Spectral stability of periodic wavetrains

Lecture 3. Evans function techniques

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1. The Evans function
2. The monodromy matrix for periodic problems
3. Application: non-linear Klein-Gordon wavetrains

Outline

- The **Evans function** is an analytic function of the spectral parameter designed to locate point spectra within its natural domain
- It has analytic extensions beyond its natural domain, for example, within some distance into the essential spectrum
- Plays a fundamental role in understanding bifurcations associated with point spectra that are ejected from the essential spectrum of the linearized operator under perturbation
- Conceptually lies at the interface of dynamical systems and functional analysis

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- The **zeroes of the Evans function coincide with the complex numbers in the point spectrum**
- The order of the zero is the same as the algebraic multiplicity of the eigenvalue
- The construction of the Evans function has a geometric motivation, associated with the analysis of the spectrum of the exponentially asymptotic linear operators
- The eigenvalue problem is recast as a **dynamical system**, and look at the eigenvalue problem not as an existence problem, but as an intersection problem for stable/unstable subspaces associated with the dynamical system

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- This perspective is highly versatile, it allows a simple generalization to unbounded domains, to higher-order systems, to eigenvalue pencils, and to multi-dimensional problems
- The Evans function affords insight that is not easily motivated by the classical formulation of the eigenvalue problem
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History

- The concept first appeared in the work of **J. W. Evans** on the stability of pulses in nerve axons in the mid 70s
- **C.K.R.T. Jones**, in the early 80s, realized the importance of the Evans function and applied it to the stability of the FitzHugh-Nagumo pulse
- **Alexander, Gardner and Jones** (1990) finally formalized the concepts and recast the Evans function in a dynamical systems language
- **Pego and Weinstein** (1994) showed that the Evans function plays a key role in determining the stability of the Korteweg - de Vries soliton

- **Gardner** (1994) set the foundations for applying Evans function to periodic problems
- Independently, **Gardner and Zumbrun** (1998) and **Kapitula and Sandstede** (1998) proved the **Gap Lemma**, a technical result that allows to extend the Evans function beyond its natural domain (and into the essential spectrum)
- **Zumbrun** (1998-2006) and collaborators pushed Evans function techniques into the stability analysis of **viscous shock profiles** via Green's function estimates
- Evans function techniques are currently applied to a great variety of structures beyond pulses and fronts: breathers, time and spatially periodic waves, discrete structures, etc.

1. The Evans function
2. The monodromy matrix for periodic problems
3. Application: non-linear Klein-Gordon wavetrains

Spectral system with periodic coefficients

Consider a generic spectral problem, arising form the linearization around a periodic traveling wave solution $f = f(z)$, with fundamental period T, $f(z+T) = f(z)$.

First order formulation:

$$\mathbf{w}_z = \mathbf{A}(z,\lambda)\mathbf{w}, \qquad \mathbf{w} \in H^1(\mathbb{R};\mathbb{C}^n),$$

Coefficients $\mathbf{A}(z,\lambda)$ are L^∞ and periodic in $z \in \mathbb{R}$, analytic in $\lambda \in \mathbb{C}$.

Family of closed, densely defined operators:

$$\mathcal{T}(\lambda) : \mathcal{D} \subset X \to X$$

$$\mathcal{T}(\lambda)\mathbf{w} := \mathbf{w}_z - \mathbf{A}(z,\lambda)\mathbf{w}.$$

$$\mathcal{D} = H^n(\mathbb{R};\mathbb{C}^n), \quad X = L^2(\mathbb{R};\mathbb{C}^n),$$

Spectral stability of periodic waves with respect to **localized perturbations**.

Monodromy matrix

Let $\mathbf{F}(z,\lambda)$ denote the fundamental solution matrix for the first order system,

$$\mathbf{F}_z(z,\lambda) = \mathbf{A}(z,\lambda)\mathbf{F}(z,\lambda),$$

with initial condition $\mathbf{F}(0,\lambda) = \mathbf{I}$, $\forall\lambda \in \mathbb{C}$. The T-periodicity in z of the coefficient matrix $\mathbf{A}$ then implies that

$$\mathbf{F}(z+T,\lambda) = \mathbf{F}(z,\lambda)\mathbf{M}(\lambda), \quad \forall z \in \mathbb{R}, \quad \text{where} \quad \mathbf{M}(\lambda) := \mathbf{F}(T,\lambda)$$

The matrix $\mathbf{M}(\lambda)$ is the **monodromy matrix**.

Important feature: $\mathbf{A}$ entire in λ, Picard iterates converge for $\mathbf{F}$ in z bounded $\Rightarrow$ $\mathbf{M}$ is an **entire** (analytic) function of $\lambda \in \mathbb{C}$.

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Floquet multipliers:

$\lambda \in \sigma$ if and only if there exists at least one $\mu \in \mathbb{C}$ (Floquet multiplier) with $|\mu| = 1$ such that

$$\hat{D}(\lambda, \mu) := \det(\mathbf{M}(\lambda) - \mu \mathbf{I}) = 0.$$

$\mu = \mu(\lambda) = e^{i\theta(\lambda)}$ are the eigenvalues of $\mathbf{M}(\lambda)$. $\theta = \theta(\lambda)$ are called the Floquet exponents.

Periodic Evans function

Definition (Gardner, 1997)

The **periodic Evans function** $D : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is defined as

$$D(\lambda, \theta) := \hat{D}(\lambda, e^{i\theta}),$$

that is, the restriction of the determinant $\hat{D}(\lambda, \mu)$ to the unit circle $S^1 \subset \mathbb{C}$ in the second argument, which is to be regarded as a unitary parameter. Thus, for each fixed $\theta \in \mathbb{R} \pmod{2\pi}$, is an entire function of $\lambda \in \mathbb{C}$ whose (isolated) zeros are particular points of the (continuous) spectrum σ.

Properties: (Gardner 1997, 1998)

- σ is the set of all $\lambda \in \mathbb{C}$ such that $D(\lambda, \theta) = 0$ for some real θ.
- D is analytic in λ and θ.
- The order of the zero in λ is the multiplicity of the eigenvalue.
- $\hat{D}(\lambda, 1) = D(\lambda, 0)$ detects spectra corresponding to perturbations which are T-periodic.

