TRANSVERSE ORBITAL STABILITY OF PERIODIC TRAVELING WAVES FOR NONLINEAR KLEIN-GORDON EQUATIONS

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Abstract. In this paper we establish the orbital stability of a class of spatially periodic wavetrain solutions to multidimensional nonlinear Klein-Gordon equations with periodic potential. We show that the orbit generated by the one-dimensional wavetrain is stable under the flow of the multi-dimensional equation under perturbations which are, on one hand, co-periodic with respect to the translation or Galilean variable of propagation, and, on the other hand, periodic (but not necessarily co-periodic) with respect to the transverse directions. That is, we show their transverse orbital stability. The class of periodic wavetrains under consideration is the family of subluminal rotational waves, which are periodic in the momentum but unbounded in their position.

1. Introduction

The one dimensional sine-Gordon equation in laboratory coordinates [41],

$$u_{tt} - u_{xx} + \sin u = 0, \tag{1.1}$$

where $u = u(x,t)$ is a scalar and $x \in \mathbb{R}$, $t > 0$, is one of the most important equations in mathematical physics. It is used to model mechanical oscillations of the "ribbon" pendulum [11], the propagation of the magnetic flux on a Josephson line [40, 43], and the dynamics of crystal dislocations [13], among other physical phenomena. For a detailed account of these and other applications the reader is referred to [6]. The natural generalization of equation (1.1) to more spatial dimensions, namely

$$u_{tt} - \Delta u + \sin u = 0, \tag{1.2}$$

with $(x,t) \in \mathbb{R}^d \times (0,+\infty)$, $d \geqq 2$, arises as a model of physical interest, as well. For example, equation (1.2) in dimension $d = 2$ was proposed to describe the electrodynamics of extended rectangular Josephson junctions, which consist of two layers of superconducting materials separated by an isolating barrier (cf. [27]; see also §5.1 below).

In this paper, we consider nonlinear Klein-Gordon equations of the form

$$u_{tt} - \Delta u + F(u) = 0, \tag{1.3}$$

with $(x,t) \in \mathbb{R}^d \times (0,+\infty)$, $d \geqq 1$. The nonlinearity, $F = F(u)$, is supposed to derive from a bounded, periodic potential $V = V(u)$ of class C^2 as

$$F(u) = V'(u), \qquad \text{for all } u \in \mathbb{R}.$$

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This class of potentials is designed as a generalization of the sine-Gordon case with $V(u) = -\cos u$. Specializing equation (1.3) to one spatial dimension, $d = 1$, we obtain

$$u_{tt} - u_{xx} + F(u) = 0, \tag{1.4}$$

for $(x, t) \in \mathbb{R} \times (0, +\infty)$.

This paper studies the orbital stability of a class of one-dimensional periodic traveling waves (or periodic wavetrains) as solutions to the multi-dimensional wave equation (1.3), that is, their *transverse orbital stability*. More precisely, if we consider a one-dimensional periodic wavetrain $u(x, t) = \varphi(x - ct)$, solution to the one-dimensional equation (1.4), then the function

$$u(x_1, x_2, ..., x_d, t) = \varphi(x_1 - ct) \tag{1.5}$$

is a particular solution to the multi-dimensional equation (1.3). The problem we pose is the following: Is the orbit generated by this solution stable under the flow of (1.3)? We answer this question in the positive for a certain class of periodic wavetrains known as *subluminal rotations* (see [25, 26], or §2 below). As a first step we verify their *transverse spectral stability*, or the property that the Floquet spectrum associated to the linearization around the periodic profile remains in the imaginary axis for any transverse wavelength. Whence, we are able to show that the orbit generated by the "line" of the rotational subluminal wave profile (1.5) remains orbitally stable under co-periodic perturbations in the Galilean variable, $z = x_1 - ct$, and under periodic (but not necessarily co-periodic) perturbations in the transverse coordinates, by the flow of equation (1.3). We start the exposition by showing that the orbit generated by any rotational subluminal wave φ, namely $\mathcal{O}_\varphi = \{\varphi(\cdot + y) : y \in \mathbb{R}\}$, remains stable by the flow of the one-dimensional equation (1.4) in the sense that, for initial data close enough to $\mathcal{O}_\varphi$, the corresponding solution to will remain close enough to $\mathcal{O}_\varphi$ for all times. This result is obtained by using a classical Lyapunov stability analysis (cf. [2, 16, 17]). Although there exist already related results for Hamiltonian equations of this type (cf. [9, 7]), we include the one-dimensional study here in order to outline the subsequent multi-dimensional analysis. As a corollary, we mention the stability of subluminal rotations for the sine-Gordon equation (1.1) under co-periodic perturbations, a fact that, surprisingly, had not been previously reported in the literature, up to our knowledge.

Following these ideas, we are able to prove that subluminal rotations are indeed transverse orbitally stable under co-periodic perturbations in the translation variable, and periodic (with any period L) in the transverse direction. This is a remarkable property, inasmuch as there are many examples of orbitally stable one-dimensional wavetrains which fail to be transverse orbitally stable (see [2] and the references therein). The detailed spectral properties of a related Hill's operator is a key element for developing the theory.

The stability of spatially periodic traveling wave solutions of nonlinear wave equations is a matter of fundamental interest. Recent advances pertain to both spectral and nonlinear stability, as well as to their relation to classical modulation theory (see, for example, the following abridged list of recent references [3, 4, 7, 8, 9, 14, 19, 21, 22, 24, 25, 26, 28, 33, 37]). Regarding the non-linear Klein-Gordon case under consideration, Jones *et al.* [25, 26] (see also [33]) recently provided a complete characterization of the spectral stability of periodic waves for the one-dimensional equations (1.1) and (1.4). This paper is part of the on-going effort to understand the stability properties of these one-dimensional spatially periodic waves as solutions to the multi-dimensional equation (1.3), by considering their orbital stability under transverse perturbations.

The interest in one-dimensional traveling fronts as solutions to equations in more than one spatial dimension goes beyond the mere mathematical challenge of studying them (see, e.g., the transverse (in)stability analysis of periodic wavetrains in [18, 23]). From a physical viewpoint and besides the aforementioned extended Josephson layer (see §5.1 below), the two-dimensional sine-Gordon equation has also been proposed by Xin [47] as a model to explain the emergence and evolution of very short optical pulses known as "light bullets". Moreover, recent potential applications of wave solutions (like periodic wavetrains) to the multi-dimensional sine-Gordon equation seem to arise as models to effectively describe nonlinear oscillations in intra cellular living structures such as DNA chains or folding proteins (for a recent review see [20] and the references therein). Thus, the transverse (in)stability of these periodic coherent one-dimensional structures is the first step in the development of a stability theory for more complex multi-dimensional wave solutions underlying various physical interpretations.

Plan of the paper. In §2 we describe the one-dimensional subluminal rotational periodic wavetrains for the equations under consideration: we establish the assumptions on the potential, describe some of the features of the waves, and recall the spectral stability result of [26], motivating the choice of this class of solutions as a subject of study. In §3 we show the one-dimensional orbital stability of this class of waves as solutions to (1.4), with respect to co-periodic perturbations. The central §4 extrapolates the theory to the study of orbital stability of the subluminal rotational wavetrains under the flow of the multi-dimensional equation (1.3), under co-periodic perturbations in the direction of translation and periodic (with any period) in the transverse direction. The final section 5 contains a physical interpretation of the results in terms of the multi-dimensional Josephson model, as well as a list of remaining open problems. Appendix A is devoted to the well-posedness of the nonlinear perturbation system of equations.

Notation. We denote the real and imaginary parts of any complex number $\lambda \in \mathbb{C}$ as $\operatorname{Re}\lambda$ and $\operatorname{Im}\lambda$, respectively. For any $s \in \mathbb{R}$, $s \geqq 0$, $H^s_{\mathrm{per}}([0,T])$ is the following space of (real, except where it is otherwise explicitly stated) functions,

$$H^s_{\mathrm{per}}([0,T]) = \{u \in H^s([0,T];\mathbb{R}) \, : \, u(x+T) = u(x) \ \text{a.e.}\},$$

endowed with the standard norm $\|u\|^2_{H^s_{\mathrm{per}}} = T\sum_{k\in\mathbb{Z}}(1+k^2)^s|\hat{u}(k)|^2$. According to custom we denote $H^0_{\mathrm{per}}([0,T]) = L^2_{\mathrm{per}}([0,T])$.

2. Preliminaries

2.1. **Periodic wavetrains.** First, we make the following assumptions on the potential function V.

Assumption 2.1. The potential function $V : \mathbb{R} \to \mathbb{R}$ satisfies:

(a) $V = V(u)$ is a periodic function of class C^2, with fundamental period 2π.
(b) V is uniformly bounded with $\min_{\mathbb{R}} V(u) = -1$ and $\max_{\mathbb{R}} V(u) = 1$; furthermore, we assume that V', V'' are also uniformly bounded.

Remark 2.2. Since equation (1.3) involves V' and not V directly, we are free to choose the period and amplitude of V via scalings $u \mapsto \alpha u$ and $(x,t) \mapsto (\beta x, \beta t)$, with $\alpha, \beta > 0$. Thus, assumptions (a) and (b) are made with no loss of generality.

