attention to the elliptic PDE problems, where we apply Theorem 2.1 in an infinite dimensional setting.

3 Radially symmetric solutions of a nonlinear Laplace-Beltrami operator on the sphere

We consider the partial differential equation

$$\Delta u + \lambda u + u^2 = 0 \tag{9}$$

posed on the sphere $S^2 \subset \mathbb{R}^3$, where Δ is the Laplace-Beltrami operator on the manifold (the natural geometric generalization of the Laplace operator). Here $\lambda \geq 0$ is a parameter. The PDE (9) describes a classical nonlinear elliptic problem [17], often studied on the unit ball in arbitrary dimension and with a variety of nonlinearities. Here we restrict attention to a quadratic nonlinearity, and we pose the problem on a *sphere*, cf. [9, 31].

Letting $x = r\cos\phi\sin\theta$, $y = r\sin\phi\sin\theta$ and $z = r\cos\theta$, letting $u(x, y, z) = u(r, \phi, \theta)$,

$$\Delta u = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial u}{\partial\theta} \right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 u}{\partial\phi^2}.$$

We look for solutions of (9) that are radially symmetric (with respect to rotations around the z-axis) *and* symmetric in the equator. This reduces the PDE to an ODE and leads to the following boundary value problem (BVP) for $u = u(\theta)$:

$$\begin{cases} u''(\theta) + \cot(\theta)u'(\theta) + \lambda u(\theta) + u(\theta)^2 = 0, & \text{for } \theta \in (0, \frac{\pi}{2}], \\ u'(0) = 0, \\ u'(\frac{\pi}{2}) = 0. \end{cases} \tag{10}$$

The goal is to prove existence of solutions of (10) via the radii polynomial approach (see Theorem 2.1). The first step in doing so is to introduce a zero finding problem of the form $F(x) = 0$ on a Banach space.

3.1 The zero finding problem for the Laplace-Beltrami problem

Our approach will be based on Taylor series, and it turns out that it has advantages (see Remark 3.1 below) to work on a domain $[0, 1]$. Hence we rescale the independent variable $\theta = \frac{\pi}{2}\vartheta$. The algebra is nicer if we also scale the dependent variable, as well as the parameter λ:

$$v(\vartheta) = \frac{\pi^2}{4} u\left(\frac{\pi}{2}\vartheta\right) \qquad \text{and} \qquad \tilde{\lambda} = \frac{\pi^2}{4}\lambda.$$

The BVP (10) in the new variables becomes

$$\begin{cases} v''(\vartheta) + \dfrac{\pi}{2}\vartheta \cot\left(\dfrac{\pi}{2}\vartheta\right) \dfrac{v'(\vartheta)}{\vartheta} + \tilde{\lambda} v(\vartheta) + v(\vartheta)^2 = 0 & \text{for } \vartheta \in (0, 1], \\ v'(0) = 0, \\ v'(1) = 0. \end{cases} \tag{11}$$

For the sake of presentation, here we have anticipated that it will be convenient to split off a factor ϑ in the second term of the differential equation, as it will allow us to deal with smooth functions that are even in ϑ only. We search for v as a power series of ϑ around zero: $v(\vartheta) = \sum_{n=0}^{\infty} a_n \vartheta^n$. Let us assume for the moment that the radius of convergence of our power series is larger than 1, then the coefficients are in a Banach space of geometrically decaying coefficients. More precisely, for $\nu > 1$, denote

$$\ell^1_\nu \stackrel{\text{def}}{=} \Big\{ a = (a_n)_{n\geq 0} : \|a\|_{1,\nu} \stackrel{\text{def}}{=} \sum_{n\geq 0} |a_n| \nu^n < \infty \Big\}.$$

Given $a, c \in \ell^1_\nu$, denote by $a * c$ the Cauchy product given component-wise by

$$(a * c)_n = \sum_{\substack{n_1+n_2=n \\ 0\leq n_1, n_2 \leq n}} a_{n_1} c_{n_2} = \sum_{n_1=0}^{n} a_{n_1} c_{n-n_1}.$$

An important property of the space ℓ^1_ν is that it is a Banach algebra under the Cauchy product:

$$\|a * c\|_{1,\nu} \leq \|a\|_{1,\nu} \|c\|_{1,\nu}. \tag{12}$$

The expansion for the cotangent is

$$\cot(\theta) = \frac{1}{\theta} - 2\sum_{n=1}^{\infty} \frac{\theta^{2n-1}}{\pi^{2n}} \zeta(2n), \quad \text{where } \zeta(2n) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \text{ is the Riemann zeta function.}$$

Hence we get

$$\frac{\pi}{2}\vartheta\cot\left(\frac{\pi}{2}\vartheta\right) = 1 - 2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2^{2n}}\vartheta^{2n} = \sum_{j=0}^{\infty} b_j\vartheta^j,$$

where the b_j are the Taylor coefficients defined by

$$b_0 = 1, \qquad b_j = -2\frac{\zeta(j)}{2^j} \quad \text{if } j \geq 1 \text{ is even}, \qquad b_j = 0 \quad \text{if is odd}.$$

The decay rate of these coefficients shows that we should restrict attention to $1 < \nu < 2$. After expanding all terms in (11) as Taylor series, using the Cauchy product, and equating powers, we arrive at the operator $F(a) = (F_n(a))_{n\geq 0}$ defined as

$$F_n(a) \stackrel{\text{def}}{=} \begin{cases} a_1 & \text{for } n = 0, \\ \sum_{j=1}^{\infty} j a_j & \text{for } n = 1, \\ n(n-1)a_n + (Ja * b)_n + \tilde{\lambda}a_{n-2} + (a * a)_{n-2} & \text{for } n \geq 2. \end{cases} \tag{13}$$

Here the multiplication operator J on sequence spaces is defined by

$$(Ja)_j \stackrel{\text{def}}{=} j a_j \qquad \text{for } j \geq 0.$$

Remark 3.1. *Without the rescaling of the independent variable, the formula for F_1 corresponding to the boundary condition at $\theta = \frac{\pi}{2}$ would have become $\sum_{j\geq 1} j a_j (\frac{\pi}{2})^j$. Evaluating this sum (say up to some finite N) is numerically unstable because of the high powers of $\frac{\pi}{2}$. Hence the rescaling.*

The following result expresses that to solving $F(a) = 0$ leads to a solution of (11).