Floquet spectrum

Boundary value problem of the form

$$\mathbf{w}_z = \mathbf{A}(z,\lambda)\mathbf{w}, \qquad \mathbf{w}\text{bounded},$$

$$\mathbf{w}(T) = e^{i\theta}\mathbf{w}(0), \qquad \theta \in \mathbb{R}$$

For a given $\theta \in \mathbb{R}$ we define $\sigma_\theta \subset \mathbb{C}$ to be the set of complex λ for which there exists a nontrivial, **bounded** solution. The Floquet spectrum σ_F is defined then as the union over θ of these sets:

$$\sigma_F := \bigcup_{-\pi<\theta\leq\pi} \sigma_\theta.$$

Observations:

- We have shown that $\sigma = \sigma_F$ (**Lecture 1**)
- Each set σ_θ is discrete: zero set of the entire function $\det(\mathbf{M}(\lambda) - e^{i\theta} I)$
- The set σ_0 (with $\theta = 0$) is the part of the spectrum corresponding to perturbations which are co-periodic (**periodic partial spectrum**)
- θ - local coordinate; **curves of spectrum**: if $D_\lambda(\lambda_0, \theta_0) \neq 0$, $D_\theta(\lambda_0, \theta_0) \neq 0$ then σ is a smooth local curve
- At points where derivatives vanish: spectral analytic arcs (e.g. at $\lambda = 0$!)

Applications

Curves of spectrum

The real angle parameter θ is typically a local coordinate for the spectrum σ as a real subvariety of the complex λ-plane. This explains the intuition that the L^2-spectrum is purely "continuous", and gives rise to the notion of **curves of spectrum**:

Lemma (Kapitula and Promislow (2013))

Suppose that $\lambda_0 \in \sigma$ corresponding to a Floquet multiplier $\mu_0 \in S^1$, and suppose that $\hat{D}_\lambda(\lambda_0, \mu_0) \neq 0$ and $\hat{D}_\mu(\lambda_0, \mu_0) \neq 0$. Then there is a complex neighborhood Ω of λ_0 such that $\sigma \cap \Omega$ is a smooth curve through λ_0.

Proof: We work with the determinant $\hat{D}$ in terms of the Floquet multiplier μ. Since $\hat{D}_\lambda(\lambda_0, \mu_0) \neq 0$, it follows from the Analytic Implicit Function Theorem that the characteristic equation $\hat{D}(\lambda, \mu) = 0$ may be solved locally for λ as an analytic function $\lambda = l(\mu)$ of $\mu \in \mathbb{C}$ near $\mu = \mu_0 = e^{i\theta_0}$ with $l(\mu_0) = \lambda_0$. The spectrum near λ_0 is therefore the image of the map l restricted to the unit circle near μ_0, that is, $\lambda = l(e^{i\theta})$ for $\theta \in \mathbb{R}$ near θ_0. But then $D_\mu(\lambda_0, \mu_0) \neq 0$ implies that $dl(e^{i\theta})/d\theta \neq 0$ at $\theta = \theta_0$, which shows that the parametrization is regular, i.e., the image is a smooth curve (in fact, an analytic arc) passing through the point λ_0.

□

Formation of "islands"

The Evans function can help to prove spectral properties for the Bloch-wave decomposition. We first establish that the spectra of $\mathcal{L}$ consists of closed curves.

Theorem (Kapitula and Promislow (2013))

Let C be a simple closed curve oriented in the positive sense, which does not intersect σ. Then the winding number

$$W(\theta) = \frac{1}{2\pi i} \oint_C \frac{D_\lambda(\lambda,\theta)}{D(\lambda,\theta)}\, d\lambda,$$

is constant for $\theta \in (-\pi,\pi]$. Moreover, if $W(0) = 1$ then the spectra inside of C forms a smooth, closed curve

Proof sketch: Let $\mathcal{C} \subset \mathbb{C}$ be a positively oriented simple closed curve that does not intersect σ. Let $W(\theta)$ be the winding number and suppose that $W(\theta_0) = m$. That is, there are m eigenvalues (counting multiplicity) of $\mathcal{L}_{\theta_0}$ inside $\mathcal{C}$. Since the integrand is analytic in both λ and θ, we have that $W(\theta)$ is analytic in θ as long as the curve $\mathcal{C}$ does not intersect any spectra. As the winding number is integer-valued, it must be constant as long as a zero of the Evans function does not intersect the curve $\mathcal{C}$, which is precisely excluded by the assumption.

If in addition $W(0) = 1$, then $W(\theta) = 1$ for all $\theta \in (-\pi, \pi]$; whence, for each θ there is a unique $\lambda = \lambda(\theta)$ for which $D(\lambda(\theta), \theta) = 0$. Since the eigenvalue is simple it can be shown that

$$D_\lambda(\lambda(\theta), \theta) \neq 0.$$

The implicit function theorem implies that the curve $\lambda = \lambda(\theta)$ is of class C^∞. To show that the curve is closed, notice that $\lambda(-\pi) = \lambda(\pi)$, inasmuch as the eigenvalue problems are identical, with boundary condition $\mathbf{w}(T, \lambda) = e^{i\theta}\mathbf{w}(0, \lambda)$.

□

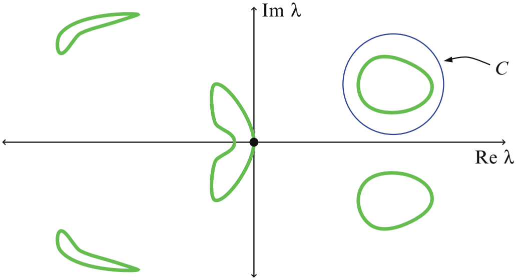

Figure : Typical spectra (green curves) for periodic problems. The curve C is the thin closed curve in blue. Image credit: Kapitula and Promislow (Springer, 2013).

Further comments: Numerics

Possible numerical procedure:

- First, compute all solutions to for $\theta = 0$ (periodic perturbations). This can be done by discretizing the operator $\mathcal{L}_0$ or $\mathcal{T}$ with periodic boundary conditions using finite differences or pseudo-spectral methods (Ascher et al., 1988), and to compute the spectrum of the resulting large matrix using eigenvalue-solvers (LAPACK)
- Second, once we have calculated all eigenvalues for $\theta = 0$, we can **utilize continuation codes** (e.g., AUTO97) to compute the solutions for $\theta \neq 0$ by using path-following of the solutions for $\theta = 0$ in θ (example, Sandstede and Scheel (2000))

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- This approach works well if for each solution (λ_0, θ_0), there is a curve of spectrum passing through it. "Islands" which are not connected to any eigenvale at $\theta = 0$ **cannot be reached by continuation**.

Rigorous justification of Whitham modulation system

- Whitham's theory has been used by physicists for more than 50 years
- Surprisingly, its relation to spectral stability of periodic waves has been elucitaded only recently
- The connection is made through the **Evans function**: allows to analyze $\lambda = 0$ as a point in the curve spectrum, and to define an **index** (related to the hyperbolicty of the Whitham system) which determines its behaviour inside the curve (arc splitting)
- The analysis rigorously justifies calculations by Whitham's methods and clarifies the connection to spectral stability

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Main references:

- Serre (2005); Oh, Zumbrun (2006) **(viscous conserv. laws)**
- Bronski, Johnson (2010); Johnson, Zumbrun (2010); Bronski, Johnson, Zumbrun (2010) **(gKdV)**
- Johnson (2010) **(BBM)**
- Noble, Rodrigues (2013) **(Kuramoto-Sivashinski)**
- Benzoni, Noble, Rodrigues (2014) **(Hamiltonian PDEs)**
- Jones *et. al* (2013, 2014) **(sine-Gordon and non-linear Klein-Gordon)**

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The non-linear Klein-Gordon equation

Non-linear Klein-Gordon with periodic potential:

$$u_{tt} - u_{xx} + V'(u) = 0. \qquad \text{(nKG)}$$

for $(x,t) \in \mathbb{R} \times [0,+\infty)$, u scalar, $V \in C^2$, periodic.

