Spectral stability of periodic wavetrains

Lecture 2. Introduction to modulation theory

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1. History

2. Modulation theory for non-linear Klein-Gordon periodic wavetrains

3. Modulation theory for KdV cnoidal periodic waves

Outline

Whitham (1965 through 1974):
Modulation theory: well established (formal) physical method based on WKB expansions. Relating to its stability, for an exact periodic wave, $f = f(x-ct) = \tilde{f}(kx-\omega t)$, allowing dependence of physical parameters $k = k(x,t), \omega = \omega(x,t)$, under "slow modulations", if the PDE system on (k,ω) is well-posed then the wave is "stable".

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References

- Whitham, Proc. Roy. Soc. Ser. A (1965)
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- Kamchatnov, *Nonlinear periodic waves and their Modulations* (2000)

Main features

- Provides an asymptotic method for studying **slowly varying** periodic waves
- It is essentially a nonlinear WKB theory
- Can be applied to any nonlinear wave equation which has a (known) periodic traveling wave solution (KdV, sine-Gordon, NLS equations, Boussinesq equations, etc.)
- Derivation of equations which describe slow evolution of governing parameters of the waves (amplitude, wavelength, frequency, etc.), called **Whitham's modulation system**.

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- The Whitham system if of first-order: well-posedness is related to its hyperbolicity/ellipticity
- Rich analytical tool: for a broad class of integrable nonlinear wave equations, a simple universal method has been developed by Kamchatnov (2000), enabling the construction of periodic solutions and the Whitham modulation equations directly in terms of Riemann invariants
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The non-linear Klein-Gordon equation

Non-linear Klein-Gordon with periodic potential:

$$u_{tt} - u_{xx} + V'(u) = 0. \qquad \text{(nKG)}$$

for $(x,t) \in \mathbb{R} \times [0,+\infty)$, u scalar, $V \in C^2$, periodic.

Sine-Gordon equation:

$$u_{tt} - u_{xx} + \sin u = 0, \qquad \text{(SG)}$$

$$V(u) = 1 - \cos u.$$

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Applications (sine-Gordon):

- Surfaces with negative Gaussian curvature (Eisenhart, 1909)
- Propagation of crystal dislocations (Frenkel and Kontorova, 1939)
- Elementary particles (Perring and Skyrme, 1962)
- Propagation of magnetic flux on a Josephson line (Scott, 1969)
- Dynamics of fermions in the Thirring model (Coleman, 1975)
- Oscillations of a rigid pendulum attached to a stretched rubber band (Drazin, 1983)

Assumptions on the potential:

(a) $V : \mathbb{R} \to \mathbb{R}$ is of class C^2 in all its domain and it is periodic with fundamental period P.

(b) V has only non-degenerate critical points.

(c) $V'(u)^4 (V(u)/V'(u)^2)'' \geq 0$ for all u under consideration.

Assumption (c) implies monotonicity of the period map with respect to the energy.

Traveling waves

$u(x,t) = f(x-ct)$, $z = x - ct$, solution to the nonlinear pendulum equation:

$$(c^2 - 1) f_{zz} + V'(f(z)) = 0,$$

Sine-Gordon case:

$$(c^2 - 1) f_{zz} + \sin(f(z)) = 0,$$

$c \in \mathbb{R}$ (wave speed), $c^2 \neq 1$.

Upon integration:

$$\tfrac{1}{2}(c^2-1)f_z^2 = E - V(f),$$

E = constant (energy). Under assumptions:
$0 < E < E_0 = \max V(u)$

Sine-Gordon case: $V(u) = 1 - \cos u$, $E_0 = 2$,

$$\tfrac{1}{2}(c^2-1)f_z^2 = E - 1 + \cos f(z).$$

W.l.o.g.

(d) V has fundamental period $P = 2\pi$ and

$$\min_{u\in\mathbb{R}} V(u) = 0, \qquad \max_{u\in\mathbb{R}} V(u) = 2.$$

Classification

First dichotomy (wave speed):

- **Subluminal** waves: $c^2 < 1$
- **Superluminal** waves: $c^2 > 1$

Second dichotomy (energy E):

- **Librational** wavetrain: $f(z+T) = f(z)$. Closed trajectory inside the separatrix in the phase portrait.
- **Rotational** wavetrain: $f(z+T) = f(z) \pm 2\pi$. Open trajectory outside the separatrix in the phase plane. Sign f_z is fixed.

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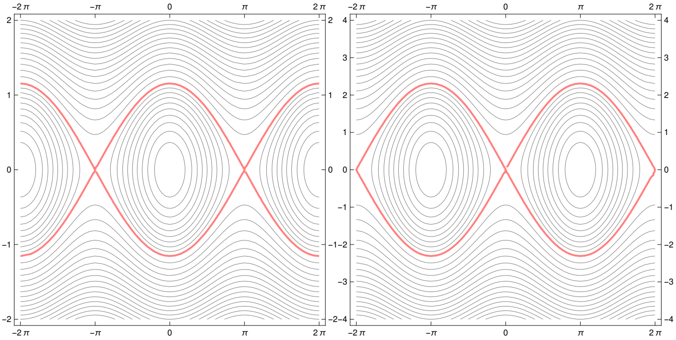

Figure : Phase portrait sine-Gordon case: $V(u) = 1 - \cos u$: superluminal $c^2 > 1$ (left); subluminal $c^2 < 1$ (right).