Periodic wavetrains are one-dimensional solutions to (1.3) of the form

$$u(x,t) = \varphi(z), \qquad z = x_1 - ct \in \mathbb{R},$$

where $c \in \mathbb{R}$ is the speed of the wave, and $\varphi : \mathbb{R} \to \mathbb{R}$ represents the wave profile function. Substituting into (1.3) one easily finds that the profile function φ must satisfy the nonlinear pendulum equation

$$(c^2 - 1)\varphi_{zz} + V'(\varphi) = 0. \tag{2.1}$$

In the sequel, we assume that $c \neq \pm 1$. Integrating (2.1) once yields

$$\tfrac{1}{2}(c^2 - 1)\varphi_z^2 = E + V(\varphi), \tag{2.2}$$

where $E \in \mathbb{R}$ is a constant of integration associated to the total energy. There are four types of solutions to (2.1) for which the momentum φ_z is periodic. The first classification pertains to the speed: traveling waves propagating with speed such that $c^2 > 1$ are called *superluminal* waves; in the case when $c^2 < 1$ they are called *subluminal* waves. The second classification involves parameter values of the energy E. We call *rotational* waves to those solutions of the pendulum equation (2.1) whose orbits in the phase plane lie outside the separatrix. Solutions whose orbits in the phase plane lie inside the separatrix are called *librational* waves. It is easy to see that librational waves correspond to energies in the range $|E| < 1$, for which $\varphi(z+T) = \varphi(z)$ for all $z \in \mathbb{R}$ and some fundamental period $T > 0$. Likewise, rotational waves correspond to energies with either $E > 1$ (in the superluminal case), or $E < -1$ (in the subluminal case). Rotational waves satisfy $\varphi(z+T) = \varphi(z) \pm 2\pi$ for all $z \in \mathbb{R}$.

2.2. **Subluminal rotations.** Subluminal periodic waves of rotational type are non-equilibrium solutions $\varphi = \varphi(z)$ to the following problem,

$$\begin{cases} \tfrac{1}{2}(c^2 - 1)\varphi_z^2 = E + V(\varphi) \\ \varphi(z+T) = \varphi(z) \pm 2\pi, \qquad \text{for all } \ z \in \mathbb{R}, \\ c^2 < 1 \text{ and } E < -1. \end{cases} \tag{2.3}$$

The fundamental period is defined as the smallest positive value of T for which the second relation in (2.3) holds (the choice of sign corresponds to the fixed sign of φ_z). Therefore φ changes by 2π over a fundamental period. We immediately note from (2.3) that φ_z and $V(\varphi(z))$ are periodic functions with a minimal period T, $\varphi_z(z+T) = \varphi_z(z)$ and $V(\varphi(z+T)) = V(\varphi(z))$ for all $z \in \mathbb{R}$, respectively. Under assumptions 2.1, together with $c^2 < 1$ and $E < -1$, we notice that the effective potential $\overline{V}(\varphi) = -(1-c^2)^{-1}(E - V(\varphi))$ is strictly positive, yielding

$$\varphi_z^2 = -\frac{2(E - V(\varphi))}{1 - c^2} > 0. \tag{2.4}$$

Integrating $dz/d\varphi$ over $0 \leqq \varphi \leqq 2\pi$ we obtain an expression for the fundamental period T,

$$T = \frac{\sqrt{1-c^2}}{\sqrt{2}} \int_0^{2\pi} \frac{d\zeta}{\sqrt{V(\zeta) - E}}, \qquad E < -1, \, c^2 < 1.$$

Finally, to single out a unique solution to (2.3) for each (E,c) in the open set $\{(E,c) \in \mathbb{R}^2 \,:\, E < -1,\, c^2 < 1\}$, we fix the condition at $z = 0$ as follows:

$$\varphi(0) = \pi, \qquad \varphi_z(0) > 0. \tag{2.5}$$

In this fashion, the periodic wavetrain is uniquely determined for given $E < -1$, $c^2 < 1$. The choice of $\varphi(0)$ provides a maximizer for the potential V. The choice of the sign of the momentum is implicitly the selection of a particular limiting kink train as $E \to -1^-$ and corresponds to selecting the positive sign in the second equation of (2.3). Therefore, in order to facilitate the exposition, in the sequel we shall assume that *the sign of the momentum φ_z is positive*, without loss of generality. The results of this paper hold for the symmetric case as well, for which $\varphi_z < 0$.

2.3. **A note on one-dimensional spectral stability.** Let us now consider a one-dimensional perturbation of the periodic wavetrain, $\varphi = \varphi(z)$, as a solution of the one-dimensional equation (1.4). Substituting $u(x,t) = \varphi(z) + v(z,t)$, with $z = x - ct$, into (1.4), the perturbation v satisfies the nonlinear equation

$$v_{tt} - 2cv_{zt} + (c^2-1)v_{zz} + F(\varphi(z)+v) - F(\varphi(z)) = 0, \tag{2.6}$$

in view of the profile equation (2.1). Specializing to perturbations of the form $v(z,t) = e^{\lambda t}w(z)$, where $\lambda \in \mathbb{C}$ and $w \in X$, for some Banach space X, and after linearizing the equation around the profile, we arrive at the following scalar spectral equation of second order (quadratic spectral pencil [9, 30]):

$$(c^2-1)w_{zz} - 2c\lambda w_z + \big(\lambda^2 + F'(\varphi(z))\big)w = 0. \tag{2.7}$$

Formally, a necessary condition for the stability of φ is the absence of L^∞ (bounded) solutions $w \in X$ to equation (2.7) with $\operatorname{Re}\lambda > 0$ (and hence, growing exponentially in time). The growth rate $\lambda \in \mathbb{C}$ plays the role of the eigenvalue, and the definition of spectrum depends upon the choice of the space X. Equation (2.7) is quadratic in λ, and therefore it does not appear as the standard (linear) eigenvalue problem. The transformation $v_1 = w$, $v_2 = \lambda w$, however, defines a cartesian product of the appropriate base space X which allows to write the spectral equation (2.7) as a customary eigenvalue problem in the form

$$\lambda \mathbf{v} = \begin{pmatrix} 0 & 1 \\ (1-c^2)\partial_z^2 - F'(\varphi) & 2c\partial_z \end{pmatrix} \mathbf{v} =: \mathcal{L}\mathbf{v}, \qquad \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

If $w \in X = L^2(\mathbb{R};\mathbb{C})$ then the spectral analysis corresponds to stability under *localized perturbations*. The standard definition of the spectrum σ of the operator $\mathcal{L}$, with dense domain $D(\mathcal{L}) = H^2(\mathbb{R};\mathbb{C}) \times H^2(\mathbb{R};\mathbb{C})$ in the product space $L^2(\mathbb{R};\mathbb{C}) \times L^2(\mathbb{R};\mathbb{C})$, applies. Since the coefficients of $\mathcal{L}$ are T-periodic, Floquet theory implies that its L^2-spectrum is purely continuous with no isolated eigenvalues with finite multiplicity (see [15, 26, 29, 30] and the references therein).

Actually, the spectrum can be continuously parametrized [26] according to Floquet multipliers $\mu = e^{i\theta}$ with $\theta \in \mathbb{R}$ (mod 2π), by defining σ_θ as the set of complex numbers λ such that there exists a non-trivial solution w to the equation (2.7) with boundary conditions

$$\begin{pmatrix} w(T) \\ w_z(T) \end{pmatrix} = e^{i\theta} \begin{pmatrix} w(0) \\ w_z(0) \end{pmatrix}, \qquad -\pi < \theta \leqq \pi. \tag{2.8}$$

Thus, the spectrum is simply determined by the union $\sigma = \bigcup_{-\pi<\theta\leqq\pi}\sigma_\theta$, (cf. [26], Proposition 3.4). It can be shown that σ is indeed a curve of spectrum, parametrized by θ, and that σ coincides with the continuous L^2-spectrum of the operator $\mathcal{L}$. Each set σ_θ is the zero set of an entire function in λ (see section 3.1 in [26]) and, hence, is purely discrete. In particular, the set σ_0 (with $\theta = 0$) is the part of the spectrum corresponding to perturbations which are co-periodic. We refer the reader to [29] and to [26], section 3.1, for further information.

Definition 2.3. The periodic wavetrain φ is *spectrally stable* if there are no curves of spectrum, $\lambda \in \sigma$, with $\operatorname{Re}\lambda > 0$.

Since equation (1.4) is Hamiltonian, the spectrum is symmetric with respect to reflection in the real and imaginary axes, that is, if $\lambda \in \sigma$ then $\lambda^*, -\lambda^* \in \sigma$, as well (see Proposition 3.9 in [26]). In this fashion, *spectral stability is equivalent to* $\sigma \subset i\mathbb{R}$.

Theorem 2.4 (Spectral (in)stability [26])**.** *Let V be a potential satisfying Assumption 2.1. Then the following statements hold:*

(a) Subluminal rotational periodic wavetrains for the nonlinear Klein-Gordon equation (1.4) *are spectrally stable (that is, $\sigma \subset i\mathbb{R}$).*
(b) Superluminal rotational waves, as well as all librational waves (provided the period $T = T(E, c)$ satisfies $\partial_E T \neq 0$) are spectrally unstable, that is, there exist values $\lambda_* \in \hat{\sigma}$ such that $\operatorname{Re}\lambda_* > 0$.

For details, see Theorems 8.1 and 10.1 in Jones *et al.* [26].

Remark 2.5. Theorem 2.4 was first stated by Scott [42] in the special case of sine-Gordon wavetrains, with $V(u) = -\cos u$. His conclusion, however, was based upon a claim which is incorrect. Scott observed that, under the change of variables $h = e^{-c\lambda z/(c^2-1)}w$, equation (2.7) is transformed into Hill's equation

$$h_{zz} + (c^2-1)^{-1}F(\varphi)h = \omega h, \tag{2.9}$$

where the spectral parameter is now $\omega = \lambda^2/(c^2-1)^2$ (see section 4 in [26]). The spectrum σ_H of Hill's equation is well understood [32, 34], and with this information at hand, Scott attempted to locate the spectrum σ of equation (2.7). Unfortunately, Scott's argument (see [42, 25]) assumes that the transformation is isospectral, and his calculation of σ is wrong in general. Moreover, Jones *et al.* showed that $\lambda \in \sigma_H \cap \sigma$ only if $\lambda \in i\mathbb{R}$ (see Lemma 3.3 in [25]). There is, however, one case in which the study of Hill's equation provides the complete spectral information for the original problem, and it is precisely the subluminal rotational wavetrain case: from Theorem 2.4, $\sigma \subset i\mathbb{R}$, and therefore σ and σ_H coincide only in this case. This implies that *the stability properties of the linearization around the subluminal rotational wave are essentially captured by Hill's operator.* This is the technical reason why, in the subsequent analysis, the study of the (co-periodic) Bloch-type operator with Floquet exponent $\theta = 0$,

$$\widetilde{\mathcal{L}}_0 = \begin{pmatrix} (c^2-1)\partial_z^2 + F'(\varphi) & 0 \\ 0 & 1 \end{pmatrix},$$

which is basically an extended Hill's operator, is sufficient to perform a complete orbital stability analysis of the subluminal rotational wave.