Sine-Gordon equation:

$$u_{tt} - u_{xx} + \sin u = 0, \qquad \text{(SG)}$$

$$V(u) = 1 - \cos u.$$

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Traveling waves

$u(x,t) = f(x-ct)$, $z = x - ct$, solution to the nonlinear pendulum equation:

$$(c^2-1)f_{zz} + V'(f(z)) = 0,$$

Sine-Gordon case:

$$(c^2-1)f_{zz} + \sin(f(z)) = 0,$$

$c \in \mathbb{R}$ (wave speed), $c^2 \neq 1$.

Upon integration:

$$\tfrac{1}{2}(c^2-1)f_z^2 = E - V(f),$$

E = constant (energy). Under assumptions:
$0 < E < E_0 = \max V(u)$

Sine-Gordon case: $V(u) = 1 - \cos u$, $E_0 = 2$,

$$\tfrac{1}{2}(c^2-1)f_z^2 = E - 1 + \cos f(z).$$

W.l.o.g.

(d) V has fundamental period $P = 2\pi$ and

$$\min_{u\in\mathbb{R}} V(u) = 0, \qquad \max_{u\in\mathbb{R}} V(u) = 2.$$

Classification

First dichotomy (wave speed):

- **Subluminal** waves: $c^2 < 1$
- **Superluminal** waves: $c^2 > 1$

Second dichotomy (energy E):

- **Librational** wavetrain: $f(z+T) = f(z)$. Closed trajectory inside the separatrix in the phase portrait.
- **Rotational** wavetrain: $f(z+T) = f(z) \pm 2\pi$. Open trajectory outside the separatrix in the phase plane. Sign f_z is fixed.

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Domain for parameters (E, c)

$$\mathbb{G}_{<}^{\text{lib}} = \{c^2 < 1,\ 0 < E < E_0\}, \text{ (subluminal librational)},$$
$$\mathbb{G}_{<}^{\text{rot}} = \{c^2 < 1,\ E < 0\}, \text{ (subluminal rotational)},$$
$$\mathbb{G}_{>}^{\text{lib}} = \{c^2 > 1,\ 0 < E < E_0\}, \text{ (superluminal librational)},$$
$$\mathbb{G}_{>}^{\text{rot}} = \{c^2 > 1,\ E > E_0\}, \text{ (superluminal rotational)},$$

$$(E,c) \in \mathbb{G} := \mathbb{G}_{<}^{\text{lib}} \cup \mathbb{G}_{<}^{\text{rot}} \cup \mathbb{G}_{>}^{\text{lib}} \cup \mathbb{G}_{>}^{\text{rot}}$$

For each fixed $z \in \mathbb{R}$, $f(z;E,c)$ is of class C^2 in $(E,c) \in \mathbb{G}$

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Spectral problem for non-linear Klen Gordon periodic wavetrains

Boundary value problem of the form

$$(c^2-1)w_{zz} - 2c\lambda w_z + (\lambda^2 + V''(f(z)))w = 0, \qquad \text{(P)}$$

$$\begin{pmatrix} w(T) \\ w_z(T) \end{pmatrix} = e^{i\theta} \begin{pmatrix} w(0) \\ w_z(0) \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

For a given $\theta \in \mathbb{R}$ we define $\sigma_\theta \subset \mathbb{C}$ to be the set of complex λ for which there exists a nontrivial solution. The Floquet spectrum σ_F is defined then as the union over θ of these sets:

$$\sigma_F := \bigcup_{-\pi<\theta\leq\pi} \sigma_\theta.$$

First order system

Problem (P) (quadratic pencil) can be written as a first order system:

$$\mathbf{w}_z = \mathbf{A}(z,\lambda)\mathbf{w},$$

$$\mathbf{w} := \begin{pmatrix} w \\ w_z \end{pmatrix},$$

$$\mathbf{A}(z,\lambda) := \begin{pmatrix} 0 & 1 \\ -\dfrac{(\lambda^2 + V''(f(z)))}{c^2-1} & \dfrac{2c\lambda}{c^2-1} \end{pmatrix}.$$

Monodromy matrix $\mathbf{M}(\lambda) := \mathbf{F}(T,\lambda)$, with $\mathbf{F}$ identity-normalized fundamental matrix.

Symmetries of the spectrum

The nonlinear Klein-Gordon equation is a real Hamiltonian system, and this forces certain elementary symmetries on the spectrum σ.

Lemma

The spectrum σ is symmetric with respect to reflection in the real and imaginary axes, i.e., if $\lambda \in \sigma$, then also $\lambda^* \in \sigma$ and $-\lambda \in \sigma$ (and hence also $-\lambda^* \in \sigma$).

Proof: Let $\lambda \in \sigma$. Then there exists $\theta \in \mathbb{R}$ for which $\lambda \in \sigma_\theta$, that is, there is a nonzero solution $w(z)$ of the first order boundary-value problem.

Since $V''(f(z))$ is a real-valued function, it follows by taking complex conjugates that $w(z)^*$ is a nonzero solution of the same boundary-value problem but with $e^{i\theta}$ replaced by $e^{-i\theta}$ and λ replaced by λ^*. It follows that $\lambda^* \in \sigma_{-\theta} \subset \sigma$. The fact that $-\lambda \in \sigma$ follows from the fact that the spectral problem is invariant under the transformation $(z, \lambda) \to -(z, \lambda)$, and it is easy to show that $\mathbf{M}(-\lambda) = \sigma_3^{-1}\mathbf{M}(\lambda)\sigma_3$, where $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the Jacobi matrix.

