Spectral stability of periodic wavetrains

Lecture 1. Preliminaries. Floquet-Bloch spectrum

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1. Introduction
2. Periodic traveling waves
3. The Floquet-Bloch spectrum

Spatially periodic traveling waves

- Waves, patterns, and other coherent structures have always played a pivotal role in applied mathematics
- A **spatially periodic traveling wave (or wavetrain)** is a periodic function of one-dimensional space traveling with constant speed
- They appear as special solutions in many natural phenomena, ranging from Biology, Chemistry and Physics to even Economics (Kondratiev waves)
- Periodic wavetrains appear typically in water waves, self-oscillatory systems, excitable systems and reaction-diffusion-advection systems

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- The mathematical theory of periodic traveling waves is most fully developed for PDEs, although they appear also as special solutions for discrete systems, cellular automata, integro-differential equations, and other models
- The existence of periodic traveling waves usually depends on the parameter values in a mathematical equation
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Examples

Stokes waves

A **Stokes wave** is a and periodic surface non-linear wave on an inviscid fluid layer of constant mean depth. The first fluid dynamics modelling was done in the mid 19th century by G. Stokes using a perturbation series approach, now known as the **Stokes expansion**, obtaining approximate solutions for non-linear wave motion. The expansion works for **deep and intermediate** water waves. For **shallow** water waves a better approximation was done by Korteweg and de Vries, and by Boussinesq.

Figure : Undular bore near the mouth of Araguari River in north-eastern Brazil. The undulations following behind the bore front appear as slowly-modulated Stokes waves (source: Wikipedia).

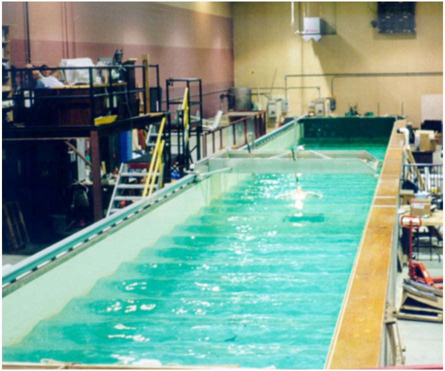

Figure : Model testing with periodic waves in the WaveTow Tank of the Jere A. Chase Ocean Engineering Laboratory, University of New Hampshire (source: Wikipedia).

Stationary structures in materials sciences

Periodic wavetrains appear also as equilibrium configurations in certain silicon-based materials after stretching and compression procedures to large levels of strain without damaging the silicon

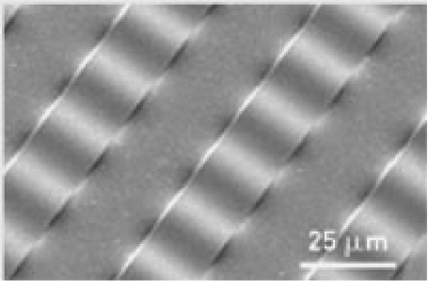

Figure : Stretchable single-crystal silicon (source: J. Rogers, U. Urbana-Champaign).

Superconductivity and quantum-tunneling

Josephson won the 1973 Nobel Prize in Physics for his discovery of the Josephson effect, describing the emergence of a supercurrent through a Josephson junction. The phase difference of wave functions of electrons in the super-conductors satisfy the **sine-Gordon equation**.

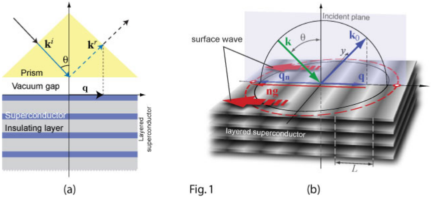

Figure : Two dimensional Josehphson junction: infinite plates of superconductors separated by a thin dielectric barrier (image credit: AIST-NT, California, USA.)

Stability of periodic wavetrains

- Stability is a fundamental issue to validate an specific mathematical model
- Periodic waves are observable in Nature: any reasonable mathematical model must pass a simple "stability" test
- In general we are interested in what happens to solutions whose initial conditions are small perturbations of the traveling wave under consideration
- If any such solution stays close to the set of all "translates" of the traveling wave for all positive times, then we say that the traveling wave is **stable**

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- The notion of “stability” depends on the mathematical model
- In these lectures we will focus on the notion of **spectral stability**
- Spectral stability can be loosely defined as the property that, if we linearize the equations around the periodic wave solution and consider the eigenvalue problem of the resulting linearized operator, then its spectrum is “well-behaved” and precludes explosive behavior

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Examples

Korteweg de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0,$$

Periodic wave solution (cnoidal wave): $u(x,t) = f(x - ct)$, where

$$f(z) = b_2 + (b_3 - b_2)\mathrm{cn}^2\left(\sqrt{2(b_3 - b_1)}z; m\right),$$

$$c = 2(b_1 + b_2 + b_3), \;\; m = \frac{b_3 - b_2}{b_3 - b_2} \text{ elliptic modulus,}$$

decsribed in terms of elliptic functions.