3. One-dimensional orbital stability

In this section we establish the stability of the orbit generated by any rotational subluminal wave φ as a solution to the one-dimensional nonlinear Klein-Gordon equation (1.4). More precisely, we study the dynamics of the set $\mathcal{O}_\varphi = \{\varphi(\cdot + y) : y \in \mathbb{R}\}$ under the flow generated by equation (1.4). For this study, we consider the following space of functions,

$$\mathcal{P}_\pm(T) = \{u : \mathbb{R} \to \mathbb{R} : u(z+T) = u(z) \pm 2\pi, \text{ for all } z \in \mathbb{R}\}, \tag{3.1}$$

which, after a period T, produce a translation with the fundamental period of V. If we suppose that for $u = u(x,t) = u(z+ct,t)$, solution to (1.4), we have $u(t) \in \mathcal{P}_\pm(T)$ for a $T > 0$ fixed and a specific choice of the sign, $+$ or $-$, then the evolution variable,

$$v(z,t) := u(z+ct,t) - \varphi(z) \tag{3.2}$$

satisfies the nonlinear perturbation equation (2.6) and $v(\cdot,t)$ is a T-periodic function for all $t \geqq 0$. Hence, we are interested in the study of the nonlinear stability of the trivial solution $v \equiv 0$ of (2.6) under T-periodic perturbations.

We start our stability study by writing the evolution equation (2.6) as a first order system of the form

$$\mathbf{v}_t = JA(\mathbf{v}), \tag{3.3}$$

where $\mathbf{v} = (v, v_t)^\top =: (v,w)^\top$, and

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 2c\partial_z \end{pmatrix}, \quad A(\mathbf{v}) = \begin{pmatrix} (c^2-1)\partial_z^2 v + F(\varphi(z)+v) - F(\varphi(z)) \\ w \end{pmatrix}. \tag{3.4}$$

Notice that J is a skew-adjoint operator with respect to the $L^2_{\mathrm{per}}([0,T]) \times L^2_{\mathrm{per}}([0,T])$-inner product.

Equation (3.3) has a conservation law determined by the smooth functional

$$\begin{aligned} \mathcal{E} \ &: H^1_{\mathrm{per}}([0,T]) \times L^2_{\mathrm{per}}([0,T]) \to \mathbb{R}, \\ \mathcal{E}(v,w) &= \frac{1}{2}\int_0^T (1-c^2)v_z^2 + w^2 + 2G(v)\, dz \end{aligned} \tag{3.5}$$

with $G'(v(z)) = F(\varphi(z)+v) - F(\varphi(z))$. Now, since for z fixed we have

$$G(s) = \int_0^s F(\varphi(z)+\tau) - F(\varphi(z))\, d\tau,$$

from the mean-value theorem we obtain $|G(s)| \leqq \frac{1}{2}s^2$. Therefore, the functional $\mathcal{E}$ is well-defined:

$$|\mathcal{E}(v,w)| \leqq \tfrac{1}{2}(1-c^2)\|v_z\|^2_{L^2([0,T])} + \tfrac{1}{2}\|v\|^2_{L^2([0,T])} + \tfrac{1}{2}\|w\|^2_{L^2([0,T])}.$$

Moreover, from (3.5) we obtain immediately that

$$\mathcal{E}'(v,w) = \begin{pmatrix} (c^2-1)\partial_z^2 v + G'(v) \\ w \end{pmatrix}, \tag{3.6}$$

and we can re-write equation (3.3) as the Hamiltonian equation,

$$\mathbf{v}_t = J\mathcal{E}'(\mathbf{v}). \tag{3.7}$$

Let us denote the Hilbert space $X = H^1_{\mathrm{per}}([0,T]) \times L^2_{\mathrm{per}}([0,T])$, endowed with the inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle_X = \langle v, u \rangle_{L^2([0,T])} + \langle v_x, u_x \rangle_{L^2([0,T])} + \langle w, \zeta \rangle_{L^2([0,T])}, \tag{3.8}$$

for each $\mathbf{v} = (v, w)^\top$, $\mathbf{w} = (u, \zeta)^\top \in X$. Here, $\langle \cdot, \cdot \rangle_{L^2([0,T])}$ denotes the standard inner-product in $L^2_{\mathrm{per}}([0,T])$. In this fashion,

$$\|\mathbf{v}\|_X^2 = \langle (v, w) \rangle_X = \|v\|^2_{L^2([0,T])} + \|v_x\|^2_{L^2([0,T])} + \|w\|^2_{L^2([0,T])}.$$

In the study of the stability of the trivial zero solution for (2.6) we need to have information about the global well-posedness of the Cauchy problem for the vector equation (3.3). This is provided by the following

Theorem 3.1. *The initial value problem associated to equation* (3.3) *is globally well-posed in* $(X, \langle \cdot, \cdot \rangle_X)$*. Moreover, the global solution* $\mathbf{v}$ *satisfies* (3.3) *in the* X*-norm, that is, in a strong sense.*

By convenience of the reader, we outline the main steps of the proof of Theorem 3.1 in Appendix A.

In order to study the nonlinear stability of the trivial solution $\mathbf{v} = (0,0)^\top$ of equation (3.7) in X, we need to analyze the following self-adjoint operator

$$\mathcal{E}''(v, w) = \begin{pmatrix} (c^2-1)\partial_z^2 + F'(\varphi(z) + v) & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.9}$$

evaluated at $(v, w) = (0, 0)$. More precisely, we have the following result.

Lemma 3.2 (spectral analysis of $\mathcal{E}''(0,0)$)**.** *We consider the linear self-adjoint operator* $\mathcal{E}''(0,0)$ *with dense domain* $D = H^2_{\mathrm{per}}([0,T]) \times L^2_{\mathrm{per}}([0,T]) \subset X$*. Then the spectrum* $\sigma = \sigma(\mathcal{E}''(0,0))$ *of* $\mathcal{E}''(0,0)$ *is real and discrete,* $\sigma = \{0, \lambda_1, \lambda_2, ...\}$*, where*

$$0 < \lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \lambda_4 \leqq \cdots$$

and $\ker \mathcal{E}''(0,0) = \mathrm{span}\,\{(\varphi_z, 0)^\top\}$*. Moreover, there exists* $\beta > 0$ *such that for every* $\mathbf{h} \in X$ *satisfying* $\mathbf{h} \perp (\varphi_z, 0)^\top$ *we obtain*

$$\langle \mathcal{E}''(0,0)\mathbf{h}, \mathbf{h} \rangle_X \geqq \beta \|\mathbf{h}\|_X^2. \tag{3.10}$$

Proof. We consider the following Hill's type scalar operator $\mathcal{L}$ in $L^2_{\mathrm{per}}([0,T])$,

$$\mathcal{L}u = (c^2-1)u_{zz} + F'(\varphi(z))u, \tag{3.11}$$

with domain $D(\mathcal{L}) = H^2_{\mathrm{per}}([0,T])$. From (2.1) we notice that $\mathcal{L}\varphi_z = 0$. Moreover, since φ is a rotational subluminal traveling wave, it follows that φ_z has a fixed sign (without loss of generality, we consider $\varphi_z > 0$) and $\varphi_z \in H^2_{\mathrm{per}}([0,T])$. Therefore, from the oscillation theory for Hill operators (see [12, 32]) we have that zero is the first eigenvalue of $\mathcal{L}$ and it is simple, with eigenfunction φ_z. Moreover, $\sigma(\mathcal{L}) = \{0, \gamma_1, \gamma_2, ...\}$, where

$$0 < \gamma_1 \leqq \gamma_2 < \gamma_3 \leqq \gamma_4 < \cdots$$

(notice that from Proposition 4.4 in [26] (see Fig. 2), it can be also be verified that $\mathcal{L}$ is a non-negative self-adjoint operator). This implies that zero is a simple eigenvalue of the operator $\mathcal{E}''(0,0)$, with eigenfunction $(\varphi_z, 0)^\top$. Moreover, since $\mathcal{E}''(0,0)$ is a diagonal operator with positive entries in the diagonal, we have that there are not negative eigenvalues for $\mathcal{E}''(0,0)$, and so $\mathcal{E}''(0,0)$ is a nonnegative operator with zero being a simple eigenvalue.