□

Solutions at $\lambda = 0$

$$f = f(z;E,c),\ (E,c) \in \mathbb{G}.$$

w solution to pencil (P), with initial conditions:

$$\begin{aligned} w(0;E,c) &= f(0;E,c) \\ &= \begin{cases} f(T;E,c), & E \in (0,E_0), \quad \text{(lib)}, \\ f(T;E,c) - \pi, & E \in (-\infty,0) \cup (E_0,+\infty), \quad \text{(rot)}, \end{cases} \end{aligned}$$

$$\partial_z w(0;E,c) = f_z(0;E,c) = f_z(T;E,c)$$

System at $\lambda = 0$:

$$\mathbf{w}_z = \mathbf{A}(z,0)\mathbf{w},$$

$$\mathbf{A}(z,0) = \begin{pmatrix} 0 & 1 \\ -V''(f(z))/(c^2-1) & 0 \end{pmatrix}.$$

Lemma

The two-dimensional complex vector space of solutions is spanned by

$$\mathbf{w}_0(z) = \begin{pmatrix} f_z \\ f_{zz} \end{pmatrix}, \quad \textit{and} \quad \mathbf{w}_1(z) = \begin{pmatrix} f_E \\ f_{Ez} \end{pmatrix}.$$

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The two-dimensional complex vector space of solutions is spanned by

$$\mathbf{w}_0(z) = \begin{pmatrix} f_z \\ f_{zz} \end{pmatrix}, \quad \text{and} \quad \mathbf{w}_1(z) = \begin{pmatrix} f_E \\ f_{Ez} \end{pmatrix}.$$

Proof: f is a C^2 function of E and z. Differentiating equation with respect to z yields $(c^2-1)f_{zzz}+V''(f)f_z=0$. This proves that $\mathbf{w}=\mathbf{w}_0(z)$ is a solution when $\lambda=0$. On the other hand, differentiating the equation with respect to E one gets $(c^2-1)f_{zzE}+V''(f)f_E=0$, proving that $\mathbf{w}=\mathbf{w}_1(z)$ is a solution for $\lambda=0$ as well. To verify independence of $\mathbf{w}_0$ and $\mathbf{w}_1$, differentiate equation for f with respect to E:

$$(c^2-1)f_z f_{zE} = 1 - V'(f)f_E.$$

Combining this with $(c^2-1)f_{zz}+V'(f)=0$, one obtains

$$\det(\mathbf{w}_0(z),\mathbf{w}_1(z)) = f_z f_{Ez} - f_E f_{zz} = \frac{1}{c^2-1} \neq 0.$$

Hence the Wronskian never vanishes and they are linearly independent for all z and for all $(E,c)\in\mathbb{G}$.

□

Solution matrix:

$$\mathbf{Q}(z,0) := (\mathbf{w}_0(z), \mathbf{w}_1(z))$$

$$\mathbf{F}(z,0) = \mathbf{Q}(z,0)\mathbf{Q}(0,0)^{-1}.$$

$$\mathbf{M}(0) = \mathbf{F}(T,0) = \mathbf{Q}(T,0)\mathbf{Q}(0,0)^{-1}$$

$$\mathbf{Q}(z,0)^{-1} = (c^2-1)\begin{pmatrix} f_{Ez} & -f_E \\ -f_{zz} & f_z \end{pmatrix}.$$

Lemma

If $T_E \neq 0$, there exists a basis in $\mathbb{R}^2$ such that the monodromy map $\mathbf{M}(\lambda)$ at $\lambda = 0$ has the Jordan form

$$\mathbf{M}(0) \sim \begin{pmatrix} 1 & -T_E \\ 0 & 1 \end{pmatrix}.$$

Proof: The matrix $\mathbf{Q}_0(0)$ may be expressed in terms of the initial functions u_0 and v_0 defined on $\mathbb{G}$

$$\mathbf{Q}_0(0) = \begin{pmatrix} v_0 & \partial_E u_0 \\ 0 & \partial_E v_0 \end{pmatrix} = \begin{pmatrix} v_0 & 0 \\ 0 & \partial_E v_0 \end{pmatrix},$$

where ∂_E denotes the partial derivative with respect to E, because u_0 is piecewise constant on $\mathbb{G}$.

Similarly, we have

$$\mathbf{Q}_0(0)^{-1} = \begin{pmatrix} (c^2-1)\partial_E v_0 & 0 \\ 0 & (c^2-1)v_0 \end{pmatrix}$$

The identity $(c^2-1)v_0\partial_E v_0 = 1$ can be obtained by differentiation of the profile equation for f with respect to E at $z=0$. Hence, the fundamental solution matrix $\mathbf{F}(z,0)$ is

$$\mathbf{F}(z,0) = \frac{1}{v_0}\begin{pmatrix} f_z(z) & (c^2-1)v_0^2 f_E(z) \\ f_{zz}(z) & (c^2-1)v_0^2 f_{Ez}(z) \end{pmatrix}.$$

The corresponding monodromy matrix $\mathbf{M}(0)$ is obtained by setting $z = T$ in $\mathbf{F}(z,0)$. To simplify the resulting formula, express $f_z(T), f_{zz}(T), f_E(T)$ and $f_{Ez}(T)$ in terms of the functions u_0 and v_0. Since f_z and f_{zz} are periodic functions with period T, we have $f_z(T) = f_z(0) = v_0$ and $f_{zz}(T) = f_{zz}(0) = 0$. To express $f_E(T)$ and $f_{Ez}(T)$ in terms of u_0 and v_0, first note that since $f(T) = f(0) \pmod{2\pi}$, we may write u_0 as $u_0 = f(T)$; differentiation with respect to E and taking into account that the period T depends on E yields

$$\begin{aligned} \partial_E u_0 &= f_E(T) + T_E f_z(T) \\ &= f_E(T) + T_E f_z(0) \quad \text{(because } f_z \text{ has period } T) \\ &= f_E(T) + T_E v_0. \end{aligned}$$

Therefore, since u_0 is piecewise constant on $\mathbb{G}$, $f_E(T) = -T_E v_0$. Similarly, since f_z has period T, we can write $v_0 = f_z(T)$, and then differentiation yields $\partial_E v_0 = T_E f_{zz}(T) + f_{Ez}(T)$, and therefore as $f_{zz}(T) = 0$, $f_{Ez}(T) = \partial_E v_0$. This yields the form of $\mathbf{M}(0)$.

□

Observation: $\mathbf{Q}(T,0) - \mathbf{Q}(0,0)$ is a rank-one matrix provided that $T_E \neq 0$:

$$\mathbf{Q}(T,0) = \mathbf{Q}(0,0) + \begin{pmatrix} 0 & -T_E v_0(E,c) \\ 0 & -T_E \frac{V'(u_0(E,c))}{c^2-1} \end{pmatrix}$$

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Under assumptions on V, we have monotonicity of the period map (Chicone, 1987: criterion for planar Hamiltonian systems):

Lemma

Under assumptions there holds $T_E \neq 0$. More precisely we have:

(i) $T_E > 0$ *in the rotational subluminal and librational superluminal cases.*

(ii) $T_E < 0$ *in the rotational superluminal and librational subluminal cases.*

Proof: See Jones *et. al* (2014).