Lemma 3.2. *Let $\nu \in (1,2)$ and let $a = (a_n)_{n\geq 0} \in \ell^1_\nu$ be such that $F_n(a) = 0$ for all $n \geq 0$. Then $v(\vartheta) = \sum_{n\geq 0} a_n \vartheta^n$ is a solution of (11).*

Proof. Let $\nu \in (1,2)$ and $a \in \ell^1_\nu$. Then the power series $v(\vartheta) = \sum_{n\geq 0} a_n\vartheta^n$ converges uniformly for $\vartheta \in [0,1]$, and similarly for the derivatives of v. The first two equations $F_0(a) = 0$ and $F_1(a) = 0$ imply that v satisfies the boundary conditions in (11), whereas the remaining equations $F_n(a) = 0$, $n \geq 2$ imply that v satisfies the differential equation in (11). □

Now that we have identified the zero finding problem $F(a) = 0$ to be solved in the Banach space $X = \ell^1_\nu$ with $\nu \in (1,2)$ to be chosen later, we are ready to apply the radii polynomial approach as introduced in Theorem 2.1. The first ingredient is a numerical approximation $\bar{a}$ of $F(a) = 0$. Given $N \in \mathbb{N}$ and $a = (a_n)_{n\geq 0} \in X = \ell^1_\nu$, denote by $a^{(N)} = (a_n)_{n=0}^N \in \mathbb{R}^{N+1}$ the finite dimensional projection of a, and by $F^{(N)} : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ the finite dimensional projection of F defined by

$$F^{(N)}(a^{(N)}) \stackrel{\text{def}}{=} \left(F_n(a^{(N)}, 0, 0, 0, \dots)\right)_{0\leq n\leq N}.$$

Assume that a numerical approximation $\bar{a}^{(N)} \in \mathbb{R}^{N+1}$ has been computed. We abuse slightly the notation by identifying $\bar{a}^{(N)} \in \mathbb{R}^{N+1}$ with $\bar{a} = (\bar{a}_0, \bar{a}_1, \cdots, \bar{a}_N, 0, 0, 0, \cdots) \in X = \ell^1_\nu$. Denote by $DF^{(N)}(\bar{a})$ the Jacobian of $F^{(N)}$ at $\bar{a}$. The radii polynomial approach as introduced in Theorem 2.1 requires defining the operators $A^\dagger$ and A. Let

$$(A^\dagger h)_n \stackrel{\text{def}}{=} \begin{cases} \left(DF^{(N)}(\bar{a})h^{(N)}\right)_n & \text{for } 0 \leq n \leq N, \\ n^2 h_n & \text{for } n > N, \end{cases}$$

where the diagonal tail is chosen in view of the dominant term in (13) for large n. Consider an $(N+1)\times(N+1)$ matrix $A^{(N)}$ computed so that $A^{(N)} \approx DF^{(N)}(\bar{a})^{-1}$. Define A as

$$(Ah)_n \stackrel{\text{def}}{=} \begin{cases} \left(A^{(N)}h^{(N)}\right)_n & \text{for } 0 \le n \le N, \\ n^{-2}h_n & \text{for } n > N. \end{cases} \tag{14}$$

One way to visualize the operator A is as

$$A = \begin{bmatrix} A^{(N)} & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \frac{1}{(N+1)^2} & 0 & \dots \\ \mathbf{0} & 0 & \frac{1}{(N+2)^2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

The following lemma states that condition (2) of Theorem 2.1 holds.

Lemma 3.3. *Let $X = \ell^1_\nu$ with $\nu \in (1,2)$, A as in (14) and F as in (13). Then $AF : X \to X$.*

We leave the proof to the reader.

We now introduce the formulas for the bounds Y_0, Z_0, Z_1 and Z_2 satisfying respectively (3), (4), (5) and (6). These bounds are derived in Sections 3.2–3.5. We first introduce an auxiliary result, used for the Z_0 and Z_2 bounds. We again leave the proof to the reader.

Lemma 3.4. *Consider a linear operator $Q\colon \ell^1_\nu \to \ell^1_\nu$ of the form*

$$Q = \begin{bmatrix} Q^{(N)} & & & 0 \\ & q_{N+1} & & \\ & & q_{N+2} & \\ 0 & & & \ddots \end{bmatrix}$$

where $Q^{(N)} = \left(Q^{(N)}_{m,n}\right)_{0\le m,n\le N}$ and $q_n \in \mathbb{R}$. Assume that $|q|_\infty = \sup_{n>N}|q_n| < \infty$. Then

$$\|Q\|_{B(\ell^1_\nu)} = \max\left(\max_{0\le n\le N}\frac{1}{\nu^n}\sum_{m=0}^{N}|Q^{(N)}_{m,n}|\nu^m\,,\,|q|_\infty\right).$$

3.2 The Y_0 bound

We look for a bound Y_0 satisfying $\|AF(\bar{a})\|_{1,\nu} \le Y_0$. We note that since $\bar{a}_n = 0$ for $n > N$, we have $(\bar{a}*\bar{a})_n = 0$ for all $n > 2N$. Hence, recalling the definition of A in (14),

$$\|AF(\bar{a})\|_{1,\nu} \le \sum_{n=0}^{N}\left|(A^{(N)}F^{(N)}(\bar{a}))_n\right|\nu^n + \sum_{n=N+1}^{2N+2}\frac{1}{n^2}\left|(J\bar{a}*b)_n + (\bar{a}*\bar{a})_{n-2}\right|\nu^n + \sum_{n=2N+3}^{\infty}\frac{1}{n^2}\left|(J\bar{a}*b)_n\right|\nu^n.$$

When calculating the finite sums in this expression, computing any b_l involves evaluating the zeta function, which is itself an infinite series. We approximate this series by summing finitely many terms and control the remainder via a straightforward integral estimate.

Concerning the final term in the expression above, since $\zeta(s) = \sum_{k=1}^{\infty} k^{-s} \le \sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}$ for all $s \ge 2$, we have $|b_l| \le \frac{\pi^2}{3} 2^{-l}$ for all $l \ge 1$. Hence

$$\begin{aligned}
\sum_{n=2N+3}^{\infty} \frac{1}{n^2} \left|(J\bar{a} * b)_n\right| \nu^n &= \sum_{n=2N+3}^{\infty} \left(\frac{1}{n^2} \sum_{j=0}^{N} |j\bar{a}_j b_{n-j}| \, \nu^n \right) \\
&\le \frac{1}{(2N+3)^2} \sum_{j=0}^{N} \left(|j\bar{a}_j| \, \nu^j \sum_{n=2N+3}^{\infty} |b_{n-j}| \, \nu^{n-j} \right) \\
&\le \frac{1}{(2N+3)^2} \sum_{j=0}^{N} |j\bar{a}_j| \, \nu^j \sum_{l=N+3}^{\infty} |b_l| \, \nu^l \\
&\le \frac{\pi^2 \|J\bar{a}\|_{1,\nu}}{3(2N+3)^2} \sum_{l=N+3}^{\infty} \left(\frac{\nu}{2}\right)^l = \frac{\pi^2 \|J\bar{a}\|_{1,\nu}}{3(2N+3)^2} \left(\frac{\nu}{2}\right)^{N+3} \frac{1}{1-\frac{\nu}{2}} \stackrel{\text{def}}{=} Y_{\text{tail}}.
\end{aligned}$$

We thus set

$$Y_0 \stackrel{\text{def}}{=} \sum_{n=0}^{N} \left| (A^{(N)} F^{(N)}(\bar{a}))_n \right| \nu^n + \sum_{n=N+1}^{2N+2} \frac{1}{n^2} \left| (J\bar{a} * b)_n + (\bar{a} * \bar{a})_{n-2} \right| \nu^n + Y_{\text{tail}}. \tag{15}$$

3.3 The Z_0 bound

Let $B \stackrel{\text{def}}{=} I - AA^\dagger$. We remark that the tails of A and $A^\dagger$ are exact inverses, hence $B_{m,n} = 0$ when $m > N$ or $n > N$. Letting

$$Z_0 \stackrel{\text{def}}{=} \max_{0 \le n \le N} \frac{1}{\nu^n} \sum_{0 \le m \le N} |B_{m,n}| \nu^m, \tag{16}$$

we get from Lemma 3.4 that $\|I - AA^\dagger\|_{B(\ell^1_\nu)} \le Z_0$.