Next we establish inequality (3.10). From the spectral theorem applied to the self-adjoint operator $\mathcal{L}$ we obtain that there is a $\beta_0 > 0$ such that, for every $g \in D(\mathcal{L})$ with $g \perp \varphi_z$, there holds

$$\langle g, \mathcal{L}g \rangle_{L^2([0,T])} \geqq \beta_0 \|g\|^2_{L^2([0,T])}. \tag{3.12}$$

Integration by parts yields,

$$\langle g, \mathcal{L}g\rangle_{L^2([0,T])} = (1-c^2)\|g_z\|^2_{L^2([0,T])} + \int_0^T F'(\varphi)|g|^2\, dz.$$

Therefore, for fixed constants $a, b > 0$, we obtain from (3.12) and from boundedness of $|F'| \leqq C$, that

$$(1-c^2)a\|g_z\|^2_{L^2([0,T])} + b\|g\|^2_{L^2([0,T])} \leqq \Big(a + \frac{b}{\beta_0}\Big)\langle g, \mathcal{L}g\rangle_{L^2([0,T])} + aC\|g\|^2_{L^2([0,T])}.$$

Hence, choosing $b > aC$ yields the existence of $\beta_1 > 0$ such that

$$\langle g, \mathcal{L}g\rangle_{L^2([0,T])} \geqq \beta_1 \|g\|^2_{H^1([0,T])}. \tag{3.13}$$

Therefore, for $\mathbf{h} = (g, h)^\top \in X$ and $g \perp \varphi_z$, we deduce immediately from (3.13) the estimate

$$\langle \mathbf{h}, \mathcal{E}''(0,0)\mathbf{h}\rangle_X = \langle g, \mathcal{L}g\rangle_{L^2([0,T])} + \|h\|^2_{L^2([0,T])} \geqq \beta_1\|g\|^2_{H^1([0,T])} + \|h\|^2_{L^2([0,T])}, \tag{3.14}$$

yielding (3.10). This finishes the proof of the Lemma. □

The following result establishes the coerciveness of the functional $\mathcal{E}$, which is the main tool for obtaining the nonlinear stability of the zero solution for model (2.6).

Lemma 3.3. *There exist $C_0 > 0$ and $\epsilon > 0$ such that*

$$\mathcal{E}(\mathbf{h}) \geqq C_0\|\mathbf{h}\|^2_X,$$

for all $\mathbf{h} \in B(0;\epsilon) = \{\mathbf{h} \in X \,:\, \|\mathbf{h}\|_X < \epsilon\}$.

Proof. Since $\mathcal{E}(0,0) = \mathcal{E}'(0,0) = 0$, it follows from Taylor expansion that

$$\mathcal{E}(\mathbf{h}) = \tfrac{1}{2}\langle \mathbf{h}, \mathcal{E}''(0,0)\mathbf{h}\rangle + o(\|\mathbf{h}\|^2_X) \tag{3.15}$$

for every $\mathbf{h} \in B(0;\epsilon)$. Hence, from the spectral theorem and Lemma 3.2 we obtain that for every $\mathbf{h} \in X$,

$$\begin{cases} \mathbf{h} = \gamma(\varphi_z, 0)^\top + \mathbf{h}^\perp, & \mathbf{h}^\perp \perp (\varphi', 0)^\top, \\ \langle \mathbf{h}, \mathcal{E}''(0,0)\mathbf{h}\rangle_X = \langle \mathbf{h}^\perp, \mathcal{E}''(0,0)\mathbf{h}^\perp\rangle_X \geqq \beta\|\mathbf{h}^\perp\|^2_X. \end{cases} \tag{3.16}$$

Therefore, since $\gamma = O(\|\mathbf{h}\|_X)$ we obtain from (3.15) and (3.16) that, for ϵ sufficiently small,

$$\mathcal{E}(\mathbf{h}) \geqq \beta\|\mathbf{h}^\perp\|^2_X + o(\|\mathbf{h}\|^2_X) \geqq C_0\|\mathbf{h}\|^2_X,$$

for some $C_0 > 0$ and $\|\mathbf{h}\|_X < \epsilon$. □

From Lemmas 3.1 and 3.3 we obtain the following stability result in $X = H^1_{\mathrm{per}}([0,T]) \times L^2_{\mathrm{per}}([0,T])$ associated to equation (3.7).

Theorem 3.4. *The trivial solution $\mathbf{v} \equiv (0,0)$ for (3.7) is stable in X by the periodic flow generated by equation (3.7). That is, for every $\epsilon > 0$ there exists $\delta > 0$ such that for $\mathbf{v}_0 \in X$ and $\|\mathbf{v}_0\|_X < \delta$, we have that the global solution $\mathbf{v}(t) \in X$ of (3.7) with $\mathbf{v}(0) = \mathbf{v}_0$ satisfies $\sup_{t\geqq 0}\|\mathbf{v}(t)\|_X < \epsilon$.*

Proof. Suppose that $\mathbf{v} = (0,0)$ is X-unstable. Then we can choose initial data $\mathbf{v}_k(0) \in X$ with $\|\mathbf{v}_k(0)\|_X < 1/k$ and $\epsilon > 0$, such that

$$\sup_{t\geqq 0}\|\mathbf{v}_k(t)\|_X \geqq \epsilon, \tag{3.17}$$

where $\mathbf{v}_k(t)$ is the solution to (3.7) with initial datum $\mathbf{v}_k(0)$. Now, by continuity in t, we can select the first time t_k such that $\|\mathbf{v}_k(t_k)\|_X = \frac{\epsilon}{2}$. Since $\mathcal{E}$ is continuous over X and is a conservation law for (3.7), we get from Lemma 3.3, that

$$0 \leftarrow \mathcal{E}(\mathbf{v}_k(0)) = \mathcal{E}(\mathbf{v}_k(t_k)) \geqq C_0 \|\mathbf{v}_k(t_k)\|_X^2,$$

as $k \to \infty$, which contradicts (3.17). This finishes the proof. □

From Theorem 3.4 and relation (3.2), we immediately obtain the following stability theorem for the rotational subluminal traveling wavetrain established in (2.3) by the flow of the nonlinear Klein-Gordon equation (1.4).

Theorem 3.5. *The rotational subluminal traveling wave solution φ determined by* (2.3) *is orbitally stable in X by the flow generated by the nonlinear Klein-Gordon equation* (1.4) *in the following sense: For every $\epsilon > 0$ there exists $\delta > 0$ such that for every $(u_0, u_1) \in (\mathcal{P}_\pm(T) \cap H^1_{\rm per}([0,T])) \times L^2_{\rm per}([0,T]) \subset X$, satisfying*

$$\|u_0 - \varphi\|_{H^1_{\rm per}([0,T])} + \|cu_0' + u_1\|_{L^2_{\rm per}([0,T])} < \delta, \tag{3.18}$$

then the solution $u = u(t)$ to equation (1.4) *with initial conditions $u(\cdot,0) = u_0(\cdot)$ and $u_t(\cdot,0) = u_1(\cdot)$ satisfies, for all $t \geqq 0$,*

$$\begin{cases} t \to u(\cdot + ct, t) - \varphi(\cdot) \in H^1_{\rm per}([0,T]) \\ t \to cu_x(\cdot + ct, t) + u_t(\cdot + ct, t) \in L^2_{\rm per}([0,T]), \end{cases} \tag{3.19}$$

and for all $t > 0$,

$$\|u(\cdot + \gamma, t) - \varphi(\cdot)\|_{H^1_{\rm per}([0,T])} + \|cu_x(\cdot,t) + u_t(\cdot,t)\|_{L^2_{\rm per}([0,T])} < \epsilon. \tag{3.20}$$

Here the modulation parameter γ is given explicitly by $\gamma(t) = ct$. Moreover, we have $t \in \mathbb{R} \to u(t) \in \mathcal{P}_\pm(T)$, for all $t > 0$.

Proof. From the relation $v(x,t) = u(z + ct, t) - \varphi(z)$ and from the assumptions $(u_0, u_1) \in X \subset L^2_{\rm per}([0,T]) \times L^2_{\rm per}([0,T])$, we obtain

$$\begin{cases} v(z,0) = u_0(z) - \varphi(z) \in H^1_{\rm per}([0,T]), \\ v_t(z,0) = cu'(z,0) + u_t(z,0) \equiv cu_0' + u_1 \in L^2_{\rm per}([0,T]). \end{cases} \tag{3.21}$$

Therefore, from the definition of the X-norm, from (3.18) and Theorem 3.4, relations (3.19) and (3.20) follow. This finishes the proof. □

As a corollary of Theorem 3.5 we deduce the orbital stability of subluminal rotational wavetrains of the sine-Gordon equation, a result that warrants note because of its importance in the nonlinear wave literature.

Corollary 3.6. *Subluminal rotational periodic traveling wave solutions to the sine-Gordon equation* (1.1) *are orbitally stable with respect to co-periodic perturbations.*

Remark 3.7. Theorem 3.5 can be alternatively established by an application of the work of Bronski *et al.* [9] (see Theorem 4.1 in the cited reference), provided that global well-posedness is verified (see Appendix A) and that the instability index count (equation (4.8) in [9]) is equal to zero (we leave such calculation in the nonlinear Klein-Gordon case to the dedicated reader). As we commented in the introduction, the purpose of proving Theorem 3.5 directly is to outline the arguments for the multi-dimensional analysis of the following section, a case which is not considered in the one-dimensional study of [9].

4. Transverse orbital stability

We now turn our attention to the stability of one-dimensional periodic wavetrains subject to transverse perturbations, that is, as solutions of the multi-dimensional nonlinear Klein-Gordon equation (1.3), with $d \geqq 2$. To facilitate the exposition we specialize the details of the analysis to the two-dimensional equation,

$$u_{tt} - u_{xx} - u_{yy} + F(u) = 0, \tag{4.1}$$

where $u = u(x, y, t)$, $(x, y) \in \mathbb{R}^2$ and $t \geqq 0$, meaning no loss of generality. Straightforward extensions to more dimensions follow immediately (see Remark 4.9 below).