Lemma

If we define

$$\bar{\Delta} := -\frac{T_E}{c^2 - 1}$$

then

(a) $\bar{\Delta} > 0$ *for rotational waves.*

(b) $\bar{\Delta} < 0$ *for librational waves.*

Solutions series expansions

The Picard iterates for the fundamental solution matrix $\mathbf{F}(z,\lambda)$ converge uniformly on $(z,\lambda) \in [0,T] \times K$, $K \subset \mathbb{C}$ an arbitrary compact set. The coefficient matrix $\mathbf{A}(z,\lambda)$ is entire in λ for each z, thus $\mathbf{F}(z,\lambda)$ is an entire analytic function of $\lambda \in \mathbb{C}$ for every $z \in [0,T]$. Hence the fundamental solution matrix $\mathbf{F}(z,\lambda)$ has a convergent Taylor expansion about every point of the complex λ-plane. In particular, the series about the origin has the form

$$\mathbf{F}(z,\lambda) = \sum_{n=0}^{\infty} \lambda^n \mathbf{F}_n(z), \quad z \in [0,T]$$

for some coefficient matrices $\{\mathbf{F}_n(z)\}_{n=0}^{\infty}$, and this series has an infinite radius of convergence. Setting $\lambda = 0$ gives $\mathbf{F}_0(z) = \mathbf{F}(z,0)$, which has already been computed.

For computational purposes we seek expansions for $\mathbf{Q} = \mathbf{Q}(z,\lambda)$, solution to

$$\frac{d\mathbf{Q}}{dz} = \mathbf{A}(z,\lambda)\mathbf{Q}.$$

$$\mathbf{Q}(0,\lambda) = \mathbf{Q}(0,0) = \left(\mathbf{w}_0(0),\ \mathbf{w}_1(0)\right), \qquad \text{for all } \lambda \in \mathbb{C},$$

Therefore, $\mathbf{F}(z,\lambda) = \mathbf{Q}(z,\lambda)\mathbf{Q}(0,0)^{-1}$. By analyticity, one seeks series expansions of the form:

$$\mathbf{Q}(z,\lambda) = \sum_{n=0}^{+\infty} \lambda^n \mathbf{Q}_n(z)$$

Collecting like powers of λ we obtain a hierarchy of equations:

$$(c^2-1)\frac{d\mathbf{Q}_1}{dz} = \mathbf{A}_0(z)\mathbf{Q}_1 + \mathbf{A}_1\mathbf{Q}_0$$

$$(c^2-1)\frac{d\mathbf{Q}_n}{dz} = \mathbf{A}_0(z)\mathbf{Q}_n + \mathbf{A}_1\mathbf{Q}_{n-1} + \mathbf{A}_2\mathbf{Q}_{n-2}, \;\; n = 2, 3, \ldots$$

Solution by variation of parameters:

$$\mathbf{Q}_1(z) = \frac{\mathbf{Q}_0(z)}{c^2-1}\int_0^z \mathbf{Q}_0(y)^{-1}\mathbf{A}_1\mathbf{Q}_0(y)\,dy$$

$$\mathbf{Q}_n(z) = \frac{\mathbf{Q}_0(z)}{c^2-1}\int_0^z \mathbf{Q}_0(y)^{-1}\big(\mathbf{A}_1\mathbf{Q}_{n-1}(y) + \mathbf{A}_2\mathbf{Q}_{n-2}\big)\,dy, \;\; n \geq 2$$

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By Abel's identity:

Lemma

For all $z \in \mathbb{R}$, $\lambda \in \mathbb{C}$, *there holds*

$$\det \mathbf{Q}(z,\lambda) = \frac{\exp\left(2c\lambda z/(c^2-1)\right)}{c^2-1}.$$

After (tedious) computations (Jones *et al.* (2014)):

Lemma

$$\operatorname{tr} \mathbf{Q}_0(T)\mathbf{Q}_0(0)^{-1} = 2.$$

$$\operatorname{tr} \mathbf{Q}_1(T)\mathbf{Q}_0(0)^{-1} = \frac{2cT}{c^2-1}.$$

$$\operatorname{tr} \mathbf{Q}_2(T)\mathbf{Q}_0(0)^{-1} = \frac{c^2T^2}{(c^2-1)^2} - \frac{T_E}{c^2-1}\int_0^T f_z(y)^2\,dy.$$

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Perturbation of the Jordan block

By analyticity of the monodromy map:

$$\mathbf{M}(\lambda) = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} \frac{d^n \mathbf{M}}{d\lambda^n}(0).$$

(Standard perturbation theory, Kato.) In general, the Floquet multipliers bifurcate from $\lambda = 0$ in Pusieux series.

Fundamental matrix:

$$\mathbf{F}(z,\lambda) = \mathbf{Q}(z,\lambda)\mathbf{Q}_0(0)^{-1} = \sum_{n=0}^{+\infty} \lambda^n \mathbf{Q}_n(z)\mathbf{Q}_0^{-1} =: \sum_{n=0}^{+\infty} \lambda^n \mathbf{F}_n(z)$$

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In view that:

$$\frac{d^n \mathbf{M}}{d\lambda^n}(0) = n!\mathbf{Q}_n(T)\mathbf{Q}_0(0)^{-1},$$

Lemma

We have convergent series expansions

$$\mathbf{M}(\lambda) = \sum_{n=0}^{+\infty} \lambda^n \mathbf{Q}_n(T)\mathbf{Q}_0(0)^{-1},$$

$$\operatorname{tr}\mathbf{M}(\lambda) = \sum_{n=0}^{+\infty} \lambda^n \operatorname{tr}\mathbf{Q}_n(T)\mathbf{Q}_0(0)^{-1},$$

and $$\det\mathbf{M}(\lambda) = \sum_{n=0}^{+\infty} \left(\frac{2cT}{c^2-1}\right)^n \frac{\lambda^n}{n!},$$

First application: the parity index

$\lambda = 0$ belongs to the spectrum σ. (In fact, $\lambda = 0$ belongs to the periodic partial spectrum σ_0, as both Floquet multipliers coincide at $\mu = 1$, with the formation of a Jordan block in the monodromy matrix $\mathbf{M}(0)$ in the generic case $T_E \neq 0$). At a physical level, this is related to the **translation invariance of the periodic traveling wave**.

Recall that the periodic eigenvalues (the points of the periodic partial spectrum σ_0) are the roots of the (entire) periodic Evans function $D(\lambda, 0)$ with $\theta = 0$. Expanding out the determinant and setting $\theta = 0$ gives the formula

$$D(\lambda, 0) = 1 - \operatorname{tr}(\mathbf{M}(\lambda)) + \det(\mathbf{M}(\lambda)).$$

To define the parity index we will consider the restriction of this formula to $\lambda \in \mathbb{R}$.