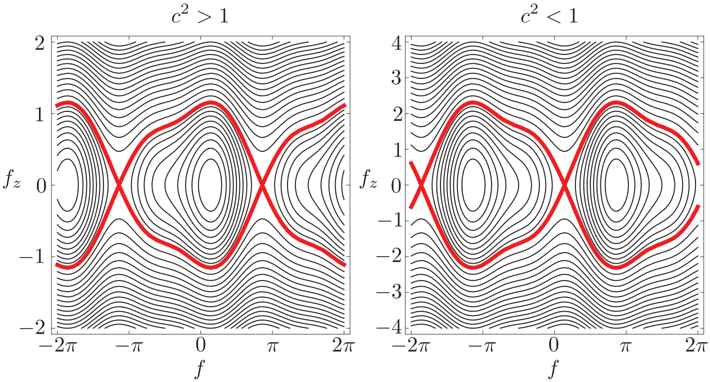

Figure : Phase portrait for $V(u) = -(0.861)(\cos u + \frac{1}{3}\sin(2u))$: superluminal $c^2 > 1$ (left); subluminal $c^2 < 1$ (right).

Superluminal librational: $c^2 > 1$, $0 < E < E_0$.
$\mathcal{K}(E) = \{u \in \mathbb{R} : (E - V(u))/(c^2 - 1) \geq 0\}$ = disjoint union of intervals in $(0, \pi)$. In (v_1, v_2), only one non-degenerate zero of V'. Librational (closed) periodic orbit.

$$f_z = \frac{\sqrt{2}}{\sqrt{c^2 - 1}} \sqrt{E - V(f)},$$

where $f \in (v_1, v_2) \subset \mathcal{K}(E)$.

$$T = \sqrt{2}\sqrt{c^2 - 1} \int_{v_1}^{v_2} \frac{d\eta}{\sqrt{E - V(\eta)}}.$$

Sine-Gordon: wave oscillates around $f = 0$, in $(v_1, v_2) = (-\text{Arc}\cos(-E+1), \text{Arc}\cos(-E+1))$

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Superluminal rotational: $c^2 > 1$, $E > E_0$, $E - V(f) > 0$ and $\mathcal{K}(E) = \mathbb{R}$. Rotation, f_z has fixed sign. Orbit outside the separatrix and $f(z+T) = f(z) \pm \pi$ for all z.

$$f_z^2 = \frac{2(E - V(f))}{c^2 - 1} > 0,$$

$$T = \frac{\sqrt{c^2-1}}{\sqrt{2}} \int_0^{\pi} \frac{d\eta}{\sqrt{E - V(\eta)}}$$

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Domain for parameters (E, c)

$$\mathbb{G}^{\text{lib}}_{<} = \{c^2 < 1,\ 0 < E < E_0\},\ \text{(subluminal librational)},$$
$$\mathbb{G}^{\text{rot}}_{<} = \{c^2 < 1,\ E < 0\},\ \text{(subluminal rotational)},$$
$$\mathbb{G}^{\text{lib}}_{>} = \{c^2 > 1,\ 0 < E < E_0\},\ \text{(superluminal librational)},$$
$$\mathbb{G}^{\text{rot}}_{>} = \{c^2 > 1,\ E > E_0\},\ \text{(superluminal rotational)},$$

$$(E,c) \in \mathbb{G} := \mathbb{G}^{\text{lib}}_{<} \cup \mathbb{G}^{\text{rot}}_{<} \cup \mathbb{G}^{\text{lib}}_{>} \cup \mathbb{G}^{\text{rot}}_{>}$$

Lemma

For each fixed $z \in \mathbb{R}$, $f(z;E,c)$ *is of class* C^2 *in* $(E,c) \in \mathbb{G}$.

Proof sketch: Rotational wave: $f = f(z;E,c)$ may be obtained by inverting the relation

$$z = \frac{\sqrt{|c^2 - 1|}}{\sqrt{2}} \int_{f(0)}^{f} \frac{d\eta}{\sqrt{|E - V(\eta)|}}$$

where either $f(0) = \underline{u}$ or $f(0) = \overline{u}$. Right-hand side is a C^3 function of (f,E,c), and its derivative with respect to f is strictly positive. By the Implicit Function Theorem, f can be solved for uniquely as a function of class C^3 in (E,c).

Librational wave: There is a maximal open interval containing $z = 0$ on which f is strictly increasing, and for z in this interval exactly the same argument given above for rotational waves applies. Adding half the period T produces another open interval of z-values for which f is strictly decreasing, a case again handled by a simple variation of the same argument. By periodicity of f with period T it remains to consider the values of z for which $f_z = 0$. But for such z we have $V(f) = E$ while $V'(f) \neq 0$ and we see by the Implicit Function Theorem that f is a C^2 function of E (and is independent of c). This completes the proof of the Lemma.

$\square$

Spectral problem

Solution $f(z) + u(z,t)$, with $u =$ perturbation:

$$u_{tt} - 2cu_{zt} + (c^2 - 1)u_{zz} + V'(u + f) - V'(f) = 0.$$

Linearized equation:

$$u_{tt} - 2cu_{zt} + (c^2 - 1)u_{zz} + V''(f(z))u = 0.$$

Specializing to sulutions of the form $u = w(z)e^{\lambda t}$, $\lambda \in \mathbb{C}$, $w \in X$ Banach:

$$(c^2 - 1)w_{zz} - 2c\lambda w_z + (\lambda^2 + V''(f(z)))w = 0. \qquad \text{(P)}$$

Quadratic "pencil" in λ. Loosely speaking, $\lambda \in \sigma_F$ is a Floquet eigenvalue if there exists a bounded solution w to (P).