4.1. Transverse spectral stability.

Consider a solution to (4.1) of the form

$$u(x, y, t) = \varphi(z) + e^{\lambda t} e^{i\xi y} w(z), \tag{4.2}$$

with $\lambda \in \mathbb{C}$, $\xi \in \mathbb{R}$, and $z = x - ct$. Linearizing around $\varphi = \varphi(z)$ we obtain the following second order equation parametrized by λ for each fixed $\xi \in \mathbb{R}$:

$$(c^2 - 1)w_{zz} - 2c\lambda w_z + (\lambda^2 + F'(\varphi(z)) + \xi^2)w = 0. \tag{4.3}$$

Genuine two-dimensional perturbations correspond to values $\xi \neq 0$. Extrapolating from the discussion in §2.3, we have that λ will belong to the spectrum for each fixed ξ if and only if there exists a Floquet multiplier on the unit circle, $|\mu| = 1$, or equivalently, if there exists a non-trivial, uniformly bounded solution w to (4.3) satisfying the boundary conditions (2.8) for some Floquet multiplier $\mu = e^{i\theta}$, $-\pi < \theta \leqq \pi$. We denote this set as σ_ξ. It can be proved (cf. [26, 29]) that

$$\sigma_\xi = \bigcup_{-\pi < \theta \leqq \pi} \sigma_{\theta,\xi},$$

for each fixed $\xi \in \mathbb{R}$.

Definition 4.1. We say the periodic wavetrain $\varphi = \varphi(z)$ is *transverse spectrally stable* if the spectrum σ_ξ is contained in the imaginary axis for each $\xi \in \mathbb{R}$.

The following lemma is a straightforward extension of Lemma 4.1 in [25], which we reproduce here for convenience.

Lemma 4.2. *Let $\varphi = \varphi(z)$ be a subluminal ($c^2 < 1$) rotational ($E < -1$) wave. Then* $\operatorname{Re} \sigma_{\theta,\xi} = 0$ *for all $\theta \in \mathbb{R}$ ($\operatorname{mod} 2\pi$) and all $\xi \in \mathbb{R}$.*

Proof. Suppose $\lambda \in \sigma_{\theta,\xi}$, for some values $\xi \in \mathbb{R}$, $-\pi < \theta \leqq \pi$. This implies that there exists a non-trivial, bounded solution w to equation (4.3), subject to the boundary conditions (2.8). One can write equation (4.3) in the form

$$(c^2 - 1)\overline{\mathcal{H}}w(z) - 2c\lambda w_z + (\lambda^2 + \xi^2)w = 0, \tag{4.4}$$

where

$$\overline{\mathcal{H}} = \partial_z^2 + \frac{F'(\varphi(z))}{c^2 - 1} \tag{4.5}$$

is Hill's operator. Observe that $(c^2 - 1)\overline{\mathcal{H}}$ is the operator in (3.11). Denoting

$$\langle u, v \rangle := \int_0^T u(z)^* v(z)\, dz,$$

it is known that, for a rotational wave, $\langle u, \overline{\mathcal{H}}u\rangle \leqq 0$ (see Proposition 4.4 in [26]). Moreover, $\langle w, w_z\rangle$ is purely imaginary, inasmuch as

$$\mathrm{Re}\,\langle w, w_z\rangle = \mathrm{Re}\int_0^T w^* w_z\,dz = \frac{1}{2}\int_0^T \frac{d}{dz}|w|^2\,dz = \frac{1}{2}\big(|w(T)|^2 - |w(0)|^2\big) = 0,$$

in view of the boundary conditions (2.8). Whence, take the inner product of equation (4.4) with w to obtain

$$(c^2-1)\langle w, \overline{\mathcal{H}}w\rangle - 2im\lambda + (\lambda^2+\xi^2)\|w\|^2 = 0,$$

where $m = -ic\langle w, w_z\rangle \in \mathbb{R}$, and $\|w\|^2 = \langle w, w\rangle$. Last equation is a second order equation for λ, with solutions given by

$$\lambda = \frac{1}{\|w\|^2}\left(im \pm \sqrt{-(m^2 + (c^2-1)\|w\|^2\langle w, \overline{\mathcal{H}}w\rangle + \xi^2\|w\|^4)}\right),$$

which are purely imaginary in the subluminal rotational case. □

Remark 4.3. Notice that, in view of the fact that determining the continuous spectrum σ_ξ is equivalent to locating the one-parameter family of *discrete point spectra* $\sigma_{\theta,\xi}$ (parametrized by θ) for each frequency $\xi \in \mathbb{R}$, Lemma 4.2 remarkably establishes transverse spectral stability under localized, bounded and/or co-periodic (in the z -Galilean- variable) perturbations.

4.2. **Transverse orbital stability under periodic perturbations.** In the last subsection we showed that the rotational subluminal wavetrains given by (2.3) are transverse spectrally stable, for instance, with respect to co-periodic perturbations for any transverse wavelength $\xi \in \mathbb{R}$ determined in (4.2). Hence, this specific behaviour for the one-dimensional rotational subluminal wave as a solution to the two-dimensional equation (4.1) in the form

$$u(x,y,t) = \varphi(x-ct),$$

induce us to conjecture that the profile is orbitally stable with respect to two-dimensional "periodic" perturbations. Indeed, the following theorem establishes more precisely this exceptional behaviour of the rotational subluminal wave.

Let us denote the Hilbert space

$$Y := H^1_{\mathrm{per}}([0,T]\times[0,L]) \times L^2_{\mathrm{per}}([0,T]\times[0,L]),$$

to represent perturbations which are square integrable, T-periodic in z and L-periodic in y, with $L>0$ arbitrary. The space Y is endowed by the standard norm

$$\|(u,v)\|_Y^2 = \|u\|_{H^1}^2 + \|v\|_{L^2}^2, \qquad \text{for all } (u,v)\in Y,$$

where

$$\|u\|_{H^1}^2 = \|u_z\|_{L^2}^2 + \|u_y\|_{L^2}^2 + \|u\|_{L^2}^2, \qquad \|u\|_{L^2} = \int_0^T\int_0^L |u(z,y)|^2\,dy\,dz.$$

Let us state the main result of the paper.

Theorem 4.4 (transverse orbital stability)**.** *The rotational subluminal traveling wave profile*

$$\Phi(z,y) = \varphi(z), \qquad (z,y)\in\mathbb{R}^2,$$

determined by (2.3) *is orbitally stable in* Y *by the flow generated by the two-dimensional nonlinear Klein-Gordon equation* (4.1) *in the following sense: For* *every $\epsilon > 0$ there exists $\delta > 0$ such that for $u_0 = u_0(\cdot,\cdot) \in \mathcal{P}_\pm(T) \times H^1_{\text{per}}([0,L])$ (see Remark 4.5 below), and $u_1 \in L^2_{\text{per}}([0,T]\times[0,L])$ satisfying*

$$\|u_0 - \Phi\|_{H^1_{\text{per}}([0,T]\times[0,L])} + \|c\partial_z u_0 + u_1\|_{L^2_{\text{per}}([0,T]\times[0,L])} < \delta,$$

then the solution $u = u(z,y,t)$ of the nonlinear Klein-Gordon equation (4.1) with initial conditions $u(\cdot,\cdot,0) = u_0(\cdot,\cdot)$ and $u_t(\cdot,\cdot,0) = u_1(\cdot,\cdot)$ satisfies, for all $t \geqq 0$,

$$\begin{cases} t \to u(\cdot + ct, \cdot, t) - \Phi(\cdot,\cdot) \in H^1_{\text{per}}([0,T]\times[0,L]) \\ t \to c\partial_z u(\cdot + ct, y, t) + u_t(\cdot + ct, y, t) \in L^2_{\text{per}}([0,T]\times[0,L]), \end{cases} \tag{4.6}$$

and, for all $t > 0$,

$$\|u(\cdot + \gamma, \cdot, t) - \Phi(\cdot,\cdot)\|_{H^1_{\text{per}}([0,T]\times[0,L])} + \|c\partial_z u(\cdot,\cdot,t) + u_t(\cdot,\cdot,t)\|_{L^2_{\text{per}}([0,T]\times[0,L])} < \epsilon.$$

Here the modulation parameter γ is given explicitly by $\gamma(t) = ct$. Moreover, we have $t \in \mathbb{R} \to u(\cdot,y,t) \in \mathcal{P}_\pm(T)$, for all y fixed and all $t > 0$.

Remark 4.5. The notation $u_0(\cdot,\cdot) \in \mathcal{P}_\pm(T) \times H^1_{\text{per}}([0,L])$ means the following:

$$\begin{cases} z \to u_0(z,y) \in \mathcal{P}_\pm(T), & \text{for every } \; y \in \mathbb{R} \\ u(z,\cdot) \in H^1_{\text{per}}([0,L]), & \text{for every } \; z \in \mathbb{R}. \end{cases} \tag{4.7}$$

In order to prove Theorem 4.4, first we need to establish some preliminary information. We start by considering, for any solution $u = u(x,y,t)$ to (4.1), the following perturbation variable $v = v(\cdot,\cdot,t)$, defined as

$$v(z,y,t) = u(z + ct, y, t) - \varphi(z). \tag{4.8}$$

Suppose $x \to u(x,\cdot,t) \in \mathcal{P}_\pm(T)$ and $y \to u(\cdot,y,t) \in L^2_{\text{per}}([0,L])$ for all $t \in \mathbb{R}$, then we have that v is a doubly-periodic function on $\mathbb{R}^2$,

$$v(z+T, y+L, t) = v(z,y,t), \qquad \text{for all } \; z,y,t,$$

and satisfies the nonlinear equation

$$v_{tt} - 2cv_{zt} + (c^2-1)v_{zz} - v_{yy} + F'(\varphi(z) + v) - F'(\varphi(z)) = 0. \tag{4.9}$$

In order to prove Theorem 4.4 we need to study the nonlinear stability of the trivial solution $v \equiv 0$ for (4.9).