Lemma

The restriction of the periodic Evans function $D(\lambda,0)$ to $\lambda \in \mathbb{R}$ is a real-analytic function. Moreover, for $\lambda \in \mathbb{R}_+$ with $\lambda \gg 1$ sufficiently large, we have

$$\operatorname{sgn}(D(\lambda,0)) = \operatorname{sgn}(c^2 - 1)$$

Proof: The system has real coefficients whenever $\lambda \in \mathbb{R}$. Therefore the fundamental solution matrix $\mathbf{F}(z,\lambda)$ is real for real λ and $z \in [0,T]$. By evaluation at $z = T$ the same is true for the elements of the monodromy matrix $\mathbf{M}(\lambda)$, and this proves the real-analyticity. When λ is large in magnitude, then $\lambda^2 + V''(f(z)) \approx \lambda^2$, and hence the first-order system can be approximated by a constant-coefficient one:

$$\mathbf{w}_z = \mathbf{A}^\infty(\lambda)\mathbf{w}, \quad \mathbf{A}^\infty := \begin{pmatrix} 0 & 1 \\ -\dfrac{\lambda^2}{c^2-1} & \dfrac{2c\lambda}{c^2-1} \end{pmatrix}.$$

The fundamental solution matrix of this approximating system is the matrix exponential $\mathbf{F}^\infty(z,\lambda) = e^{z\mathbf{A}^\infty(\lambda)}$, and the corresponding monodromy matrix is $\mathbf{M}^\infty(\lambda) = e^{T\mathbf{A}^\infty(\lambda)}$. The eigenvalues of $\mathbf{A}^\infty(\lambda)$ are $\lambda/(c \pm 1)$, and hence those of $\mathbf{M}^\infty(\lambda)$ are $e^{\lambda T/(c\pm 1)}$. The periodic (with $\theta = 0$) Evans function associated with the approximating system is therefore

$$\begin{aligned} D^\infty(\lambda,0) &= 1 - \operatorname{tr}(\mathbf{M}^\infty(\lambda)) + \det(\mathbf{M}^\infty(\lambda)) \\ &= (e^{\lambda T/(c+1)} - 1)(e^{\lambda T/(c-1)} - 1). \end{aligned}$$

This real-valued function of $\lambda \in \mathbb{R}$ clearly has the same sign as does $c^2 - 1$ for large positive λ. The coefficient matrix $\mathbf{A}^\infty(\lambda)$ is an accurate approximation of that of the original system uniformly for $z \in [0, T]$, so the respective Evans functions $D^\infty(\lambda, 0)$ and $D(\lambda, 0)$ are close to each other in the limit $\lambda \to \infty$. This shows that $D(\lambda, 0)$ has the same sign as does $c^2 - 1$ for $\lambda \gg 1$.

□

Observation: This behaviour of the Evans function on the real line seems to be generic, a powerful analytical and numerical tool to detect **real** unstable eigenvalues in many situations (pulses, fronts, periodic waves, etc.)

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$\square$

Observation: This behaviour of the Evans function on the real line seems to be generic, a powerful analytical and numerical tool to detect **real** unstable eigenvalues in many situations (pulses, fronts, periodic waves, etc.)

Typical behaviour for equations with Hamiltonian structure: the first derivative of the Evans function vanishes at $\lambda = 0$.

Lemma

The periodic Evans function $D(\cdot, 0) : \mathbb{R} \to \mathbb{R}$ *for the non-linear Klein-Gordon equation waves vanishes to even order at* $\lambda = 0$ *and satisfies,*

$$D(0,0) = D_\lambda(0,0) = 0, \quad D_{\lambda\lambda}(0,0) = 2(q^2 - \kappa),$$

where $q = cT/(c^2 - 1)$ *and where*

$$\kappa = \frac{M_{12}(0)}{(c^2-1)^2} \int_0^T F_{11}(y,0)^2 \, dy.$$

Proof: Follows by direct expansion of the Floquet multipliers (in a moment).

The parity index

Definition (parity index γ_P)

Suppose $D(\cdot, 0)$ vanishes to (even) order $2n \geq 2$ at $\lambda = 0$. The *parity (or orientation) index* is given by

$$\gamma_P := \text{sgn}\left((c^2 - 1)\partial_\lambda^{2n} D(0,1) \right).$$

This yields the following instability criterion with respect to (real) periodic eigenvalues:

Theorem

If $\gamma_P = 1$ (resp., $\gamma_P = -1$) then the number of positive real points in the periodic partial spectrum $\sigma_0 \subset \sigma$, i.e., periodic eigenvalues, is even (resp., odd) when counted according to multiplicity. In particular, if $\gamma_P = -1$ there is at least one positive real periodic eigenvalue and hence the underlying periodic wave f solving the Klein-Gordon equation is spectrally unstable, with the corresponding exponentially growing solution of the linearized equation having the same spatial period T as f_z.

Proof: If $\gamma_P = 1$, then $D(\lambda, 0)$ has the same sign for sufficiently small and sufficiently large strictly positive λ, while if $\gamma_P = -1$ the signs are opposite for small and large λ. Since $D(\lambda, 0)$ is real-analytic for real λ it clearly has an even number of positive roots for $\gamma_P = 1$ and an odd number of positive roots for $\gamma_P = -1$, with the roots weighted by their multiplicities. These roots correspond to points in the spectrum σ, and since $\theta = 0$, they are periodic eigenvalues.

□

Observation: Notice that the case $\gamma_P = 1$ is **inconclisive** for spectral stability: it only guarantees that the number of real periodic eigenvalues is even (possibly zero). Hence, its name (parity index).

Applying the result to the non-linear Klein-Gordon case we immediately have:

Theorem

Under the assumptions on the potential V. Then, subluminal librational periodic traveling wave solutions of the Klein-Gordon equation for which $T_E < 0$ are spectrally unstable, having a positive real periodic eigenvalue $\lambda \in \sigma_0 \subset \sigma$.

Proof: Follows directly from tha fact that $c^2 - 1 < 0$, $(c^2 - 1)T_E > 0$ in the subluminal librational case, and thus $M_{12}(0) = -T_E(c^1 - 1)v_0^2 < 0$ and $\kappa < 0$, yielding $D_{\lambda\lambda}(0,0) > 0$. Hence, $D(\cdot, 0)$ vanishes precisely at second order and $\gamma_P = -1$.

□

Corollary

Subluminal librational periodic traveling wave solutions of the sine-Gordon equation ($V(u) = -\cos(u)$) are spectrally unstable, having a positive real ***periodic*** *eigenvalue* $\lambda \in \sigma_0 \subset \sigma$.