3.4 The Z_1 bound

We look for a bound $\left\|A\left[DF(\bar{a}) - A^\dagger\right]\right\|_{B(\ell^1_\nu)} \le Z_1$. Given $h \in \ell^1_\nu$ with $\|h\|_{1,\nu} \le 1$, we set $z \stackrel{\text{def}}{=} [DF(\bar{a}) - A^\dagger]h$. For the finite part ($0 \le n \le N$) we see that

$$z_0 = 0, \qquad z_1 = \textstyle\sum_{j \ge N+1} j h_j, \qquad \text{and} \qquad z_n = 0 \quad \text{for } n = 2, \dots N.$$

For the tail, i.e. $n > N$, we find

$$z_n = n(n-1)h_n + (Jh * b)_n + \tilde{\lambda} h_{n-2} + 2(\bar{a} * h)_{n-2} - n^2 h_n.$$

Using the "chain rule" identity $J(h * b) = (Jh * b) + (h * Jb)$ we obtain

$$z_n = n(\tilde{b} * h)_n - (Jb * h)_n + \tilde{\lambda} h_{n-2} + 2(\bar{a} * h)_{n-2},$$

where $\tilde{b}$ is the sequence defined by $\tilde{b}_0 = 0$ and $\tilde{b}_n = b_n$ for $n \ge 1$. Next we estimate

$$\|Az\|_{1,\nu} = \sum_{n=0}^{N} \left| \sum_{j=0}^{N} A^{(N)}_{n,j} z_j \right| \nu^n + \sum_{n \ge N+1} \frac{1}{n^2} |z_n| \nu^n. \tag{17}$$

For any $\|h\|_{1,\nu} \le 1$, it is a small exercise to show that $|z_1| \le \frac{N+1}{\nu^{N+1}}$ provided $N+1 \ge (\ln \nu)^{-1}$. Hence the first term in (17) is bounded by $\frac{N+1}{\nu^{N+1}} \sum_{k=0}^{N} |A_{k,1}^{(N)}|$. For the infinite tail we first estimate

$$\|\tilde{b}\|_{1,\nu} = \sum_{l\ge 2} |b_l| \nu^l \le \frac{\pi^2}{3} \sum_{n=1}^{\infty} \left(\frac{\nu^2}{4}\right)^n = \frac{\pi^2 \nu^2}{3(4-\nu^2)} \stackrel{\text{def}}{=} C_1,$$

$$\|Jb\|_{1,\nu} = \sum_{l\ge 2} l|b_l| \nu^l \le \frac{2\pi^2}{3} \sum_{n=1}^{\infty} n \left(\frac{\nu^2}{4}\right)^n = \frac{4\pi^2 \nu^2}{3(4-\nu^2)^2} \stackrel{\text{def}}{=} C_2.$$

Then, using the Banach algebra property (12), we find, for any $\|h\|_{1,\nu} \le 1$,

$$\sum_{n\ge N+1} \frac{1}{n^2} |z_n| \nu^n \le \frac{C_1}{N+1} + \frac{C_2}{(N+1)^2} + \frac{\nu^2}{(N+1)^2} (|\tilde{\lambda}| + 2\|\bar{a}\|_{1,\nu}).$$

Hence, with the requirement that $N+1 \ge (\ln \nu)^{-1}$, we set

$$Z_1 \stackrel{\text{def}}{=} \frac{N+1}{\nu^{N+1}} \sum_{n=0}^{N} |A_{n,1}^{(N)}| + \frac{C_1}{N+1} + \frac{C_2}{(N+1)^2} + \frac{\nu^2}{(N+1)^2} (|\tilde{\lambda}| + 2\|\bar{a}\|_{1,\nu}). \tag{18}$$

3.5 The Z_2 bound

Let $c \in B_r(\bar{a})$, that is $\|c - \bar{a}\|_{1,\nu} \le r$. Given $h \in B_1(0)$, note that

$$([DF(c) - DF(\bar{a})]h)_n = \begin{cases} 0 & \text{for } n = 0, 1, \\ 2\left((c-\bar{a}) * h\right)_{n-2} & \text{for } n \ge 2, \end{cases}$$

so that

$$\begin{aligned} \|A[DF(c) - DF(\bar{a})]\|_{B(\ell^1_\nu)} &= \sup_{h\in B_1(0)} \|A[DF(c) - DF(\bar{a})]h\|_{1,\nu} \le 2\nu^2 \sup_{h\in B_1(0)} \|A[(c-\bar{a}) * h]\|_{1,\nu} \\ &\le 2\nu^2 \sup_{h\in B_1(0)} \|A\|_{B(\ell^1_\nu)} \|c - \bar{a}\|_{1,\nu} \|h\|_{1,\nu} \le 2\nu^2 \|A\|_{B(\ell^1_\nu)} r. \end{aligned}$$

Using Lemma 3.4, we get that

$$2\nu^2 \|A\|_{B(\ell^1_\nu)} \le Z_2 \stackrel{\text{def}}{=} 2\nu^2 \max\left(\max_{0\le \ell \le N} \frac{1}{\nu^\ell} \sum_{0\le k\le N} \left| A_{k,\ell}^{(N)} \right| \nu^k, \frac{1}{(N+1)^2} \right). \tag{19}$$

3.6 Computer-assisted proofs

Combining the bounds (15), (16), (18) and (19), we define the radii polynomial $p(r)$ as in (7). We prove the existence of three different types of solutions by verifying the hypothesis of Theorem 2.1 with the routine `script_three_proofs_LB.m` available at [4]. The data of each proof can be found in the following table. We find that $p(r) < 0$ for $r \in [r_{\min}, r_{\max}]$.

solution	#1	#2	#3
$\lambda = 4\tilde{\lambda}/\pi^2$	5.67	20	19.961
ν	1.06	1.04	1.058
N	250	410	360
Y_0	$4.8014 \cdot 10^{-9}$	$3.1548 \cdot 10^{-7}$	$3.9179 \cdot 10^{-7}$
Z_0	$4.408 \cdot 10^{-11}$	$1.4085 \cdot 10^{-11}$	$2.6315 \cdot 10^{-9}$
Z_1	$4.212 \cdot 10^{-1}$	$2.7293 \cdot 10^{-2}$	$1.1063 \cdot 10^{-1}$
Z_2	$7.6773 \cdot 10^{3}$	$4.897 \cdot 10^{3}$	$3.90307 \cdot 10^{6}$
$r_{\min}$	$8.3 \cdot 10^{-9}$	$3.249 \cdot 10^{-7}$	$5.9684 \cdot 10^{-7}$
$r_{\max}$	$7.538 \cdot 10^{-5}$	$1.983 \cdot 10^{-4}$	$1.6818 \cdot 10^{-6}$

The three solutions can be seen in Figures 2, 3 and 4. One can observe the different qualitative behaviour of these three solutions.