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Floquet spectrum

Boundary value problem of the form

$$(c^2-1)w_{zz} - 2c\lambda w_z + (\lambda^2 + V''(f(z)))w = 0,$$

$$\begin{pmatrix} w(T) \\ w_z(T) \end{pmatrix} = e^{i\theta} \begin{pmatrix} w(0) \\ w_z(0) \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

Reminder from Lecture 1: For a given $\theta \in \mathbb{R}$ we define $\sigma_\theta \subset \mathbb{C}$ to be the set of complex λ for which there exists a nontrivial solution. The Floquet spectrum σ_F is defined then as the union over θ of these sets:

$$\sigma_F := \bigcup_{-\pi<\theta\leq\pi} \sigma_\theta.$$

Recall: Written as a first order system with periodic coefficients, by Floquet the solutions are of form $\mathbf{w} = e^{r(\lambda)z/T}\mathbf{y}$, where $\mathbf{w} = (w, w_z)^\top$, and $\mathbf{y}$ is T-periodic, and $r = r(\lambda)$ is the Floquet exponent. Thus, $\mu = \mu(\lambda) = e^{r(\lambda)}$ is the **Floquet multiplier**. In order to have bounded solutions it is necessary that $|\mu(\lambda)| = 1$, or $\mu(\lambda) = e^{i\theta(\lambda)}$.

Spectral stability

Definition

We say the wave is **spectrally stable** if $\sigma_F \subset \{\mathrm{Re}\,\lambda \leq 0\}$. Otherwise it is **spectrally unstable.**

Since (nKG) is a real Hamiltonian system:

Lemma

σ is symmetric with respect to reflection in real and imaginary axes: $\lambda \in \sigma \;\Rightarrow\; -\lambda, \overline{\lambda} \in \sigma$.

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Summary of stability results

Wave	Whitham (1974)	Scott (1969)
Subluminal rotational	stable	stable
Superluminal rotational	stable	unstable
Subluminal librational	unstable	unstable
Superluminal librational	unstable	unstable

Scott (1969):

$$y = \exp\left(\frac{-c\lambda z}{c^2-1}\right),$$

$$y_{zz} + \frac{V''(f(z))}{c^2-1} y = \left(\frac{\lambda}{c^2-1}\right)^2 y =: \nu y. \qquad \text{(H)}$$

Hill's equation with period T. $\nu \in \sigma_H$ (Floquet spectrum of (H)) if there is a bounded solution y.

Scott assumed that the transformation is **isospectral**: ($\sigma_H = \sigma_F$). This is **not true**. Actually:

Lemma (Jones et al. (2013))

If $\lambda \in \sigma_H \cap \sigma_F$ then $\lambda \in i\mathbb{R}$.

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Formal slow modulation theory

Reference: Whitham, Proc. Roy. Soc. Ser. A (1965).

Exact periodic wave: for given (E, c), fixed,

$$u(x,t) = f(x - ct) =: \Phi(\theta(x,t)),$$

with

$$\theta(x,t) := kx - \omega t, \qquad \Phi(kz) := f(z).$$

k is the **wave number**; $\omega := ck$ is the **frequency**. Both constants. In this framework:

$$\tfrac{1}{2}(\omega^2 - k^2)\Phi_\theta^2 = E - V(\Phi)$$

Dispersion relation

Necessary condition for the wave to be a solution

$$\omega = \omega(k)$$

ω and k are not independent from each other. From inspection from expressions for T:

(a) superluminal ($\omega^2 > k^2$), librational ($0 < E < E_0$):

$$\omega^2 = k^2 + \tfrac{1}{2}k^2T^2 \left(\int_{v_1}^{v_2} \frac{d\eta}{\sqrt{E - V(\eta)}} \right)^{-2},$$

(b) subluminal ($\omega^2 < k^2$), librational ($0 < E < E_0$):

$$\omega^2 = k^2 - \tfrac{1}{2}k^2T^2\left(\int_{v_3}^{v_4}\frac{d\eta}{\sqrt{V(\eta)-E}}\right)^{-2},$$

(c) superluminal ($\omega^2 > k^2$), rotational ($E > E_0$):

$$\omega^2 = k^2 + 2k^2T^2\left(\int_0^{\pi}\frac{d\eta}{\sqrt{E-V(\eta)}}\right)^{-2},$$

(d) subluminal ($\omega^2 < k^2$), rotational ($E < 0$):

$$\omega^2 = k^2 - 2k^2T^2\left(\int_0^{\pi}\frac{d\eta}{\sqrt{V(\eta)-E}}\right)^{-2}.$$

Approximation by slow modulations

Let $\varepsilon > 0$, rescale (nKG) equation by $(x,t) \mapsto (\varepsilon x, \varepsilon t)$:

$$\varepsilon^2 u_{tt} - \varepsilon^2 u_{xx} + V'(u) = 0,$$

Now (x,t)=slow variables. Whitham's key ingredient seek solutions with a **WKB approximation** of the form

$$u(x,t) = \Phi\Big(\frac{\theta(x,t)}{\varepsilon}\Big) + O(\varepsilon) = f\Big(\frac{z(x,t)}{\varepsilon}\Big) + O(\varepsilon).$$

Suppose that k, ω **are no longer constant** (and hence, E and c). Allows the wave to be **slowly modulated**

Conservation of fluxons

In the constant case, clearly, $k = \theta_x$, $\omega = -\theta_t$, as $\theta = kx - \omega t$. First approximation: **conservation of fluxons** equation

$$k_t + \omega_x = 0$$

This will be the first of Whitham's equations. This is a scalar conservation law for the wave number, with $\omega = \omega(k)$ (nonlinear dispersion relation) as a “flux” function.

How to obtain the second? There are many approaches:

- Substitution of the WKB expansion, collect powers of ε (Whitham, 1974). (Applicable to weakly dissipative systems without Lagrangian formulation nor conserved quantities.)
- Averageing densities and fluxes of local conservation laws associated to the equation (original's Whitham approach, Proc. R. Soc. London A, 1965)
- Averaged variational principle (Whitham, J. Fluid Mech. 1965).