Expansion of the Floquet multipliers

Recall that, given $\lambda \in \mathbb{C}$, the Floquet multipliers $\mu = \mu(\lambda)$ are defined as the roots of the characteristic equation $\hat{D}(\lambda,\mu) = 0$, i.e., they are the eigenvalues of the monodromy matrix $\mathbf{M}(\lambda)$. The quadratic formula gives the multipliers in the form

$$\hat{D}(\lambda,\mu) = \det(\mathbf{M}(\lambda) - \mu\mathbf{I}) = \mu^2 - (\operatorname{tr}\mathbf{M}(\lambda))\mu + \det\mathbf{M}(\lambda) = 0$$

$$\mu_\pm(\lambda) = \frac{1}{2}\Big(\operatorname{tr}\mathbf{M}(\lambda) \pm \big((\operatorname{tr}\mathbf{M}(\lambda))^2 - 4\det\mathbf{M}(\lambda)\big)^{1/2}\Big)$$

Substituting the powers expansion series for $\operatorname{tr}\mathbf{M}$ and $\det\mathbf{M}$ we obtain:

$$\operatorname{tr}\mathbf{M}(\lambda)^2 - 4\det\mathbf{M}(\lambda) =$$

$$\Big(\operatorname{tr}\mathbf{Q}_0(T)\mathbf{Q}_0(0)^{-1} + \lambda\operatorname{tr}\mathbf{Q}_1(T)\mathbf{Q}_0(0)^{-1} + \lambda^2\operatorname{tr}\mathbf{Q}_2(T)\mathbf{Q}_0(0)^{-1}\Big)^2 +$$

$$-4\Big(1 + \frac{2cT}{c^2-1}\lambda + \frac{2c^2T^2}{(c^2-1)^2}\lambda^2\Big) + O(\lambda^3)$$

$$= 4\Delta\lambda^2 + O(\lambda^3),$$

where,

$$\Delta := -\frac{T_E}{c^2-1}\int_0^T f_z(y)^2\,dy$$

Expansion of the Floquet multipliers:

The two Floquet multipliers are analytic functions of λ at $\lambda = 0$. Asymptotic form:

$$\mu_{\pm}(\lambda) = 1 + \Big(\frac{cT}{c^2-1} \pm \Delta^{1/2}\Big)\lambda + O(\lambda^2)$$

The modulational instability index

Definition

We define the **modulational instability index** to be the quantity

$$\gamma_M := \operatorname{sgn} \Delta = \operatorname{sgn}\left(-(c^2 - 1)T_E\right),$$

with the understanding that $\gamma_M = 0$ if $T_E = 0$. In particular, $\gamma_M = 1$ for rotational waves of any speed.

Observe that it is the index that determines the hyperbolicity of the Whitham system (Lecture 2).

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Observe that it is the index that determines the hyperbolicity of the Whitham system (Lecture 2).

Observation: In particular, when $\lambda \in \mathbb{R}$, the expansions yield

$$\operatorname{tr}(\mathbf{M}(\lambda)) = 2 + 2q\lambda + \left(q^2 + \kappa\right)\lambda^2 + O(\lambda^3), \quad \lambda \to 0,$$

$$\det(\mathbf{M}(\lambda)) = 1 + 2q\lambda + 2q^2\lambda^2 + O(\lambda^3), \quad \lambda \to 0,$$

and consequently

$$D(\lambda, 0) = \left(q^2 - \kappa\right)\lambda^2 + O(\lambda^3) \quad \lambda \to 0,$$

confirming the vanishing at second order (at least) of the asymptotic expansion of D on the real line.

Expansion of D near the origin

Lemma

The periodic Evans function $D(\lambda,\theta)$, for $(\lambda,\theta) \in \mathbb{C}\times\mathbb{R}$, has the following expansion in a neighborhood of $(\lambda,\theta) = (0,0)$,

$$D(\lambda,\theta) = -\kappa\lambda^2 + (i\theta - q\lambda)^2 + O(3),$$

where $O(3)$ denotes terms of order three or higher in (λ,θ), with $q = ct/(c^2-1)$ and

$$\kappa = \frac{M_{12}(0)}{(c^2-1)^2}\int_0^T F_{11}(y,0)^2\,dy.$$

Proof: Follows immediately from the formula (with $\mu = e^{i\theta}$),

$$D(\lambda, \theta) = \hat{D}(\lambda, e^{i\theta}) = e^{2i\theta} - \operatorname{tr}(\mathbf{M}(\lambda))e^{i\theta} + \det(\mathbf{M}(\lambda)),$$

upon expanding the exponentials in power series about $\theta = 0$.

□

Observation: We may use this expansion to analyse how solutions to the spectral curve bifurcate from $(0,0)$. When $c = 0$ the spectrum is well-understood (it's the Hill's spectrum, Scott's trick). Hence, without loss of generality we assume that $c \neq 0$, and since $q = 0$ iff $c = 0$, that $q \neq 0$.

Lemma

If $\gamma_M = -1$ then the equation $D(\lambda, \theta) = 0$ parametrically describes two distinct smooth curves that cross at the origin with tangent lines making acute non-zero angles with the imaginary axis:

$$\lambda^{\pm}(\theta) = -(\alpha^{\pm} + i\beta^{\pm})\theta + O(\theta^2),$$

with $\alpha^{\pm}, \beta^{\pm} \in \mathbb{R}$, $\alpha^{\pm} \neq 0$, for $|\theta| \sim 0$.
If $\gamma_M = 1$, but $\kappa \neq q^2$, then solutions to $D(\lambda, \theta) = 0$ emerge from the origin as two curves tangential to the imaginary axis in the λ-plane:

$$\lambda^{\pm}(\theta) = -i\nu^{\pm}\theta + O(\theta^2),$$

with $\nu^{\pm} \in \mathbb{R}$, $\nu^{\pm} \neq 0$, for $|\theta| \sim 0$.

Proof sketch: Follows from a direct application of the Implicit Function Theorem on the derivative if D. For details see **Jones *et. al*** (2014).

□

Observation: The case when $\kappa = q^2$ is more complicated. It is related to how many derivatives vanish at order $2n$ (recall $\gamma_M = 1$, even). In that case the index is associated to a **strong modulational instability**.

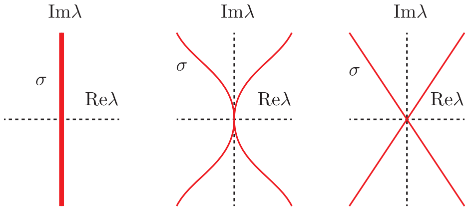

Figure : A qualitative sketch of the three generic possibilities for the spectrum σ in a neighborhood of the origin. $\gamma_M = 1$ with $\kappa \neq q^2$, in the cases $T_E \neq 0$ (left), and $T_E = 0$ (degenerate, center). The case $\gamma_M = -1$ is depicted at the right panel.

Modulational instability

Definition

A periodic traveling wave solution f of the Klein-Gordon equation is said to be **modulationally unstable** (or, to have a **modulational instability**) if for every neighborhood U of the origin $\lambda = 0$, $(\sigma \setminus i\mathbb{R}) \cap U \neq \emptyset$. Otherwise, f is said to be **modulationally stable**. For an angle $\theta \in (0, \pi/2)$, let S_θ denote the union of the open sectors given by the inequalities $|\arg(\lambda)| < \theta$ or $|\arg(-\lambda)| < \theta$ (note $0 \notin S_\theta$). A modulational instability is called **weak** if for every $\theta \in (0, \pi/2)$ and for every neighborhood U of the origin, $\sigma \cap U \cap S_\theta = \emptyset$. A modulational instability that is not weak is called **strong**.