It is not entirely straightforward to find initial guesses for solutions, i.e., starting points for applying Newton's method to the finite truncation $F^{(N)}$. To find such approximate solutions, note that the trivial solution $v = 0$ undergoes transcritical bifurcations at $\tilde{\lambda} = \pi^2 n(n + \frac{1}{2})$ for $n \in \mathbb{N}$. The solution branches that bifurcate at these parameter values can then be followed numerically (using branch following techniques) to other parameter values.

The main restriction on the Taylor series based approach presented here, is that not all solutions have an analytic extension to the complex ball of radius 1. These solutions, although real analytic on $[0, 1]$, cannot be described by a single Taylor series around the origin. One way to overcome this is via domain decomposition (i.e. matching together several power series), but we will not pursue that here, and it is also by no means the only option.

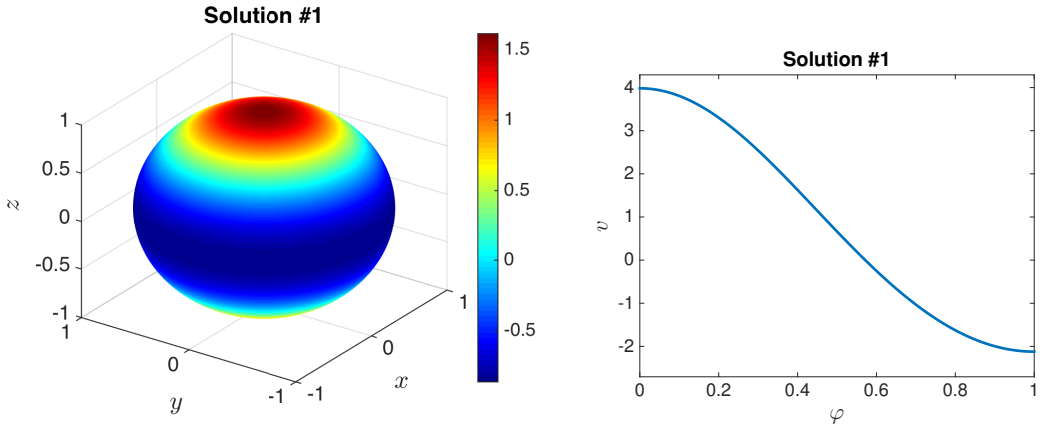

Figure 2: (Left) The first solution of (9) on the unit sphere $S^2 \subset \mathbb{R}^3$. (Right) The corresponding (numerical) solution of the BVP (11). Since $r_{\min} < 10^{-8}$, the true solution lies with the line-width by Theorem 2.1.

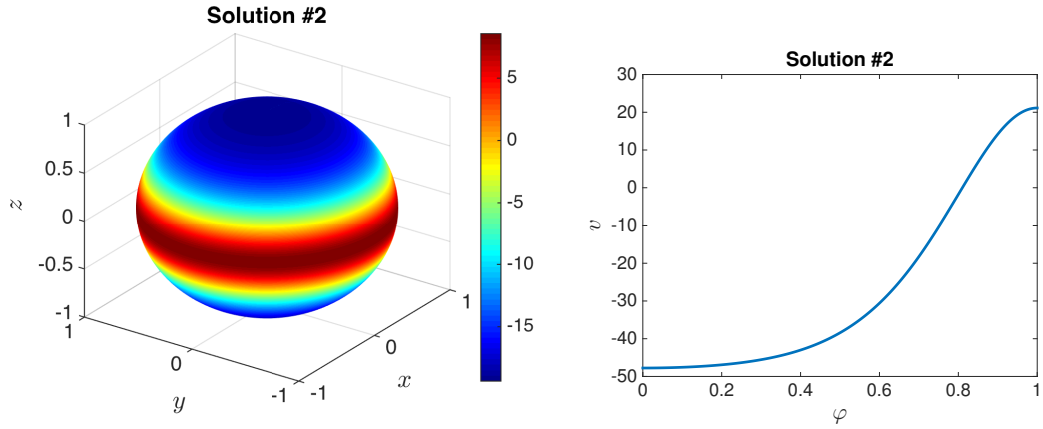

Figure 3: (Left) The second solution of (9) on the unit sphere $S^2 \subset \mathbb{R}^3$. (Right) The corresponding (numerical) solution of the BVP (11).

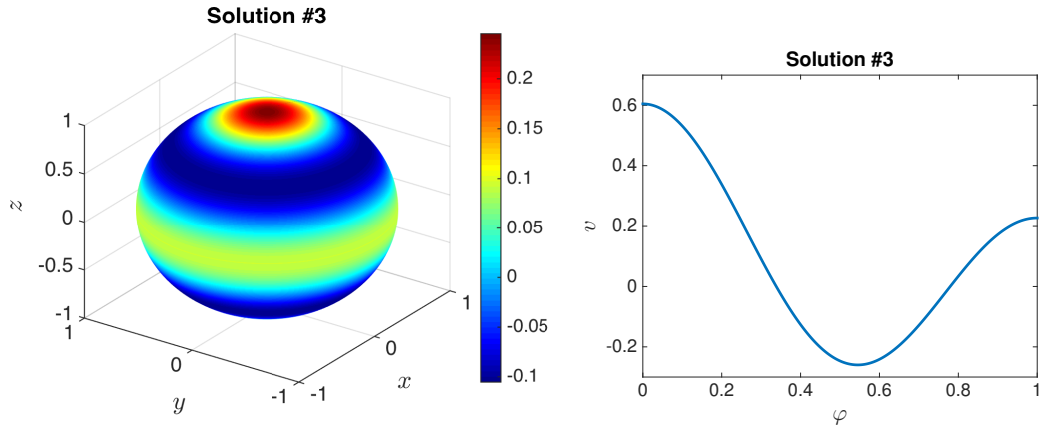

Figure 4: (Left) The third solution of (9) on the unit sphere $S^2 \subset \mathbb{R}^3$. (Right) The corresponding (numerical) solution of the BVP (11).