The evolution equation (4.9) can be seen as a first order system of the form

$$\mathbf{v}_t = J\widetilde{\mathcal{E}}'(\mathbf{v}), \tag{4.10}$$

where $\mathbf{v} = (v, v_t) \equiv (v,w)^\top$, J is defined as in (3.4), and $\widetilde{\mathcal{E}}'$ is the derivative of the well-defined smooth functional

$$\begin{aligned} &\widetilde{\mathcal{E}} : H^1_{\text{per}}([0,T]\times[0,L]) \times L^2_{\text{per}}([0,T]\times[0,L]) \to \mathbb{R}, \\ &\widetilde{\mathcal{E}}(v,w) = \frac{1}{2}\int_0^T \int_0^L (1-c^2)v_z^2 + v_y^2 + w^2 + 2G(v)\, dy\, dz, \end{aligned} \tag{4.11}$$

with $G'(v(z,y)) = F(\varphi(z) + v(z,y)) - F(\varphi(z))$. It is immediately observed that $\widetilde{\mathcal{E}}$ is a conservation law for (4.10), at least formally.

Similarly as in §3, first we need to verify the well-posedness of the Cauchy problem. By applying the same ideas as in the proof of Theorem 3.1 (see appendix A) we have the following well-posedness result for the two-dimensional Klein-Gordon (4.1) equation (details are omitted).

Theorem 4.6. *The initial value problem associated to equation* (4.10) *is globally well-posed in* $(Y, \langle\cdot\,,\,\cdot\rangle_Y)$.

The study of the stability of the trivial solution $\mathbf{v} \equiv (0,0)$ of equation (4.10) in Y requires to analyze the following self-adjoint operator

$$\widetilde{\mathcal{E}}''(v,w) = \begin{pmatrix} (c^2-1)\partial_z^2 - \partial_y^2 + F'(\varphi(z)+v) & 0 \\ 0 & 1 \end{pmatrix} : Y \to Y, \tag{4.12}$$

evaluated at $(v,w) = (0,0)$. More precisely, we have the following

Lemma 4.7 (spectral analysis of $\widetilde{\mathcal{E}}''(0,0)$)**.** *We consider the linear self-adjoint operator* $\widetilde{\mathcal{E}}''(0,0) : Y \to Y$ *with dense domain* $D = H^2_{\text{per}}([0,T]\times[0,L])\times L^2_{\text{per}}([0,T]\times[0,L])$. *Then the spectrum* $\sigma = \sigma(\widetilde{\mathcal{E}}''(0,0))$ *of* $\widetilde{\mathcal{E}}''(0,0)$ *is discrete,* $\sigma = \{0, \mu_1, \mu_2, ...\}$, *where*

$$0 < \mu_1 \leqq \mu_2 \leqq \mu_3 \leqq \mu_4 \leqq \cdots$$

and $\ker \widetilde{\mathcal{E}}''(0,0) = \operatorname{span}\{(\varphi_z, 0)\}$. *Moreover, there exists* $\beta > 0$ *such that for every* $\mathbf{h} \in Y$ *satisfying* $\mathbf{h} \perp (\varphi_z, 0)^\top$ *we obtain*

$$\langle \mathbf{h}, \widetilde{\mathcal{E}}''(0,0)\mathbf{h}\rangle_Y \geqq \beta \|\mathbf{h}\|_Y^2. \tag{4.13}$$

Proof. It is immediate from (2.1) that $\widetilde{\mathcal{E}}''(0,0)(\varphi_z,0)^\top = 0$ because $\partial_y\varphi(z) = 0$. Next, we show that $\widetilde{\mathcal{E}}''(0,0)$ does not have negative eigenvalues. Let $\mu < 0$ be an eigenvalue for $\widetilde{\mathcal{E}}''(0,0)$ and $(h,g)^\top \in H^2_{\text{per}}([0,T]\times[0,L]) \times L^2_{\text{per}}([0,T]\times[0,L])$ an associated eigenfunction. Then we obtain the relations

$$\begin{cases} \mathcal{L}_1 h \equiv (c^2-1)\partial_z^2 h - \partial_y^2 h + F'(\varphi(z))h = \mu h \\ g = \mu g. \end{cases} \tag{4.14}$$

From the later relation, it follows immediately that $\mathcal{L}_1 h_y = \mu h_y$. So, h and h_y are eigenfunctions of $\mathcal{L}_1$. Next, we see that h is a function only of the variable z, namely, $h(z,y) = A(z)$ for all $(z,y) \in \mathbb{R}^2$. Indeed, without loss of generality, we can suppose that $\mu = \inf \sigma(\mathcal{L}_1)$. From the classical result on d-dimensional Schrödinger operators, $d \geqq 2$, we conclude that μ is a simple eigenvalue for $\mathcal{L}_1$ with an eigenfunction that does not take the value zero in $[0,T]\times[0,L])$ (cf. [12]). Thus, we can suppose that $h(z,y) > 0$ for every z, y. Then, there exists $\theta > 0$ such that $h_y(z,y) = \theta h(z,y)$ for every z, y. Now, for z fixed, we defined $j(y) = h(z,y)$, and so j satisfies the following boundary problem (for fixed z),

$$\begin{cases} j'(y) = \theta j(y) \\ j(0) = h(z,0) \equiv A(z). \end{cases} \tag{4.15}$$

Therefore,

$$j(y) = h(z,y) = A(z)e^{\theta y}, \qquad \text{for all } \ y. \tag{4.16}$$

Since h is periodic in the y-variable, $\theta = 0$. Therefore, $h(z,y) = A(z)$ for all z, y, and satisfies

$$\mathcal{L}_1 A(z) = [(c^2-1)\partial_z^2 + F(\varphi(z))]A(z) = \mu A(z), \qquad \mu < 0.$$

This is a contradiction by Lemma (3.2). Hence, $\widetilde{\mathcal{E}}''(0,0)$ is a non-negative operator.

By the analysis above $\mathcal{L}_1$ has no negative eigenvalues. Moreover, $\mathcal{L}_1 G = 0$ with $G(z,y) = \varphi_z \in H^2_{\text{per}}([0,T]\times[0,L])$ and $G(z,y) > 0$ for all z, y. Therefore, zero is an simple eigenvalue for $\mathcal{L}_1$, it which implies that $\ker \widetilde{\mathcal{E}}''(0,0) = \operatorname{span}\{(\varphi_z,0)^\top\}$. The proof of inequality (4.13) follows similarly as that of (3.10). □

From Lemmas 4.6 and 4.7 we obtain the following stability result associated to equation (4.9).

Theorem 4.8. *The trivial solution* $\mathbf{v} \equiv (0,0)$ *for* (4.10) *is stable in* Y *by the periodic flow generated by the evolution equation* (4.10). *Indeed, for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that for* $\mathbf{v}_0 \in Y$*, and* $\|\mathbf{v}_0\|_Y < \delta$*, we have that the global solution* $\mathbf{v}(t)$ *of* (4.10) *with* $\mathbf{v}(0) = \mathbf{v}_0$ *satisfies* $\mathbf{v}(t) \in Y$ *and* $\|\mathbf{v}(t)\|_Y < \epsilon$ *for all* $t \geqq 0$.

Proof. Follows from arguments similar to those in the proof of Theorem 3.4. □

Finally, as a consequence of last observations, we can easily prove the main theorem of this section.

Proof of Theorem 4.4. By an analogous analysis as that of §3, we obtain from Theorem 4.8 and relation (4.8), the nonlinear stability behaviour for the wavetrain φ in (2.3) under the flow of the two-dimensional equation (4.1). □

Remark 4.9. It follows immediately from the analysis above, that the rotational subluminal traveling wavetrain profiles

$$\Phi(z, y_1, y_2, ..., y_{d-1}) = \varphi(z), \qquad (z, y_1, y_2, ..., y_{d-1}) \in \mathbb{R}^d,$$

with φ given in (2.3), are also nonlinearly stable in $H^1_{\mathrm{per}}([0,T] \times [0,L_1] \times \cdots \times [0, L_{d-1}]) \times L^2_{\mathrm{per}}([0,T] \times [0,L_1] \times \cdots \times [0,L_{d-1}])$ for any chosen wavelengths $L_i > 0$, $1 \leqq i \leqq d-1$, by the flow of the d-dimensional nonlinear Klein-Gordon equation (1.3), with $d \geqq 3$.

5. Discussion

5.1. **On the physical interpretation in multiple dimensions.** One of the most important physical interpretations (at least historically) of the two-dimensional sine-Gordon equation pertains to a large area Josephson junction (or "extended" junction), originally proposed by B. D. Josephson [27]. The model considers two infinite superconducting metal plates separated by a dielectric barrier layer parallel to the xy-plane, thin enough to allow coupling of the wave functions of electrons for the two superconductors (tunnelling of superconducting electron pairs). If φ denotes the phase difference between the two wave functions then the Josephson and Maxwell equations can be reduced (cf. [46, 6]) to a scalar equation for the phase, namely

$$\partial_x^2 \varphi + \partial_y^2 \varphi - \left(\frac{1}{v_s^2}\right) \partial_t^2 \varphi = \left(\frac{1}{\lambda_J^2}\right) \sin\varphi,$$

where λ_J is the Josephson penetration length, a constant depending on well-known physical parameters such as the applied Josephson current J_m (actually, $\lambda_J = O(J_m^{-1/2})$; see [46] for details), and $v_s = \omega_J \lambda_J$, with ω_J being the plasma frequency (with $\omega_J = O(J_m^{1/2})$). v_s is the velocity of the linear electromagnetic waves in the dielectric layer of the junction in the absence of Josephson current (notice that the right hand side of the equation vanishes as $J_m \to 0$). After rescaling $(x,y) \mapsto \lambda_J^{-1}(x,y)$, $t \mapsto \omega_J t$, one obtains the two-dimensional sine-Gordon equation

$$\varphi_{xx} + \varphi_{yy} - \varphi_{tt} = \sin\varphi. \tag{5.1}$$

If it is assumed that φ does not vary on the y-direction one obtains solutions to equation (1.1). These are special solutions to (5.1) representing the propagation of a single flux quantum through the junction. Subluminal wave solutions, for example, describe finite energy periodic coherent structures inside the junction traveling with speed below the dielectric speed $|v_s|$ of linear electromagnetic waves. Using tools from the theory of Jacobi elliptic functions (cf. [1, 31]) and for given (normalized) values of the speed $c^2 < 1$ and the energy $E < -1$, it is possible to exactly determine the profile of a one-dimensional rotational subluminal traveling wavetrain for the sine-Gordon equation (5.1), $\varphi(x, y, t) = \varphi_{c,E}(x - ct)$, where the profile function $\varphi_{c,E}$ is given by

$$\varphi_{c,E}(z) = \begin{cases} -\arccos\Big[1 - 2\,\mathrm{cn}^2\Big(\sqrt{\dfrac{1-E}{2(1-c^2)}}\,z; k\Big)\Big], & 0 \leqq z \leqq \dfrac{T}{2}, \\ \arccos\Big[1 - 2\,\mathrm{cn}^2\Big(\sqrt{\dfrac{1-E}{2(1-c^2)}}\,(T-z); k\Big)\Big], & \dfrac{T}{2} \leqq z \leqq T. \end{cases} \tag{5.2}$$