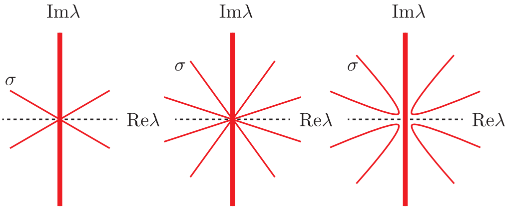

Figure : The case $\kappa = q^2$: Spectrum near the origin when $D(\cdot, 0) = 0$ to order $2n$ with $n = 2$ (left panel) and $n = 3$ (center panel). The right panel illustrates the spectrum near the origin for $\partial_\lambda^2 D(0,0)$ and $\partial_\lambda^4 D(0,0)$ both small but nonzero.

Theorem

Let V be a potential satisfying Assumptions (a), (b) and (c). A librational periodic traveling wave solution of the nonlinear Klein-Gordon equation for which $(c^2 - 1)T_E > 0$ holds (equivalently $\gamma_M = -1$) is ***strongly modulationally unstable****.*

Corollary

All librational waves satisfying $(c^2 - 1)T_E > 0$ are ***spectrally unstable****.*

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All librational traveling wave solutions of the sine-Gordon equation, with $V(u) = 1 - \cos(u)$, are ***strongly modulationally unstable*** *and hence* ***spectrally unstable****.*

We also recover Whitham modulational stability results:

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Theorem (Rigorous proof of Whitham's instability results)

*Under the non-degenerate condition $T_E \neq 0$, if the periodic traveling wave is **modulationaly unstable** in the sense defined by Whitham then it is **spectrally unstable**.*

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Theorem

Under Assumptions (a), (b) and (c) for the potential there hold:

- *Both super- and subluminal rotational waves are* ***modulationally stable****, and*
- *both super- and subluminal librational waves are* ***modulationally unstable*** *(and, therefore* ***spectrally unstable****).*

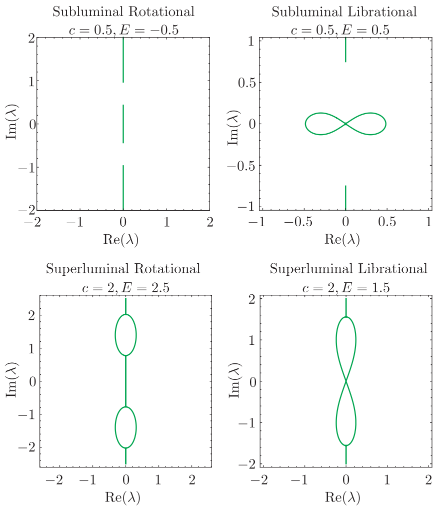

Figure : Numerical plots of the Floquet spectrum $G(\lambda) = 0$ for sine-Gordon.

(In)stability in the rotational case

Recall: Scott (1969), introduced the transformation

$$y = w \exp\left(\frac{-c\lambda z}{c^2-1}\right),$$

and obtained the related Hill's operator:

$$y_{zz} + \frac{V''(f(z))}{c^2-1} y = \left(\frac{\lambda}{c^2-1}\right)^2 y =: \nu y. \qquad \text{(H)}$$

Hill's equation with period T. $\nu \in \sigma_H$ (Floquet spectrum of (H)) if there is a bounded solution y.

Further (non-Evans function related) results

Theorem

Under assumptions on V we have:

(A) *Superluminal rotational waves are* ***spectrally unstable****.*

(B) *Subluminal rotational waves are* ***spectrally stable****. That is: if $\lambda \in \sigma$ then λ is purely imaginary.*

Proof sketch: Part (A)

Define $G : \mathbb{C} \to \mathbb{R}$ by

$$G(\lambda) = \log|\mu_+(\lambda)| \log|\mu_-(\lambda)|.$$

G continuous in $\mathbb{R}^2$ and $\lambda \in \sigma$ if and only if $G(\lambda) = 0$. Fact: if $\mu(\lambda) \in \sigma\mathbf{M}(\lambda)$ (Floquet mult. for (P) then $\eta(\lambda) = \exp(-\lambda c T/(c^2-1)) \in \sigma\mathbf{M}_H(\lambda)$ (Floquet mult. for (H)). By Abel's identity:

$$\begin{aligned} G(\lambda) &= \left(\text{Re}\,\frac{c\lambda T}{c^2-1}\right)^2 - (\log|\eta_+(\lambda)|)^2 \\ &= \left(\text{Re}\,\frac{c\lambda T}{c^2-1}\right)^2 - (\log|\eta_-(\lambda)|)^2. \end{aligned}$$

Thus, for $\lambda \in i\mathbb{R}$, $G \leq 0$. Moreover, $G(i\beta) = 0$ iff $i\beta \in \sigma \cap i\mathbb{R} = \sigma^H \cap i\mathbb{R}$. Thus,

Corollary

Suppose $\beta \in \mathbb{R}$ is such that $\left(\frac{i\beta}{c^2-1}\right)^2 \notin \sigma^H$. Then $G(i\beta) < 0$.

Moreover, we can show:

Lemma

For a superluminal rotational wave, $G(\lambda) > 0$ for $\lambda \in \mathbb{R}$, $\lambda \gg 1$, and there is a $i\beta_*$ in the spectral gap of σ_H, that is, $G(i\beta) < 0$.

By continuity, there must be an eigenvalue $\lambda = \alpha_* t + i\beta_*(1-t)$ for some $t \in (0,1)$, where $G(\alpha_*) > 0$, α_* large and real, such that $G(\lambda) = 0$. Clearly, Re $\lambda > 0$.

This shows (A).

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This shows (A).

Part (B): Spectral stability of subluminal rotations.

By energy estimates: define the Hamiltonian operator $H = d^2/dz^2 + V''(f)/(c^2-1)$ so that the spectral equation (P) is:

$$(c^2-1)Hw(z) - 2c\lambda w_z(z) + \lambda^2 w(z) = 0$$

Lemma

The operator H is negative semidefinite in the case of a rotational wave. For librations, H is indefinite.

If $\lambda \in \sigma$, multiply eq. by w^* and integrate by parts on a fundamental period $[0,T]$:

$$(c^2-1)\langle w, Hw\rangle - 2im\lambda + \|w\|^2\lambda^2 = 0,$$

$$m := -ic\int_0^T w(z)^* w_z(z)\,dz \in \mathbb{R}$$

$m \in \mathbb{R}$ using the periodicity of w and integrating by parts. The roots of the quadratic are:

$$\lambda = \frac{1}{\|w\|^2}\left[im \pm \sqrt{-m^2-(c^2-1)\|w\|^2\langle w, Hw\rangle}\right].$$

$\lambda \in i\mathbb{R}$ whenever $c^2 < 1$.

This shows (B).

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□

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Thanks!