4 Radially symmetric equilibria of the Swift-Hohenberg equation on the 3D unit ball

We consider the Swift-Hohenberg equation [25] with Dirichlet boundary conditions:

$$\begin{cases} u_t = -(\Delta - 1)^2 u + \lambda u - u^3, & \text{on } D_1, \\ u = \Delta u = 0 & \text{on } \partial D_1. \end{cases} \tag{20}$$

Here $D_1 \subset \mathbb{R}^3$ is the unit ball, and $\lambda \in \mathbb{R}$ is a parameter. The parabolic PDE (20) is a popular deterministic model for pattern formation, see e.g. [18]. It has been well studied, analytically in one spatial dimension and predominantly numerically in two spatial dimensions. Here we consider time-independent solutions in three spatial dimensions. Indeed, we will focus on radially symmetric equilibrium solutions of (20). Letting $v = (\Delta - 1)u$, these solutions also

correspond to radially symmetric equilibria of the reaction-diffusion system

$$\begin{cases} u_t = \Delta u - u - v \\ v_t = \Delta v - v - \lambda u + u^3 \end{cases} \tag{21}$$

on the unit ball with Dirichlet boundary conditions. The method works very generally for radially symmetric equilibrium solutions in reaction-diffusion systems (cf. [23]), which are ubiquitous in models in the life sciences. This motivates us to work with the system (21) rather than with the equivalent (at the level of equilibria) scalar equation (20). As an additional benefit, the analysis below illustrates how the method of radii polynomials extends naturally to systems of equations. Looking for radially symmetric equilibria of (21), i.e., time independent solutions of the form $u(x,y,z) = u(s) = u(\sqrt{x^2+y^2+z^2})$, leads to a coupled systems of ODEs:

$$\begin{cases} u''(s) + \frac{2}{s}u'(s) - u(s) - v(s) = 0 & \text{for } s \in (0,1], \\ v''(s) + \frac{2}{s}v'(s) - v(s) - \lambda u(s) + u(s)^3 = 0 & \text{for } s \in (0,1], \\ u'(0) = v'(0) = 0, \\ u(1) = v(1) = 0. \end{cases} \tag{22}$$

We expand the functions u and v as power series $u(s) = \sum_{n=0}^{\infty} a_n s^n$ and $v(s) = \sum_{n=0}^{\infty} b_n s^n$. Define the coefficient sequences as $a = (a_n)_{n\geq 0}$ and $b = (b_n)_{n\geq 0}$. Consider the Banach space

$$X = \ell^1_\nu \times \ell^1_\nu = \left\{ x = (a,b) : a, b \in \ell^1_\nu \right\},$$

endowed with the norm $\|x\|_X = \max\left\{\|a\|_{1,\nu}, \|b\|_{1,\nu}\right\}$. The equations for the Taylor coefficients are $F_1(x) = ((F_1(x))_n)_{n\geq 0} = 0$ and $F_2(x) = ((F_2(x))_n)_{n\geq 0} = 0$, given component-wise by

$$(F_1(x))_n = \begin{cases} a_1 & \text{for } n = 0, \\ \sum_{k\geq 0} a_k & \text{for } n = 1, \\ n(n+1)a_n - a_{n-2} - b_{n-2} & \text{for } n \geq 2, \end{cases}$$

$$(F_2(x))_n = \begin{cases} b_1 & \text{for } n = 0, \\ \sum_{k\geq 0} b_k & \text{for } n = 1, \\ n(n+1)b_n - b_{n-2} - \lambda a_{n-2} + (a^3)_{n-2} & \text{for } n \geq 2, \end{cases}$$

where

$$(a^3)_n = (a * a * a)_n = \sum_{\substack{n_1+n_2+n_3=n \\ n_1,n_2,n_3\geq 0}} a_{n_1}a_{n_2}a_{n_3} = \sum_{n_1=0}^{n} a_{n_1} \left(\sum_{n_2=0}^{n-n_1} a_{n_2}a_{n-n_1-n_2} \right).$$

It is not difficult to derive that all odd coefficients will vanish, but we do not exploit that here.

Denoting $F = (F_1, F_2)$, the problem is to find $x = (a,b) \in X = \ell^1_\nu \times \ell^1_\nu$ for some $\nu > 1$ such that $F(x) = 0$. To achieve this, we use the radii polynomial approach as introduced in Theorem 2.1. Given $N \in \mathbb{N}$, denote $x^{(N)} = ((a_n)_{0\leq n\leq N}, (b_n)_{0\leq n\leq N}) \in \mathbb{R}^{2N+2}$. Consider the finite dimensional projection $F^{(N)} = (F_1^{(N)}, F_2^{(N)}) : \mathbb{R}^{2N+2} \to \mathbb{R}^{2N+2}$ defined by

$$F_i^{(N)}(x^{(N)}) \stackrel{\text{def}}{=} \left((F_i(x^{(N)}))_k \right)_{0\leq k\leq N}.$$

Given $\bar{x} \in \mathbb{R}^{2N+2}$ a numerical approximation of $F^{(N)}(x) = 0$, denote by $DF^{(N)}(\bar{x})$ the Jacobian of $F^{(N)}$ at $\bar{x}$, and let us write it as

$$DF^{(N)}(\bar{x}) = \begin{pmatrix} D_a F_1^{(N)}(\bar{x}) & D_b F_1^{(N)}(\bar{x}) \\ D_a F_2^{(N)}(\bar{x}) & D_b F_2^{(N)}(\bar{x}) \end{pmatrix} \in \mathbb{R}^{(2N+2)\times(2N+2)}.$$

The radii polynomial approach requires defining the operators $A^\dagger$ and A. Let

$$A^\dagger = \begin{pmatrix} A^\dagger_{1,1} & A^\dagger_{1,2} \\ A^\dagger_{2,1} & A^\dagger_{2,2} \end{pmatrix}, \tag{23}$$

whose action on an element $h = (h_1, h_2) \in X$ is defined by $(A^\dagger h)_i = A^\dagger_{i,1} h_1 + A^\dagger_{i,2} h_2$, for $i = 1, 2$. Here the action of $A^\dagger_{i,j}$ is defined as

$$(A^\dagger_{i,1} h_1)_n = \begin{cases} \left(D_a F_i^{(N)}(\bar{x}) h_1^{(N)}\right)_n & \text{for } 0 \le n \le N, \\ \delta_{i,1} n(n+1)(h_1)_n & \text{for } n > N, \end{cases}$$
$$(A^\dagger_{i,2} h_2)_n = \begin{cases} \left(D_b F_i^{(N)}(\bar{x}) h_2^{(N)}\right)_n & \text{for } 0 \le n \le N, \\ \delta_{i,2} n(n+1)(h_2)_n & \text{for } n > N, \end{cases}$$

where $\delta_{i,j}$ is the Kronecker δ. Consider now a matrix $A^{(N)} \in \mathbb{R}^{(2N+2)\times(2N+2)}$ computed so that $A^{(N)} \approx DF^{(N)}(\bar{x})^{-1}$. We decompose it into four $(N+1)\times(N+1)$ blocks:

$$A^{(N)} = \begin{pmatrix} A^{(N)}_{1,1} & A^{(N)}_{1,2} \\ A^{(N)}_{2,1} & A^{(N)}_{2,2} \end{pmatrix}.$$

Thus we define A as

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}, \tag{24}$$

whose action on an element $h = (h_1, h_2) \in X$ is defined by $(Ah)_i = A_{i,1} h_1 + A_{i,2} h_2$, for $i = 1, 2$. The action of $A_{i,j}$ is defined as

$$(A_{i,j} h_j)_n = \begin{cases} \left(A^{(N)}_{i,j} h_j^{(N)}\right)_n & \text{for } 0 \le n \le N, \\ \delta_{i,j} \frac{1}{n(n+1)} (h_j)_n & \text{for } n > N. \end{cases}$$

Finally, we set $T(x) = x - AF(x)$, which indeed maps X into itself. As in Section 3, the next step is to derive explicit, computable expressions for the bounds (3), (4), (5) and (6).