Ref.: Korteweg, Philosophical Magazine, Series 5, **39** (1895).

sine-Gordon equation

$$u_{tt} - u_{xx} + \sin u = 0,$$

"Periodic" wave, $u(x,t) = f_{c,E}(x - ct)$, determined for $E < -1$, $c^2 < 1$ (subluminal rotation)

$$f_{c,E}(z) = \begin{cases} -\arccos^{-1}\left[1 - 2\,\mathrm{cn}^2\left(\sqrt{\frac{1-E}{2(1-c^2)}}z; k\right)\right], & 0 \leqq z \leqq \frac{T}{2}, \\ \arccos^{-1}\left[1 - 2\,\mathrm{cn}^2\left(\sqrt{\frac{1-E}{2(1-c^2)}}(T-z); k\right)\right], & \frac{T}{2} \leqq z \leqq T, \end{cases}$$

$$k^2 = \frac{2}{1-E} \in (0,1), \qquad \text{elliptic modulus,}$$

$$\mathrm{cn} = \mathrm{cn}(\cdot), \qquad \text{elliptic cnoidal function}$$

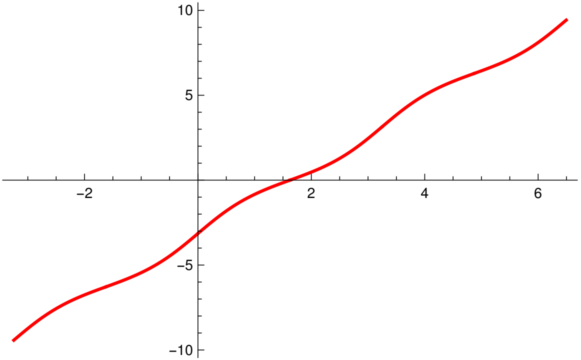

Figure : Rotational subluminal periodic wave $f = f_{c,E}(z)$ with $E = -2$, $c = 0.5$ in the interval $z \in [-T, 2T]$ (plot in red). Here the fundamental period is $T = 3.2476$.

Nonlinear Klein-Gordon equation

$$u_{tt} - u_{xx} + V'(u) = 0,$$

Potential: $V = V(u)$, nonlinear function of class C^2, periodic or not. (If periodic, we have the sine-Gordon case, essentially).

By a phase portrait analysis, one can analyze the existence of periodic waves, for example, for the quartic potential

$$V(u) = \tfrac{1}{2}u^2 - \tfrac{1}{4}u^4.$$

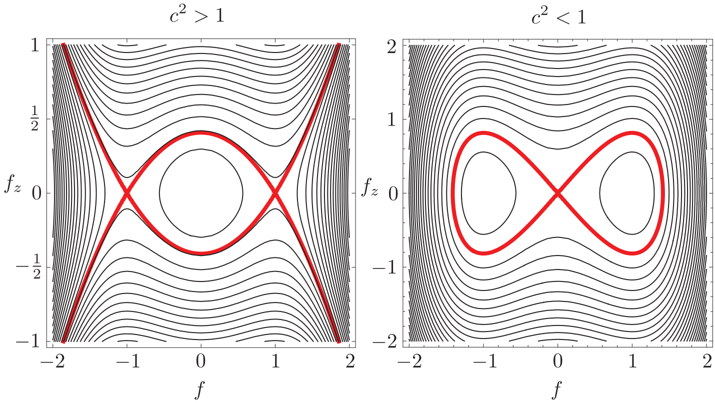

Figure : Phase portraits for $c = 2$ (left) and $c = 1/2$ (right) for a quartic potential $V(u) = \frac{1}{2}u^2 - \frac{1}{4}u^4$.

Benjamin-Ono (BO) equation

$$u_t + uu_x - \mathcal{H}u_{xx} = 0,$$

where $\mathcal{H}$ is the Hilbert transfrom:

$$\mathcal{H}g(z) = \frac{1}{2T}\text{P.V.}\int_{-T}^{T} \cot\left(\frac{\pi(z-y)}{2T}\right) g(y)\,dy$$

Periodic waves with period $2T$, for $c > \pi/T$ given by

$$f(z) = \frac{2\pi}{T}\frac{\sinh(\gamma)}{\cosh(\gamma) - \cos(\pi z/T)}$$

where γ solves $\tanh\gamma = \pi/CT$.