Averaged variational principle

The nKG equation is the Euler-Lagrange equation associated to the functional

$$I[u] = \iint L(u, u_x, u_t)\, dx\, dt,$$

with Lagrangian density

$$L(u, u_x, u_t) = \tfrac{1}{2}u_t^2 - \tfrac{1}{2}u_x^2 - V(u).$$

We may calculate the Lagrangian density at the exact wave, $u = f(x - ct) = \Phi(kx - \omega t)$:

$$L(u, u_x, u_t) = \tfrac{1}{2}(\omega^2 - k^2)\Phi_\theta(\theta)^2 - V(\Phi(\theta))$$

Since $\theta = kx - \omega t$, last expression is periodic in θ with period kT.

Integrating over one period and using equation for Φ_θ^2 we obtain the **averaged Lagrangian evaluated at the exact wave**:

$$\begin{aligned}
\langle L \rangle &= \frac{1}{kT}\int_0^{kT} \tfrac{1}{2}(\omega^2 - k^2)\Phi_\theta(\theta)^2 - V(\Phi(\theta))\, d\theta \\
&= \frac{1}{kT}\int_0^{kT} (\omega^2 - k^2)\Phi_\theta(\theta)^2\, d\theta - E \\
&= \frac{\sqrt{2}}{kT}\oint ((\omega^2 - k^2)(E - V(\eta)))^{1/2}\, d\eta - E \\
&=: \tilde{\mathcal{L}}(\omega, k, E)
\end{aligned}$$

When k, ω and E are allowed to be slowly varying functions of x and t, then Whitham proposes the **averaged variational principle**:

$$\delta \int\int \tilde{\mathcal{L}}(\omega(x,t), k(x,t), E(x,t))\, dx\, dt = 0,$$

which (when viewed as a variational principle for $E(x,t)$ and for $\theta(x,t)$, through the relations $\theta_x = k$ and $\theta_t = -\omega$) yields the following variational equations

$$\begin{aligned} \tilde{\mathcal{L}}_E &= 0, \\ (\tilde{\mathcal{L}}_\omega)_t - (\tilde{\mathcal{L}}_k)_x &= 0 \end{aligned}$$

Interpretation

- If the wavetrain is slowly modulated (slow variations of k, ω, E), then these must be solutions, to leading order, to the variation equations for the averaged Lagrangian
- The first equation, $\tilde{\mathcal{L}}_E = 0$, is the nonlinear **dispersion relation**: only two of E, k, ω are independent
- We take the second eq. and the conservation of fluxons eq. as the **Whitham modulation system**: governs two of the fields (e.g. $\theta = \theta(x,t)$, $\omega = \omega(x,t)$) as functions of (x,t), describing how these parameters vary slowly. (On the macroscopic scale only.)

Whitham's modulation system:

$$\begin{aligned} k_t + \omega_x &= 0 \\ (\tilde{\mathcal{L}}_\omega)_t - (\tilde{\mathcal{L}}_k)_x &= 0. \end{aligned} \tag{W1}$$

First order system. If the last system (W1) is hyperbolic (Cauchy problem well-posed) then the wave is **stable under slow modulations** (Whitham, 1974).

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Averageing conserved quantities method

Reference: Witham, Proc. Roy. Soc. Ser. A (1965).
Return to the unscaled equation

$$u_{tt} - u_{xx} + V'(u) = 0,$$

Use (E, c) as the two unknowns.

Objective: derive Whitham system for E and c. For fixed $(E, c) \in \mathbb{G}$ we replace the exact wave $u(x,t) = f(z; E, c)$ to compute the Lagrangian density there,

$$L(u, u_x, u_t) = \tfrac{1}{2}(c^2 - 1) f_z(z)^2 - V(f(z)).$$

Last expression is periodic in z with period T. Its average over one period is the **averaged Lagrangian**:

$$
\begin{aligned}
\langle L\rangle &= \frac{1}{T}\int_0^T \tfrac{1}{2}(c^2-1)f_z(z)^2 - V(f(z))\,dz \\
&= \frac{1}{T}\int_0^T (c^2-1)f_z(z)^2\,dz - E, \\
&= \frac{\sqrt{2}}{T}\oint ((c^2-1)(E-V(\eta)))^{1/2}\,d\eta - E \\
&=: \mathcal{L}(E,c).
\end{aligned}
$$

The contour integral is well determined for each $(E,c)\in\mathbb{G}$, and defined as twice the integral between two consecutive zeroes v_i and v_j of $E-V(u)$ in the librational case, or as the integral from 0 to π in the rotational case.

More precisely,

$$\mathcal{L}(E,c) = \frac{2\sqrt{2}}{T}\sqrt{c^2-1}\int_{v_1}^{v_2}\sqrt{E-V(\eta)}\,d\eta - E, \qquad \text{(sup, lib)},$$

$$\mathcal{L}(E,c) = -\frac{2\sqrt{2}}{T}\sqrt{1-c^2}\int_{v_3}^{v_4}\sqrt{V(\eta)-E}\,d\eta - E, \qquad \text{(sub, lib)},$$

$$\mathcal{L}(E,c) = \frac{\sqrt{2}}{T}\sqrt{c^2-1}\int_{0}^{\pi}\sqrt{E-V(\eta)}\,d\eta - E, \qquad \text{(sup, rot)},$$

$$\mathcal{L}(E,c) = -\frac{\sqrt{2}}{T}\sqrt{1-c^2}\int_{0}^{\pi}\sqrt{V(\eta)-E}\,d\eta - E, \qquad \text{(sub, rot)}.$$

Classical action (mechanics):

$$W(E,c) = \sqrt{2}\oint ((c^2-1)(E-V(\eta)))^{1/2}\,d\eta,$$

In our case, it can be expressed as:

$$W(E,c) := \operatorname{sgn}(c^2-1)\sqrt{|c^2-1|}J(E),$$

where

$$J(E) := \begin{cases} J_{\rm lib}(E), & \text{librations,} \\ J_{\rm rot}(E), & \text{rotations.} \end{cases}$$