4.1 The Y_0 bound

Observe first that the nonlinear term of $F_2(\bar{x})$ is the Cauchy product $(\bar{a} * \bar{a} * \bar{a})_{n-2}$, which vanishes for $n \ge 3N+3$. This implies that $(F_1(\bar{x}))_n = (F_2(\bar{x}))_n = 0$ for all $n \ge 3N+3$. For $i = 1, 2$, we set

$$Y_0^{(i)} \overset{\text{def}}{=} \sum_{n=0}^{N} \left| \sum_{j=1}^{2} \left(A^{(N)}_{i,j} F_j^{(N)}(\bar{x}) \right)_n \right| \nu^n + \sum_{n=N+1}^{3N+2} \left| \frac{1}{n(n+1)} (F_i(\bar{x}))_n \right| \nu^n,$$

which is a collection of finite sums that can be evaluated with interval arithmetic. We get

$$\|[T(\bar{x}) - \bar{x}]_i\|_{1,\nu} = \|[-AF(\bar{x})]_i\|_{1,\nu} = \left\| \sum_{j=1}^{n} A_{i,j} F_j(\bar{x}) \right\|_{1,\nu} \le Y_0^{(i)},$$

and we set

$$Y_0 \overset{\text{def}}{=} \max\left(Y_0^{(1)}, Y_0^{(2)} \right). \tag{25}$$

4.2 The Z_0 bound

We look for a bound of the form $\|I - AA^\dagger\|_{B(X)} \le Z_0$. Recalling the definitions of A and $A^\dagger$ given in (24) and (23), let $B \stackrel{\text{def}}{=} I - AA^\dagger$ the bounded linear operator represented as

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}.$$

We remark that $(B_{i,j})_{n_1,n_2} = 0$ for any $i,j = 1,2$ whenever $n_1 > N$ or $n_2 > N$. Hence we can compute the norms $\|B_{i,j}\|_{B(\ell^1_\nu)}$ using Lemma 3.4. Given $h = (h_1, h_2) \subset X = \ell^1_\nu \times \ell^1_\nu$ with $\|h\|_X = \max(\|h_1\|_{1,\nu}, \|h_2\|_{1,\nu}) \le 1$, we get

$$\|(Bh)_i\|_{1,\nu} = \left\| \sum_{j=1}^{2} B_{i,j} h_j \right\|_{1,\nu} \le \sum_{i=1}^{2} \|B_{i,j}\|_{B(\ell^1_\nu)}.$$

Hence we define

$$Z_0 \stackrel{\text{def}}{=} \max\left(\|B_{1,1}\|_{B(\ell^1_\nu)} + \|B_{1,2}\|_{B(\ell^1_\nu)}, \|B_{2,1}\|_{B(\ell^1_\nu)} + \|B_{2,2}\|_{B(\ell^1_\nu)} \right). \tag{26}$$

4.3 The Z_1 bound

Recall that we look for the bound $\|A[DF(\bar{x}) - A^\dagger]\|_{B(X)} \le Z_1$. Given $h = (h_1, h_2) \in X$ with $\|h\|_X \le 1$, set

$$z \stackrel{\text{def}}{=} [DF(\bar{x}) - A^\dagger]h.$$

Note that for $j = 1,2$, $(z_j)_0 = 0$, $(z_j)_1 = \sum_{k \ge N+1} (h_j)_k$, $(z_j)_n = 0$ for $n = 2, \dots, N$, and

$$(z_1)_n = -(h_1)_{n-2} - (h_2)_{n-2}, \qquad \text{and} \qquad (z_2)_n = -(h_2)_{n-2} - \lambda(h_1)_{n-2} + 3(\bar{a} * \bar{a} * h_1)_{n-2},$$

for $n \ge N+1$. It is not hard to show that $|(z_j)_1| \le \nu^{-(N+1)}$ for all $\|h\|_X \le 1$, $j = 1,2$, hence

$$\begin{aligned}
\|(Az)_1\|_{1,\nu} &\le \sum_{j=1}^{2} \|A_{1,j} z_j\|_{1,\nu} \\
&= \sum_{j=1}^{2} \sum_{n=0}^{N} \left| \left(A^{(N)}_{1,j} z^{(N)}_j\right)_n \right| \nu^n + \sum_{n \ge N+1} \frac{1}{n(n+1)} |(z_1)_n| \nu^n \\
&\le \frac{1}{\nu^{N+1}} \sum_{j=1}^{2} \sum_{n=0}^{N} \left| \left(A^{(N)}_{1,j}\right)_{n,1} \right| \nu^n + \frac{1}{(N+1)(N+2)} (\|h_1\|_{1,\nu} + \|h_2\|_{1,\nu}) \\
&\le \frac{1}{\nu^{N+1}} \sum_{j=1}^{2} \sum_{n=0}^{N} \left| \left(A^{(N)}_{1,j}\right)_{n,1} \right| \nu^n + \frac{2}{(N+1)(N+2)} \stackrel{\text{def}}{=} Z_1^{(1)},
\end{aligned}$$

and similarly, now also using the Banach algebra property (12),

$$\|(Az)_2\|_{1,\nu} \le \frac{1}{\nu^{N+1}} \sum_{j=1}^{2} \sum_{n=0}^{N} \left| \left(A^{(N)}_{2,j}\right)_{n,1} \right| \nu^n + \frac{1}{(N+1)(N+2)} (1 + \lambda + 3\|\bar{a}\|^2_{1,\nu}) \stackrel{\text{def}}{=} Z_1^{(2)}.$$

We thus define

$$Z_1 \stackrel{\text{def}}{=} \max\left(Z_1^{(1)}, Z_1^{(2)} \right). \tag{27}$$

4.4 The Z_2 bound

Let $c = (c_1, c_2) \in B_r(\bar{x})$, that is $\|c - \bar{x}\|_X = \max(\|c_1 - \bar{a}\|_{1,\nu}, \|c_2 - \bar{b}\|_{1,\nu}) \leq r$. Given $\|h\|_X \leq 1$, note that $([DF_1(c) - DF_1(\bar{x})]h)_n = 0$ and that

$$([DF_2(c) - DF_2(\bar{x})]h)_n = \begin{cases} 0 & \text{for } n = 0, 1, \\ 3\left((c_1 * c_1 - \bar{a} * \bar{a}) * h_1\right)_{n-2} & \text{for } n \geq 2, \end{cases}$$

so that

$$\begin{aligned} \|A[DF(c) - DF(\bar{x})]\|_{B(X)} &= \sup_{\|h\|_X \leq 1} \|A[DF(c) - DF(\bar{x})]h\|_X \\ &\leq \|A\|_{B(X)} \sup_{\|h\|_X \leq 1} \|[DF(c) - DF(\bar{x})]h\|_X \\ &= 3\nu^2 \|A\|_{B(X)} \sup_{\|h\|_X \leq 1} \|(c_1 - \bar{a}) * (c_1 + \bar{a}) * h_1\|_{1,\nu} \\ &\leq 3\nu^2 \|A\|_{B(X)} \sup_{\|h\|_X \leq 1} \|c_1 - \bar{a}\|_{1,\nu} \|c_1 + \bar{a}\|_{1,\nu} \|h_1\|_{1,\nu} \\ &\leq 3\nu^2 \|A\|_{B(X)} r(\|c_1\|_{1,\nu} + \|\bar{a}\|_{1,\nu}) \\ &\leq 3\nu^2 \|A\|_{B(X)} r(r + 2\|\bar{a}\|_{1,\nu}). \end{aligned}$$

Then, assuming a loose a priori bound $r \leq 1$ on the radius, we set

$$Z_2 \stackrel{\text{def}}{=} 3\nu^2 \|A\|_{B(X)}(1 + 2\|\bar{a}\|_{1,\nu}), \tag{28}$$

where $\|A\|_{B(X)}$ is computed using Lemma 3.4, see also (26).