Here $K = K(k)$ denotes the complete elliptic integral of the first kind, and the elliptic modulus $k \in (0, 1)$, associated to the Jacobian elliptic function $\mathrm{cn}(\cdot)$ (the so called cnoidal wave), is the positive root of $k^2 = 2/(1-E)$. The period T is defined by $T = 2\sqrt{\frac{2(1-c^2)}{1-E}}K(k)$. (We omit the details of this derivation.) A depiction of the periodic solution (5.2), for the parameter values $E = -2$ and $c = 0.5$, can be found in Figure 1.

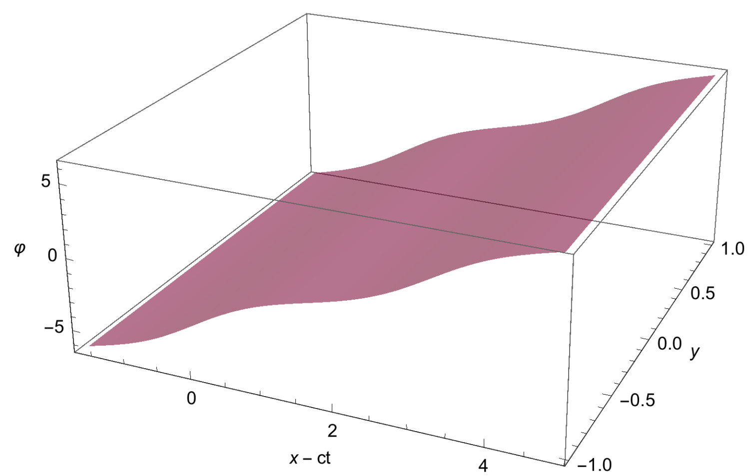

FIGURE 1. Plot of the rotational subluminal periodic wave $\varphi(x, y, t) = \varphi_{c,E}(x - ct, y)$ defined in (5.2), for the parameter values $E = -2$, $c = 0.5$ in the moving box $(x - ct, y) \in [-T/2, 3T/2] \times [-1, 1]$ (color plot online). Here the fundamental period is, approximately, $T \approx 3.2476$.

Although these waves are fundamentally one-dimensional, they emerge as important particular cases of more complex two-dimensional waves obeying dispersion relations of the form $\omega^2 = \omega_J^2(1 + \lambda_J^2(k_1^2 + k_2^2))$ (see [46]), as limits when $k_2 \to 0$. Furthermore, since the Laplace operator in the original model (5.1) is invariant under rotations, the latter are specific solutions associated to one single directed magnetic field fluxon.

Theorem 4.4 establishes that these subluminal rotational one-dimensional waves are orbitally stable under small transversal periodic perturbations under the flow of the extended Josephson junction equation (5.1).

5.2. **Open problems.** As far as we know, the orbital (in)stability with respect to co-periodic perturbations of superluminal rotational and librational waves (see the classification in §2.1) has not been established, not even in one dimension, nor in the particular sine-Gordon case, even though these classes of wavetrains are known to be spectrally unstable [26]. This task would be substantially completed if an unstable co-periodic eigenvalue could be detected, as we conjecture that orbital instability can be deduced from spectral instability under co-periodic perturbations. The theory regarding subluminal librations, however, is more developed. Indeed, Jones *et al.* [26] proved that all linearized operators around librational subluminal waves with $\partial_E T < 0$ have a real co-periodic eigenvalue. This information is consistent with the analysis of Natali [36], who proved orbital instability with respect to co-periodic perturbations in the particular case of the one-dimensional sine-Gordon equation (for which the non-degeneracy condition $\partial_E T < 0$ is automatically satisfied).

Another interesting question concerns the extension to more complicated (for example, not necessarily periodic) potentials. If periodicity is dropped then there can no longer exist periodic traveling waves of rotational type, even for bounded V, and all periodic wavetrains are librations. There exist, however, studies in the literature considering this case (see, for example, the recent one-dimensional orbital instability results of [5] for quintic potentials).

Finally, it is to be observed that, up to our knowledge, the orbital stability of periodic wavetrains for the sine-Gordon equation (1.1) or its generalization (1.4) under *localized* L^2-perturbations (in the variable of propagation) is an interesting open problem, even in one dimension. The Lyapunov method employed here fails to establish the positiveness of the linearized operator for that case. For instance, we have even studied its applicability to the case of two-dimensional perturbations which are co-periodic in the variable of propagation, but localized in the transverse direction. It can be shown that the corresponding operator $\widetilde{\mathcal{E}}''(0,0)$ (see equation (4.12)), defined in the appropriate space, has not closed range and $\lambda = 0$ belongs to the essential spectrum, precluding the existence of a spectral gap (a computation not included here). The Lyapunov method cannot guarantee the orbital stability under such perturbations, even though spectral stability does hold (see Lemma 4.2).

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Appendix A. Well-posedness theory

The well-posedness of nonlinear wave equations of the form (1.3) with bounded nonlinearity is well-understood (an abridged list of references includes [10, 35, 39, 44, 45]). In this appendix we discuss the global existence of solutions to the nonlinear perturbation equations (2.6) and (4.9) in a suitable framework needed for the stability analysis. We gloss over some details, which are classical and can be found elsewhere, and provide the main steps of the proof. We focus on the existence of solutions to (2.6) in the standard periodic (in the Galilean variable) Sobolev spaces.

For simplicity, and without loss of generality, let us suppose that $T = 2\pi$. We consider the Cauchy problem for equation (2.6), or equivalently, for system (3.3), written in the form

$$\begin{cases} \mathbf{v}_t = L\mathbf{v} + R(\mathbf{v}), & (x,t) \in [0,2\pi] \times (0,+\infty), \\ \mathbf{v}(0) = \mathbf{v}_0, & x \in [0,2\pi], \end{cases} \tag{A.1}$$

where $\mathbf{v} = (v,w)^\top$, and

$$L = \begin{pmatrix} 0 & I \\ (1-c^2)\partial_x^2 & 2c\partial_x \end{pmatrix}, \qquad R(\mathbf{v}) = \begin{pmatrix} 0 \\ F(\varphi) - F(\varphi+v) \end{pmatrix}. \tag{A.2}$$

L is a linear, closed, densely defined operator in the Hilbert space $X = H^1_{\mathrm{per}}([0,2\pi]) \times L^2_{\mathrm{per}}([0,2\pi])$, with domain $D(L) = H^2_{\mathrm{per}}([0,2\pi]) \times H^1_{\mathrm{per}}([0,2\pi])$, endowed with the inner product,

$$\langle \mathbf{v}, \mathbf{w} \rangle_X = \langle v, u \rangle + \langle v_x, u_x \rangle + \langle w, \zeta \rangle,$$

for each $\mathbf{v} = (v,w)^\top$, $\mathbf{w} = (u,\zeta)^\top \in X$. Here, $\langle \cdot, \cdot \rangle$ denotes the standard inner-product in $L^2_{\mathrm{per}}([0,2\pi])$, with $\|u\|^2 = \langle u, u \rangle$, so that,

$$\|\mathbf{v}\|_X^2 = \langle (v,w) \rangle_X = \|v\|^2 + \|v_x\|^2 + \|w\|^2.$$

Theorem 3.1 can be proved in consecutive stages.