We define:

$$J_{\mathrm{rot}}(E) := \sqrt{2}\int_0^{\pi} \sqrt{\operatorname{sgn}(c^2-1)(E-V(\eta))}\, d\eta$$

with $E > E_0 = \max V$ if $c^2 > 1$, or $E < 0 = \min V$ if $c^2 < 1$; and with

$$J_{\mathrm{lib}}(E) := 2\sqrt{2}\int_{v_i}^{v_j} \sqrt{\operatorname{sgn}(c^2-1)(E-V(\eta))}\, d\eta$$

in the case $0 < E < E_0$ (libration), with v_i, v_j being the two simple consecutive zeroes of $E - V(u)$, such that $(c^2-1)(E-V(u)) \geq 0$ for $u \in (v_i, v_j)$.

Lemma

For each of the four cases under consideration (sub- or superluminal, libration or rotation) there hold

$$W_E = T, \tag{1}$$

$$W_c = \frac{cW}{c^2 - 1}. \tag{2}$$

Proof: The second expression follows immediately from definition of W in the four cases. For **rotations**, $W_E = T$ can be verified by simple differentiation under the (proper) integral sign and by comparison with the formula for the period.

For **librations**, though, the zeroes v_i and v_j may depend on E and we need a change of variables. Let us define

$$h(u) = \begin{cases} u\Big(\frac{V(u)}{u^2}\Big)^{1/2}, & u \in \mathcal{K}(E)\backslash\{0\}, \\ 0, & u = 0, \end{cases}$$

where $\mathcal{K}(E) = \{u \in \mathbb{R} : (E - V(u))/(c^2 - 1) \geq 0\}$. h is smooth and monotonically increasing with $h' > 0$.

Let us first consider the superluminal case. Making the change of variables $\xi = h(\eta)$ we have that

$$
\begin{aligned}
\frac{\partial}{\partial E}\int_{v_i}^{v_j}\sqrt{E-V(\eta)}\,d\eta &= \frac{\partial}{\partial E}\int_{-\sqrt{E}}^{\sqrt{E}} h'(h^{-1}(\xi))\sqrt{E-V(h^{-1}(\xi))}\,d\xi \\
&= h'(h^{-1}(\sqrt{E}))\sqrt{E-Vh^{-1}(\sqrt{E})} - h'(h^{-1}(-\sqrt{E}))\sqrt{E-Vh^{-1}(-\sqrt{E})} \\
&\quad + \frac{1}{2}\int_{-\sqrt{E}}^{\sqrt{E}} \frac{h'(h^{-1}(\xi))}{\sqrt{E-V(h^{-1}(\xi))}}\,d\xi \\
&= h'(v_j)\sqrt{E-V(v_j)} - h'(v_i)\sqrt{E-V(v_i)} + \frac{1}{2}\int_{v_i}^{v_j}\frac{d\eta}{\sqrt{E-V(\eta)}} \\
&= \frac{1}{2}\int_{v_i}^{v_j}\frac{d\eta}{\sqrt{E-V(\eta)}}.
\end{aligned}
$$

Upon comparison with the expression for T we obtain

$$W_E = \sqrt{2}\sqrt{c^2 - 1} \int_{v_1}^{v_2} \frac{d\eta}{\sqrt{E - V(\eta)}} = T.$$

The calculation for the subluminal case is analogous.

□

Conserved quantities

For any quantity $F = F(u) = F(f(z))$ we denote its average over a period by

$$\langle F \rangle = \frac{1}{T}\int_0^T F(f(z))\, dz = \frac{1}{T}\oint \frac{F(f)}{f_z}\, df.$$

The non-linear Klein-Gordon equation is endowed with an infinite number of conserved quantities. Whitham takes two:

$$\left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + V(u)\right)_t - (u_t u_x)_x = 0, \quad \text{(energy)}$$

$$(-u_t u_x)_t + \left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 - V(u)\right)_x = 0, \quad \text{(linear momentum)}$$

These conservation equations have the generic form:

$$\partial_t D(u) + \partial_x F(u) = 0,$$

$D(u)$ is the conserved quatity; $F(u)$ is the flux.

Idea: for each fixed choice of $(E,c) \in \mathbb{G}$, we first replace $u(x,t)$ by the exact periodic traveling wave $u = f(z;E,c)$ in the expressions for $D(u)$ and $F(u)$ and therefore obtain T-periodic functions of z that can be averaged over a period. The corresponding averages are functions of $(E,c) \in \mathbb{G}$

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Calculating:

$$\langle \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 \rangle = \frac{c^2+1}{2T} \oint f_z \, df = \frac{1}{2}\Big(\frac{c^2+1}{c^2-1}\Big)\frac{W}{T}.$$

$$\begin{aligned}\langle \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 \rangle &= \frac{c^2+1}{2T} \int_0^T f_z^2 \, dz \\ &= \frac{1}{T}\Big(\frac{c^2+1}{c^2-1}\Big) \int_0^T E - V(f(z)) \, dz \\ &= \Big(\frac{c^2+1}{c^2-1}\Big)(E - \langle V(u) \rangle).\end{aligned}$$

$$\Rightarrow \langle V(u) \rangle = E - \frac{W}{2T}$$

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$$\begin{aligned}
\langle \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 \rangle &= \frac{c^2+1}{2T} \int_0^T f_z^2 \, dz \\
&= \frac{1}{T}\Big(\frac{c^2+1}{c^2-1}\Big) \int_0^T E - V(f(z)) \, dz \\
&= \Big(\frac{c^2+1}{c^2-1}\Big)(E - \langle V(u) \rangle).
\end{aligned}$$

$$\Rightarrow \langle V(u) \rangle = E - \frac{W}{2T}$$

Similarly,

$$\begin{aligned}\langle -u_t u_x \rangle &= \frac{c}{T}\int_0^T f_z^2\, dz = \frac{c}{T}\int_0^T \frac{2(E - V(f(z)))}{c^2-1}\, dz \\ &= \Big(\frac{2c}{c^2-1}\Big)(E - \langle V(u)\rangle) \\ &= \Big(\frac{c}{c^2-1}\Big)\frac{W}{T}\end{aligned}$$