4.5 Computer-assisted proofs

With a numerical continuation algorithm, we continued a branch of solutions of (22) that bifurcates from the zero solution at $\lambda = (\pi^2 + 1)^2$. Combining the bounds Y_0, Z_0, Z_1 and Z_2 given in (25), (26), (27) and (28), respectively, we define the radii polynomial $p(r)$ as in (7). We prove the existence of six solutions of (22) by verifying the hypotheses of Theorem 2.1 with the routine `script_proofs_SH.m` available at [4]. The following table contains values of the bounds, as well as intervals $[r_{\min}, r_{\max}]$ on which $p(r) < 0$, for sample values of λ ranging from 118.2 to 500.

λ	ν	N	Y_0	Z_1	Z_2	$r_{\min}$	$r_{\max}$
118.2	1.15	39	$8.4414 \cdot 10^{-11}$	0.44499	$1.9642 \cdot 10^{6}$	$1.5218 \cdot 10^{-10}$	$2.8242 \cdot 10^{-7}$
120	1.1	54	$1.3132 \cdot 10^{-9}$	0.15775	$2.002 \cdot 10^{5}$	$1.5598 \cdot 10^{-9}$	$4.2056 \cdot 10^{-6}$
250	1.04	114	$4.3386 \cdot 10^{-8}$	0.27014	$4.89752 \cdot 10^{5}$	$6.2026 \cdot 10^{-8}$	$1.4282 \cdot 10^{-6}$
350	1.03	136	$4.8408 \cdot 10^{-8}$	0.15617	$1.536768 \cdot 10^{6}$	$6.508 \cdot 10^{-8}$	$4.8401 \cdot 10^{-7}$
450	1.02	164	$4.7337 \cdot 10^{-8}$	0.033858	$3.274572 \cdot 10^{6}$	$6.2042 \cdot 10^{-8}$	$2.33 \cdot 10^{-7}$
500	1.009	169	$5.6167 \cdot 10^{-8}$	0.06321	$3.73724 \cdot 10^{6}$	$9.9273 \cdot 10^{-8}$	$1.5139 \cdot 10^{-7}$

We observe from these data that as we increase the parameter λ, the dimension of the projection N needs to increase while the decay parameter ν needs to decrease. This is due to the fact that the Taylor coefficients of the solutions decay slower as λ increases. Moreover, notice that the values of $r_{\min}$ and $r_{\max}$ are approaching each other, meaning that the proofs are getting harder and harder to obtain. This suggests that for larger parameter values a single Taylor expansion is not enough. The corresponding solutions can be seen in Figure 5. We plot in Figure 6 the solution (u, v) of (20) at $\lambda = 500$.

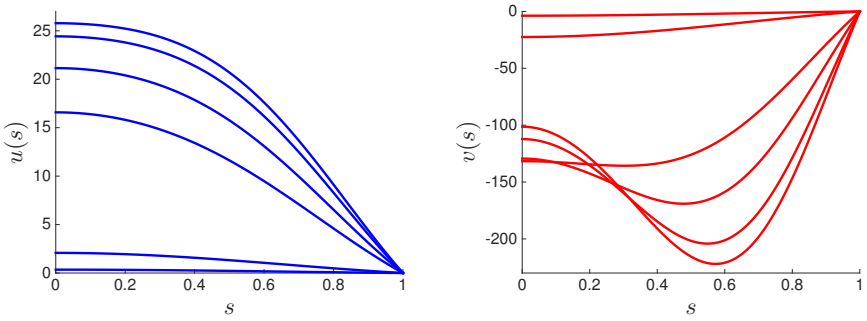

Figure 5: Six solutions of (22) for $\lambda \in \{118.2, 120, 250, 350, 450, 500\}$.

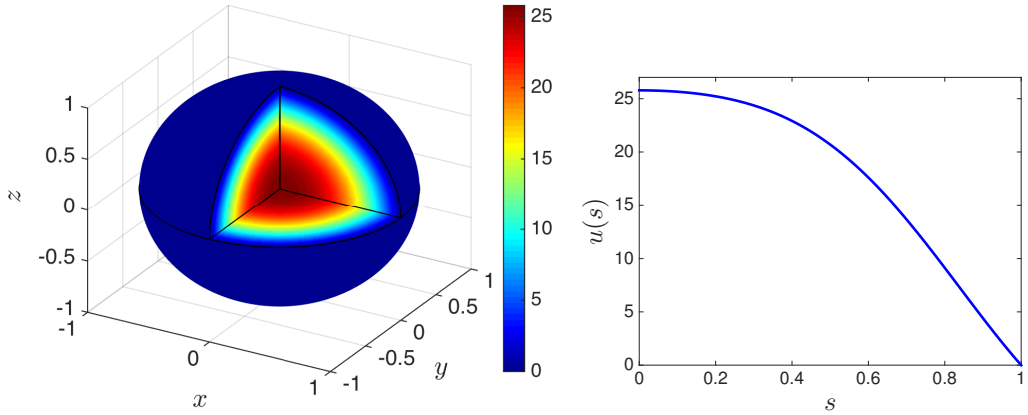

Figure 6: (Left) A stationary solution of the Swift-Hohenberg equation (20) on the unit ball in $\mathbb{R}^3$ at $\lambda = 500$. (Right) The corresponding graph of $u(s) = u(\sqrt{x^2 + y^2 + z^2})$.

5 Conclusion

We have seen that some nontrivial boundary value problems originating from nonlinear (geometric) PDEs can be solved in a Taylor series setting, one that requires relatively little technical machinery. Cutting of the Taylor series at some finite order and solving the associated finite dimensional algebraic system numerically, leads to an approximate solution, and we have proven that the true solution lies nearby. Indeed, based on Theorem 2.1 and computable bounds (using interval arithmetic), we can estimate the distance between approximate and true solution rigorously and explicitly. This turns the numerical computation into a mathematical statement about the PDE.