1. The operator $L : D(L) \subset X \to X$ is the infinitesimal generator of a C_0-group, $\{S(t)\}_{t \in \mathbb{R}}$ in X. This fact can be verified via a direct computation of the group. Indeed, by a standard Fourier analysis, the solution to the linear evolution equation

$$\begin{aligned} \mathbf{v}_t &= L\mathbf{v}, \\ \mathbf{v}(0) &= (v_0, w_0)^\top, \end{aligned} \tag{A.3}$$

with $(v_0, w_0)^\top \in X$ is given explicitly by the formula

$$(v,w)(t)^\top = S(t)(v_0,w_0)^\top := \begin{pmatrix} \sum_{n\in\mathbb{Z}} a(n,t) e^{-inx} \\ \sum_{n\in\mathbb{Z}} b(n,t) e^{-inx} \end{pmatrix}, \quad t \in \mathbb{R},$$

where the Fourier coefficients are defined, for $n = 0$, as

$$a(0,t) = \hat{w}_0(0)t + \hat{v}_0(0), \quad b(0,t) = \hat{w}_0(0), \tag{A.4}$$

and for $n \neq 0$ as,

$$\begin{aligned}
a(n,t) &= \Big(\tfrac{1}{2}(1-c)\hat{v}_0(n) - \tfrac{1}{2in}\hat{w}_0(n)\Big)e^{-in(1+c)t} + \\
&\quad + \Big(\tfrac{1}{2}(1+c)\hat{v}_0(n) + \tfrac{1}{2in}\hat{w}_0(n)\Big)e^{in(1-c)t} \\
b(n,t) &= \Big(-\tfrac{in}{2}(1-c^2)\hat{v}_0(n) + \tfrac{1}{2}(1+c)\hat{w}_0(n)\Big)e^{-in(1+c)t} + \\
&\quad + \Big(\tfrac{in}{2}(1-c^2)\hat{v}_0(n) + \tfrac{1}{2}(1-c)\hat{w}_0(n)\Big)e^{in(1-c)t},
\end{aligned} \tag{A.5}$$

where

$$\hat{v}_0(n) = \int_0^{2\pi} v_0(x)e^{inx}\,dx, \quad \hat{w}_0(n) = \int_0^{2\pi} w_0(x)e^{inx}\,dx, \quad n \in \mathbb{Z}.$$

The reader may easily verify that $S(t) : X \to X$ is a C_0-group. Moreover, we have an estimate of the form

$$\|S(t)(v_0,w_0)^\top\|_X^2 \leqq 4\max\{1,t^2\}\|(v_0,w_0)^\top\|_X^2, \tag{A.6}$$

for $t > 0$ and all $(v_0,w_0)^\top \in X$. This can be verified using Parseval's identity:

$$\begin{aligned}
\frac{1}{2\pi}\|v(t)\|^2 = \sum_{n\in\mathbb{Z}} |a(n,t)|^2 &= |a(0,t)|^2 + \sum_{n\neq 0}|a(n,t)|^2 \\
&\leqq 2(t^2|\hat{w}_0(0)|^2 + |\hat{v}_0(0)|^2) + 2\sum_{n\neq 0}(|\hat{w}_0(0)|^2 + |\hat{v}_0(0)|^2) \\
&\leqq 2\max\{1,t^2\}\frac{1}{2\pi}(\|\hat{v}_0\|^2 + \|\hat{w}_0\|^2),
\end{aligned}$$

after using (A.4) and (A.5). Likewise, since

$$|b(n,t)| \leqq (1-c^2)|n\hat{v}_0(n)| + |\hat{w}_0(n)| \leqq |\widehat{\partial_x v_0}(n)| + |\hat{w}_0(n)|,$$

and since $\widehat{\partial_x v_0}(0) = 0$, it is easy to verify that

$$\|w(t)\|^2 \leqq 2\|\partial_x v_0\|^2 + 2\|w_0\|^2.$$

This shows (A.6).

Moreover, since $V' = F$ is uniformly bounded (Assumption 2.1 (b)), then by the mean value theorem we have

$$\|F(\varphi) - F(\varphi+v)\|^2 \leqq \bar{C}\int_0^{2\pi}|v(x,t)|^2\,dx = \bar{C}\|v(t)\|^2,$$

for some uniform $\bar{C} > 0$ (that depends on F). Therefore, we obtain the estimate

$$\begin{aligned}
\|S(t)R(\mathbf{v}(s))\|_X^2 &\leqq 4\max\{1,t^2\}\|(0, F(\varphi) - F(\varphi+v))^\top\|_X^2 \\
&= 4\max\{1,t^2\}\|F(\varphi) - F(\varphi+v)\|^2 \\
&\leqq 4\bar{C}\max\{1,t^2\}\|v(s)\|^2.
\end{aligned} \tag{A.7}$$

2. Local well-posedness in X. The local existence of solutions to the nonlinear evolution (A.1) can be proved via a standard contraction mapping argument. Let T be such that $0 < T \leqq 1$. Let us define

$$Y_{T,\beta} := \Big\{\mathbf{v} \in C([0,T];X) \,:\, \sup_{t\in[0,T]}\|\mathbf{v}(t)\|_X < \beta\Big\},$$

and for fixed $\mathbf{v}_0 = (v_0, w_0)^\top \in X$, let us define the mapping

$$\Phi_{\mathbf{v}_0}(\mathbf{v})(t) := S(t)\mathbf{v}_0 + \int_0^t S(t-s)R(\mathbf{v}(s))\, ds.$$

We claim that we can choose $T > 0$ and $\beta > 0$ such that $\Phi_{\mathbf{v}_0}(\mathbf{v}(t)) \in Y_{T,\beta}$ for all $\mathbf{v} \in Y_{T,\beta}$ and that $\Phi_{\mathbf{v}_0}(\mathbf{v}(t)) : Y_{T,\beta} \to Y_{T,\beta}$ is a contraction. Indeed, since $T \leqq 1$, from (A.6) we obtain $\|S(t)\mathbf{v}_0\|_X \leqq 2\|\mathbf{v}_0\|_X$, for all $t \in [0,T]$. Also, (A.7) yields

$$\Big\| \int_0^t S(t-s)R(\mathbf{v}(s))\, ds \Big\|_X \leqq 2\bar{C}^{1/2} T \sup_{s\in[0,T]} \|v(s)\|.$$

Choosing $\beta = 4\|\hat{v}_0\|_X$ and $T < \frac{1}{4}\bar{C}^{-1/2}$ we obtain

$$\|\Phi_{\mathbf{v}_0}(\mathbf{v})\|_X \leqq 2\|\mathbf{v}_0\|_X + 2\bar{C}^{1/2} T \sup_{s\in[0,T]} \|\mathbf{v}(s)\|_X = \tfrac{1}{2}\beta + 2\bar{C}^{1/2} T\beta < \beta.$$

This shows $\Phi_{\mathbf{v}_0}(\mathbf{v}) \in Y_{T,\beta}$. Now let $\mathbf{v} = (v,w)^\top$, $\mathbf{z} = (z,u)^\top \in Y_{T,\beta}$. Thus,

$$\Phi_{\mathbf{v}_0}(\mathbf{v})(t) - \Phi_{\mathbf{v}_0}(\mathbf{z})(t) = \int_0^t S(t-s)(R(\mathbf{v}(s)) - R(\mathbf{z}(s))\, ds.$$

By a similar argument as before, we can show that $\|F(\varphi + z) - F(\varphi + v)\|^2 \leqq \bar{C}\|z - v\|^2$, and therefore it is easy to verify that

$$\|\Phi_{\mathbf{v}_0}(\mathbf{v})(t) - \Phi_{\mathbf{v}_0}(\mathbf{z})(t)\|_X \leqq 2\bar{C}^{1/2} T \sup_{s\in[0,T]} \|\mathbf{v}(s) - \mathbf{z}(s)\|_X,$$

for all $t \in [0,T]$. Since $T < \frac{1}{4}\bar{C}^{-1/2}$ we immediately obtain

$$\sup_{t\in[0,T]} \|\Phi_{\mathbf{v}_0}(\mathbf{v})(t) - \Phi_{\mathbf{v}_0}(\mathbf{z})(t)\|_X < \sup_{t\in[0,T]} \|\mathbf{v}(t) - \mathbf{z}(t)\|_X,$$

and $\Phi_{\mathbf{v}_0}$ is a contraction in the ball $Y_{T,\beta}$. By Banach's fixed point theorem there exists $\mathbf{v} \in C([0,T];X)$ such that

$$\mathbf{v}(t) = S(t)\mathbf{v}_0 + \int_0^t S(t-s)R(\mathbf{v}(s))\, ds. \tag{A.8}$$

Thus, we conclude that the Cauchy problem (A.1) is locally well-posed for every initial data $\mathbf{v}(0) = \mathbf{v}_0$ in X; by standard semigroup theory [38] the solution (A.8) satisfies (A.1) in the X-norm, i.e., in a strong sense. Moreover, the above analysis also implies continuity of the solution operator with respect to initial data in X.

3. Global well-posedness. Finally, we verify via *a priori* energy estimates, that the procedure above can be extended globally in time. Suppose $\mathbf{v} = (v,w)^\top$ is a solution to (A.1). Hence,

$$\begin{aligned} v_t &= w, \\ w_t &= (1-c^2)v_{xx} + 2cw_x + F(\varphi) - F(\varphi + v). \end{aligned}$$

Calculate,

$$\frac{1}{2}\partial_t\int_0^{2\pi} v^2\,dx = \int_0^{2\pi} vv_t\,dx = \int_0^{2\pi} vw\,dx,$$

$$\frac{1}{2}\partial_t\int_0^{2\pi} w^2\,dx = \int_0^{2\pi} ww_t\,dx = \int_0^{2\pi} w\big((1-c^2)v_{xx} + 2cw_x + F(\varphi) - F(\varphi+v)\big)\,dx,$$

$$\frac{1}{2}(1-c^2)\partial_t\int_0^{2\pi} v_x^2\,dx = (1-c^2)\int_0^{2\pi} v_x w_x\,dx = -(1-c^2)\int_0^{2\pi} wv_{xx}\,dx,$$

because $v \in H^1_{\mathrm{per}}([0,2\pi])$. Set $H(t) := \frac{1}{2}(\|v\|^2 + (1-c^2)\|v_x\|^2 + \|w\|^2)$. Hence,

$$\frac{dH}{dt} = \int_0^{2\pi} vw\,dx + \int_0^{2\pi} F(\varphi) - F(\varphi+v)\,dx,$$

because $\int_0^{2\pi} ww_x\,dx = 0$ by periodicity. Therefore,

$$\frac{dH}{dt} \leqq (1+\bar{C})\int_0^{2\pi} |v||w|\,dx \leqq C(\|v\|^2 + \|w\|^2) \leqq CH(t),$$

for some uniform $C > 0$. Thus, by Gronwall's lemma we obtain

$$H(t) \leqq e^{Ct}H(0) \leqq C(T)H(0).$$

Hence, the solution can be extended globally in time by the same procedure. We conclude that there exists a unique global solution $\mathbf{v} \in C([0,+\infty);X)$ to the Cauchy problem (A.1).

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