Yielding:

$$\begin{aligned}\langle \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + V(u)\rangle &= \frac{1}{T}(EW_E + cW_c - W), \\ \langle \tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 - V(u)\rangle &= \frac{cW_c}{T} - E, \\ \langle -u_t u_x \rangle &= \frac{W_c}{T} = \frac{c}{T}(EW_E + cW_c - W) - cE.\end{aligned}$$

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Taking average of conservation of energy and momentum equations we obtain the system:

$$\left(\frac{1}{T}(EW_E + cW_c - W)\right)_t + \left(\frac{c}{T}(EW_E + cW_c - W)\right)_x = 0,$$
$$\left(\frac{W_c}{T}\right)_t + \left(\frac{cW_c}{T} - E\right)_x = 0.$$

Simplifying (use $W_t = W_c c_t + W_E E_t$, $W_x = W_c c_x + W_E E_x$, and $W_E/T \equiv 1$):

$$\left(\frac{W_c}{T}\right)_t + \left(\frac{cW_c}{T} - E\right)_x = 0,$$
$$\left(\frac{1}{T}\right)_t + \left(\frac{c}{T}\right)_x = 0. \tag{*}$$

Recall that T is also a function of $(E,c) \in \mathbb{G}$.

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Recall that T is also a function of $(E,c) \in \mathbb{G}$.

Lemma

Whitham's system of equations (*) *is equivalent to the system:*

$$\begin{pmatrix} E \\ c \end{pmatrix}_t + \mathbf{A}(E,c) \begin{pmatrix} E \\ c \end{pmatrix}_x = 0, \qquad \text{(W2)}$$

$$\mathbf{A}(E,c) = \frac{1}{N(E,c)} \begin{pmatrix} c(J(E)J''(E)+J'(E)^2) & -J(E)J'(E) \\ (c^2-1)^2 J'(E)J''(E) & c(J(E)J''(E)+J'(E)^2) \end{pmatrix},$$

$$N(E,c) = J(E)J''(E) + c^2 J'(E)^2.$$

Proof: Follows by repeated substitution and the chain rule. For instance, it can be shown that the second equation in (*) is equivalent to

$$(W_E)_t + c(W_E)_x - W_E c_x = 0,$$

inasmuch as $W_E = T \neq 0$.

Moreover,

$$\begin{aligned}\left(\frac{W_c}{T}\right)_t + \left(\frac{cW_c}{T} - E\right)_x &= \frac{(W_c)_t}{W_E} + \frac{c(W_c)_x}{W_E} - E_x + \\ &\quad + \frac{W_c}{(W_E)^2}\big(W_E c_x - (W_E)_t - c(W_E)_x\big) \\ &= \frac{1}{W_E}\big((W_c)_t + c(W_c)_x - W_E E_x\big) = 0,\end{aligned}$$

obtaining the system:

$$\begin{aligned}(W_E)_t + c(W_E)_x - W_E c_x &= 0, \\ (W_c)_t + c(W_c)_x - W_E E_x &= 0.\end{aligned}$$

Now, from the definition of W we substitute

$$W_E = \operatorname{sgn}(c^2-1)\sqrt{|c^2-1|}J'(E),$$

$$W_c = \operatorname{sgn}(c^2-1)\frac{c}{\sqrt{|c^2-1|}}J(E),$$

the result is:

$$\mathbf{B}_1(E,c)\begin{pmatrix} E \\ c \end{pmatrix}_t + \mathbf{B}_2(E,c)\begin{pmatrix} E \\ c \end{pmatrix}_x = 0,$$

where

$$\mathbf{B}_1(E,c) = \begin{pmatrix} (c^2-1)J''(E) & cJ'(E) \\ c(c^2-1)J'(E) & -J(E) \end{pmatrix},$$

$$\mathbf{B}_2(E,c) = \begin{pmatrix} c(c^2-1)J'(E) & J'(E) \\ (c^2-1)J'(E) & -cJ(E) \end{pmatrix}.$$

Noticing that

$$\begin{aligned}\det \mathbf{B}_1(E,c) &= -(c^2-1)(J(E)J''(E)+c^2J'(E)^2)\\ &= -(c^2-1)N(E,c),\end{aligned}$$

we readily obtain

$$\mathbf{B}_1(E,c)^{-1} = \frac{1}{(c^2-1)N(E,c)}\begin{pmatrix} J(E) & cJ'(E) \\ c(c^2-1)J'(E) & -(c^2-1)J''(E)\end{pmatrix}.$$

A direct computation shows that $A(E,c) = \mathbf{B}_1(E,c)^{-1}\mathbf{B}_2(E,c)$, proving the lemma.