There are limitations to the particular choice of Taylor series as our means of describing a solution, i.e., using monomials as our basis functions. In particular, we have seen that this puts limits on the parameter range where we can apply this approach. More generally, there is a large variety of functional analytic setups and numerical algorithms, adapted to the particular problem under study, which fit into the general framework of Theorem 2.1. Successful examples include domain decomposition, Fourier series, Chebyshev series, splines, finite elements, as well as combinations of these. With these tools one is able, using the paradigm illustrated by the

two examples in Sections 3 and 4, to solve eigenvalue problems, find periodic and connecting orbits, continue solutions in parameters, and analyze bifurcations. Due to the nonlinear nature of these problems, you usually simply cannot get your hands on such solutions without the help of a computer. For further reading we refer to [1, 2, 6, 12, 20, 28, 30, 33] and the references therein. Of particular note is [3], where the authors apply a related method to find *a*symmetric solutions of a variant of (1) with $u = u(x)$ scalar and x in the unit disc in $\mathbb{R}^2$.

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References

[1] G. Arioli and H. Koch. Computer-assisted methods for the study of stationary solutions in dissipative systems, applied to the Kuramoto-Sivashinski equation. *Arch. Ration. Mech. Anal.*, 197:1033–1051, 2010.

[2] G. Arioli and H. Koch. Integration of dissipative partial differential equations: a case study. *SIAM J. Appl. Dyn. Syst.*, 9:1119–1133, 2010.

[3] G. Arioli and H. Koch. Validated numerical solutions for some semilinear elliptic equations on the disk, 2017. Preprint.

[4] I. Balázs, J.B. van den Berg, J. Courtois, J. Dudás, J.-P. Lessard, A. Vörös-Kiss, J.F. Williams, and X.Y. Yin. MATLAB code for "Computer-assisted proofs for radially symmetric solutions of PDEs", 2017. `http://www.math.vu.nl/~janbouwe/code/radialpdes/`.

[5] A.N. Baltagiannis and K.E. Papadakis. Periodic solutions in the Sun–Jupiter–Trojan Asteroid–Spacecraft system. *Planetary and Space Science*, 75:148–157, 2013.

[6] B. Breuer, J. Horák, P.J. McKenna, and M. Plum. A computer-assisted existence and multiplicity proof for travelling waves in a nonlinearly supported beam. *J. Diff. Eq.*, 224(1):60–97, 2006.

[7] J. Burgos-García and M. Gidea. Hill's approximation in a restricted four-body problem. *Celestial Mech. Dynam. Astronom.*, 122(2):117–141, 2015.

[8] CAPD: Computer assisted proofs in dynamics, a package for rigorous numerics. `http://capd.ii.uj.edu.pl/`.

[9] A. Castro and E.M. Fischer. Infinitely many rotationally symmetric solutions to a class of semilinear Laplace-Beltrami equations on spheres. *Canad. Math. Bull.*, 58(4):723–729, 2015.

[10] Lamberto Cesari. Functional analysis and periodic solutions of nonlinear differential equations. *Contributions to Differential Equations*, 1:149–187, 1963.

[11] Lamberto Cesari. Functional analysis and Galerkin's method. *Michigan Math. J.*, 11:385–414, 1964.

[12] J.-L. Figueras, M. Gameiro, J.-P. Lessard, and R. de la Llave. A framework for the numerical computation and a posteriori verification of invariant objects of evolution equations. *SIAM J. Appl. Dyn. Syst.*, 16:1070–1088, 2017.

[13] A. Hungria, J.-P. Lessard, and J.D. Mireles James. Rigorous numerics for analytic solutions of differential equations: the radii polynomial approach. *Math. Comp.*, 85(299):1427–1459, 2016.

[14] H. Koch, A. Schenkel, and P. Wittwer. Computer-assisted proofs in analysis and programming in logic: a case study. *SIAM Rev.*, 38(4):565–604, 1996.

[15] Oscar E. Lanford, III. A computer-assisted proof of the Feigenbaum conjectures. *Bull. Amer. Math. Soc. (N.S.)*, 6(3):427–434, 1982.

[16] E.S.G. Leandro. On the central configurations of the planar restricted four-body problem. *J. Diff. Eq.*, 226(1):323–351, 2006.

[17] P.-L. Lions. On the existence of positive solutions of semilinear elliptic equations. *SIAM Rev.*, 24(4):441–467, 1982.

[18] S. McCalla and B. Sandstede. Snaking of radial solutions of the multi-dimensional Swift-Hohenberg equation: a numerical study. *Phys. D*, 239(16):1581–1592, 2010.

[19] R.E. Moore, R.B. Kearfott, and M.J. Cloud. *Introduction to interval analysis*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009.

[20] M.T. Nakao. Numerical verification methods for solutions of ordinary and partial differential equations. *Numer. Funct. Anal. Optim.*, 22(3-4):321–356, 2001.

[21] S.M. Rump. INTLAB - INTerval LABoratory. In Tibor Csendes, editor, *Developments in Reliable Computing*, pages 77–104. Kluwer Academic Publishers, Dordrecht, 1999. http://www.ti3.tu-harburg.de/rump/.

[22] S.M. Rump. Verification methods: Rigorous results using floating-point arithmetic. *Acta Numerica*, 19:287–449, 2010.

[23] A. Scheel. Radially symmetric patterns of reaction-diffusion systems. *Mem. Amer. Math. Soc.*, 165(786), 2003.

[24] C. Simó. Relative equilibrium solutions in the four-body problem. *Celestial Mech.*, 18(2):165–184, 1978.

[25] J.B. Swift and P.C. Hohenberg. Hydrodynamic fluctuations at the convective instability. *Phys. Rev. A*, 15(1), 1977.

[26] W. Tucker. A rigorous ODE Solver and Smale's 14th Problem. *Foundations of Computational Mathematics*, 2(1):53–117–117, 2002.

[27] W. Tucker. *Validated numerics*. Princeton University Press, Princeton, NJ, 2011. A short introduction to rigorous computations.

[28] J.B. van den Berg, A. Deschênes, J.-P. Lessard, and J.D. Mireles James. Stationary coexistence of hexagons and rolls via rigorous computations. *SIAM J. Appl. Dyn. Syst.*, 14:942–979, 2015.

[29] J.B. van den Berg and J.-P. Lessard. Rigorous numerics in dynamics. *Notices Amer. Math. Soc.*, 62(9):1057–1061, 2015.

[30] J.B. van den Berg, J.-P. Lessard, and K. Mischaikow. Global smooth solution curves using rigorous branch following. *Math. Comp.*, 79(271):1565–1584, 2010.

[31] M.J. Ward. Spots, traps, and patches: Asymptotic analysis of localized solutions to some linear and nonlinear diffusive systems, 2017. Preprint.

[32] N. Yamamoto. A numerical verification method for solutions of boundary value problems with local uniqueness by Banach's fixed-point theorem. *SIAM J. Numer. Anal.*, 35:2004–2013, 1998.

[33] P. Zgliczyński. Rigorous numerics for dissipative partial differential equations. II. Periodic orbit for the Kuramoto-Sivashinsky PDE — a computer-assisted proof. *Found. Comput. Math.*, 4(2):157–185, 2004.

[34] P. Zgliczyński and K. Mischaikow. Rigorous numerics for partial differential equations: the Kuramoto-Sivashinsky equation. *Found. Comput. Math.*, 1(3):255–288, 2001.