□

Lemma

Whitham system (W2) *is hyperbolic if and only if*

$$J''(E) < 0.$$

Proof: The characteristic velocities $s_\pm$ of the system satisfy

$$c(J(E)J''(E) + J'(E)^2) - s_\pm = \pm|c^2 - 1|\big(-J(E)J''(E)J'(E)^2\big)^{1/2}.$$

Thus, the system is hyperbolic if and only if $J(E)J''(E) < 0$. For the nonlinear Klein-Gordon periodic waves we observe that $J(E) > 0$, as the continued inspection of formulas shows. Thus, we conclude that the system is hyperbolic if and only if J is a strictly concave function of the energy E. This proves the lemma. □

Whitham's modulational (in)stability results

Lemma

$$\operatorname{sgn} J''(E) = \operatorname{sgn}\left((c^2-1)T_E\right).$$

Proof: Differentiate twice to reckon:

$$T_E = W_{EE} = \operatorname{sgn}(c^2-1)\sqrt{|c^2-1|}J''(E).$$

□

Definition

We define the **modulational stability index** as:

$$\gamma_M = -\operatorname{sgn}\left((c^2-1)T_E\right).$$

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Corollary

The quasilinear Whitham system (W2) *is hyperbolic if and only if* $\gamma_M = 1$. *In this case we say that the underlying periodic traveling wave is modulationally stable (otherwise we say it is modulationally unstable).*

Finally we recover:

Theorem (Whitham, 1974)

- *Both super- and subluminal rotational waves are modulationally stable,*
- *Both super- and subluminal librational waves are modulationally unstable.*

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Interpretation: “Modulational” stability pertains to perturbations for which the wave parameters underlie small variations with respect to wavelength.

What is its relation to spectral stability? Whitham’s is an **instability theory**: Tomorrow we will show that modulational stability is a necessary condition for spectral stability.

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1. History

2. Modulation theory for non-linear Klein-Gordon periodic wavetrains

3. Modulation theory for KdV cnoidal periodic waves

The Korteweg-de Vries (KdV) equation

In 1895, **Korteweg and de Vries** derived a mathematical model to describe Rusell's solitary wave. Such a wave describes surface water waves whose wavelength is large compared to the water depth. It is possible to derive, from equations for shallow water waves, the **KdV model**:

$$\eta_t + \sqrt{gh}\eta_x + \frac{3}{2}\sqrt{\frac{g}{h}}\eta\eta_x + \frac{1}{6}h^2\sqrt{gh}\eta_{xxx} = 0,$$

where $\eta = \eta(x,t)$ is the surface of water elevation (opposing gravity), $g > 0$ is the gravity constant, $h > 0$ is the mean water depth. Model in one spatial dimension, $x \in \mathbb{R}$, direction of propagation; $t > 0$, time.

Non-dimensional, standard form

Making the transformations

$$u = \frac{3}{2}\frac{\eta}{h}, \quad \hat{x} = \frac{x}{h} - \frac{1}{6}\sqrt{\frac{g}{h}}t, \quad \hat{t}, = \frac{1}{6}\sqrt{\frac{g}{h}}t,$$

leads to the standard **KdV equation**:

$$u_t + 6uu_x + u_{xxx} = 0. \qquad \text{(KdV)}$$

Cnoidal waves

These have the form:

$$\eta = \eta_2 + H\,\mathrm{cn}^2\Big(\frac{2}{L}K(m)(x-ct); m\Big),$$

where: H wave height; L wave length; c phase speed; and, η_2 trough elevation.

$\mathrm{cn}(\cdot)$ is the **Jacobi elliptic cnoidal function** defined as

$$\mathrm{cn}(y;m) = \cos\phi,$$

$$y = \int_0^{\phi} \frac{ds}{\sqrt{1-m^2\sin^2 s}},$$

$m \in (0,1)$ is the elliptic modulus.

$K = K(m)$ is the **complete elliptic integral of the first kind**:

$$K(m) = \int_0^{2\pi} \frac{ds}{\sqrt{1 - m^2 \sin^2 s}}.$$

Observation:

- When $m \to 1$, $\mathrm{cn}(y;m)\mathrm{sech}(y)$, "soliton" solution
- When $m \to 0$, $\mathrm{cn}(y;m) \cos(y)$, wave of sinusoidal type

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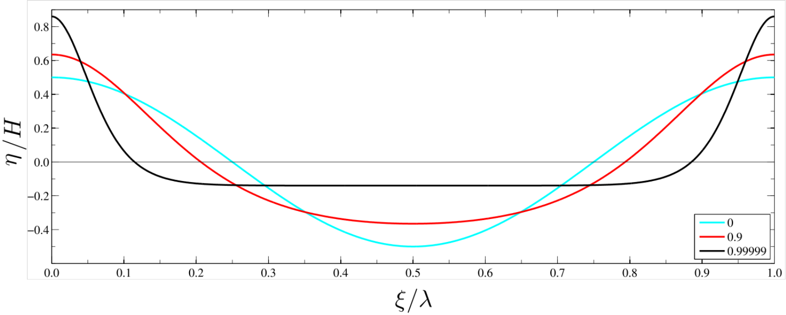

Figure : Cnoidal wave profiles for elliptic parameter values $m = 0$ (blue); $m = 0.9$ (red); $m = 0.99999 \sim 1^-$ (black).

Traveling wave solutions

Solutions of the form $u(x,t) = f(x-ct), f = f(z)$, $z = x - ct$, traveling with speed $c \in \mathbb{R}$. Upon substitution:

$$-cf_z + 6ff_z + f_{zzz} = 0.$$

Integrating:

$$-cf + 3f^2 + f_{zz} = a,$$

with $a \in \mathbb{R}$. Multiply by f_z and integrate once more:

$$\frac{1}{2}f_z^2 = E + af + \frac{c}{2}f^2 - f^3.$$

The wave is completely determined in terms of the parameters $(E,a,c) \in \mathbb{R}^3$. Soliton case: $E = 0$, $a = 0$. (Solitons are a codimension two set of periodic wavetrains.)

To assure the existence of periodic wavetrains, the effective potential

$$V(u;a,c) = u^3 - \frac{c}{2}u^2 - au,$$

must have a **local minimum**.

Parameter set:

Let us define:

$$\mathbb{G} := \{(a, E, c) \in \mathbb{R}^3 : c > 0,\, c^2 + 12a > 0,\, E_{\min} < E < E_{\max}\},$$

where, for fixed $c > 0$, the potential $V(u; a, c)$ has min and max at:

$$u_{\min} = \tfrac{1}{6}(c + \sqrt{c^2 + 12a}),\; u_{\max} = \tfrac{1}{6}(c - \sqrt{c^2 + 12a}),$$

and $E_{\max} = V(u_{\max}; a, c)$, $E_{\min} = V(u_{\min}; a, c)$.

Lemma

For each $(a,E,c) \in \mathbb{G}$ there exist a traveling T-periodic wave.

The **period** is given by

$$T(a,E,c) = 2\int_{u_-}^{u_+} \frac{du}{\sqrt{2(E - V(u;a,c))}}$$

where $u_\pm$ are the simple roots of $V(u;a,c) = E$, for each $(a,E,c) \in \mathbb{G}$, with $V(u;a,c) < E$ for all $u \in (u_-, u_+)$. It can be shown that $u_\pm \in C^1(\mathbb{G})$.

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Modulation equations

It is more convenient, for modulation purposes, to write the waves in terms of three (equivalent) elliptic parameters: (b_1, b_2, b_3), with $b_1 \leq b_2 \leq b_3$.

Cnoidal traveling wave:

$$u(x,t) = f(x - ct),$$

$$f(z) = b_2 + (b_3 - b_2)\mathrm{cn}^2\left(\sqrt{2(b_3 - b_1)}z; m\right),$$

where

$$c = 2(b_1 + b_2 + b_3), \;\; m = \frac{b_3 - b_2}{b_3 - b_2} \text{ elliptic modulus,}$$

$$L = \frac{2\sqrt{2}K(m)}{\sqrt{b_3 - b_1}} \quad \text{wavelength}$$

The wave profile equation

$$f_z^2 = 2af + 2E + cf^2 - 2f^3,$$

can be written as:

$$U_z^2 = -4\mathbb{P}(U), \quad \mathbb{P}(U) = \prod_{j=1}^{3}(U - b_j), \quad U := \frac{f}{2} - \frac{c}{4},$$

b_j are the zeroes of the cubic polynomial $\mathbb{P}$, with $b_1 \leq b_2 \leq b_3$.

Here, $L = 2\pi/k$, $c = \omega/k$, and the cnoidal wave is L-periodic in z ($T \equiv L$):

$$u(x,t) = f(z;\mathbf{b}), \quad f(z+L;\mathbf{b}) = f(z;\mathbf{b})$$

$$\mathbf{b} := (b_1, b_2, b_3)$$

Equivalent three-parameter description:

$$\mathbf{b} \in \tilde{\mathbb{G}} \iff (a, E, c) \in \mathbb{G}$$

Goal: Slowly modulate the periodic wave by considering $b_j = b_j(x,t)$ as slowly varying funcions of (x,t). Express an equivalent Whitham system of equations for b_j's and determine its hyperbolicity.

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$$\partial_t \langle D_j \rangle + \partial_x \langle F_j \rangle = 0.$$

where, for each $F = F(x,t)$,

$$\langle F \rangle = \frac{1}{L} \oint F\, dz = \frac{1}{L} \int_{b_2}^{b_3} \frac{F}{\sqrt{-\mathbb{P}(\mu)}}\, d\mu$$

Conservation laws for the KdV equation:

$$u_t + (3u^2 + u_{xx})_x = 0,$$
$$(u^2)_t + (4u^3 + 2uu_{xx} - u_x^2)_x = 0.$$

We take these two, plus the **conservation of fluxons** equation:

$$k_t + \omega_x = 0.$$

It can be shown that:

$$\langle f \rangle = 2(b_3 - b_1)\frac{\tilde{E}(m)}{K(m)} + b_1 - b_2 - b_3,$$

$$\langle f^2 \rangle = \tfrac{2}{3}c(b_3 - b_1)\frac{\tilde{E}(m)}{K(m)} + 2cb_1 + 2(b_1^2 - b_2 b_3) + \frac{c}{4},$$

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where

$$\tilde{E}(m) = \int_0^{\pi/2} \sqrt{1 - m^2 \sin^2 s}\, ds,$$

is the **complete elliptic integral of the second kind**.

We arrive at the Whitham system of 3 equations, one for each of the conserved quantities:

$$D_1 = u, \ \ D_2 = u^2, \ \ D_3 = k,$$

and fluxes:

$$F_1 = 3u^2 + u_{xx}, \ \ F_2 = 4u^3 + 2uu_{xx} - u_x^2, \ \ F_3 = \omega,$$

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Lemma

The Whitham equations for the modulated cnoidal KdV periodic wave solution have the following structure

$$\mathbf{b}_t + \mathbf{A}(\mathbf{b})_x = 0,$$

where $\mathbf{A}(\mathbf{b}) = \mathbf{D}^{-1}\mathbf{F}$*, with* $\mathbf{D}$ *and* $\mathbf{F}$ *defined via*

$$D_{ij}(\mathbf{b}) := \partial_{b_j}\langle D_i\rangle, \;\; F_{ij}(\mathbf{b}) := \partial_{b_j}\langle F_i\rangle$$

$i,j = 1,2,3.$

Lemma

The eigenvalues of the Whitham system are given by

$$\nu_1 = c + 4(b_3 - b_1)(1-m)\frac{K(m)}{\tilde{E}(m)},$$

$$\nu_2 = c - 4(b_3 - b_1)(1-m)\frac{K(m)}{(\tilde{E}(m) - (1-m)K(m))},$$

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They are all real and the KdV cnoidal wave is ***modulationally stable****.*

References: Whitham (1974), Grimshaw (2007).

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Thank you!