Ref.: Benjamin, J. Fluid Mech. **29** (1967)

General framework

System of PDEs (in one spatial dimension) of the form

$$u_t = \mathcal{A}(\partial_x)u + \mathcal{N}(u), \quad x \in \Omega \subset \mathbb{R},\ t > 0,$$

$u \in X$, X a Banach space, $\mathcal{A}(\zeta)$ is a vector valued polynomial in ζ, so that $\mathcal{A}(\partial_x) : X \to X$ is a closed, densely defined operator. $\mathcal{N} : X \to X$ denotes the nonlinearity.

Traveling wave solutions

$$u(x,t) = f(z), \qquad z := x - ct,\ c \in \mathbb{R},$$

where the profile function, $f : \mathbb{R} \to X$ is sufficiently smooth and T-periodic:

$$f(z+T) = f(z), \qquad \text{for all } z \in \mathbb{R}.$$

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$c \in \mathbb{R}$ is the **wave speed**. Introducing the coordinate $z = x - ct$ (Galilean variable) we recast the solutions in the form $u = u(z,t)$. Upon substitution:

$$u_t = \mathcal{A}(\partial_z)u + cu_z + \mathcal{N}(u), \qquad z \in \mathbb{R}, \;\; u \in X.$$

Thus, the profile f is a stationary solution:

$$0 = \mathcal{A}(\partial_z)f + cf_z + \mathcal{N}(f)$$

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The spectral problem

Linearization around f

Consider solutions of the form $u(z,t)+f(z)$, where now u represents a **perturbation**. Substituting and using the profile equation we obtain the **nonlinear equation for the perturbation**:

$$\begin{aligned} u_t &= \mathcal{A}(\partial_z)u + cu_z + \mathcal{N}(u+f) - \mathcal{N}(f) \\ &= \mathcal{A}(\partial_z)u + cu_z + \partial_u\mathcal{N}(f)u + O(\|u\|_X^2) \end{aligned}$$

Dropping the nonlinear terms we arrive at the **linearized equation for the perturbation**:

$$u_t = \mathcal{A}(\partial_z)u + cu_z + \partial_u\mathcal{N}(f)u.$$

By separation of variables, specializing to solutions of the form $u(z,t) = e^{\lambda t} w(z)$, where $\lambda \in \mathbb{C}$ and $w \in X$, we arrive at the **spectral problem** with $\lambda \in \mathbb{C}$ as an eigenvalue:

$$\mathcal{L}w = \lambda w, \qquad \lambda \in \mathbb{C}, \;\; w \in X,$$

$$\mathcal{L} : X \to X, \qquad \mathcal{L} := \mathcal{A}(\partial_z) + c\partial_z + \partial_u \mathcal{N}(f).$$

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Spectral stability (formal)

The stability of the wave is strongly related to the spectrum of $\mathcal{L}$. A **necessary condition** for the stability of f is that there are no points of spectrum with Re $\lambda > 0$, which would imply the existence of a solution of form $u = e^{\lambda t} w$ that grows exponentially in time.

This is, loosely speaking, **spectral stabilty.** (The precise definition will be made precise shortly in the case of periodic wavetrains).

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The spectral problem associated to a periodic wave

Consider the prototypical differential operator arising as the linearization of a nonlinear PDE around a periodic traveling wave,

$$\mathcal{L}w := \partial_z^n w + a_{n-1}(z)\partial_z^{n-1} w + \ldots + a_1(z)\partial_z w + a_0(z)w.$$

Recall $z = x - ct$, and the problem has been expressed in terms of the Galilean variable $z \in \mathbb{R}$. Here we assume that $\mathcal{L}$ has smooth T-**periodic coefficients**:

$$a_j(z+T) = a_j(z), \qquad j = 0, 1, \ldots, n-1,$$

for all $z \in \mathbb{R}$.

Resolvent and spectra

The definition of resolvent and spectra (and thus, that of spectral stability) depends upon the choice of the Banach space X where $\mathcal{L}$ is defined. Our choice is

$$X = L^2(\mathbb{R};\mathbb{C}), \qquad D = H^n(\mathbb{R};\mathbb{C}),$$

so that $\mathcal{L} : D \subset X \to X$ is a **closed, densely defined operator** in L^2. This choice corresponds to studying stability under **localized perturbations**.

Definition (resolvent and spectra)

We define the **resolvent**, **point spectrum** and **essential spectrum** of a closed, densely defined operator $\mathcal{L} : \mathcal{D} \subset X \to X$, with X a Banach space, as the sets:

$$\begin{aligned}
\rho(\mathcal{L}) &:= \{\lambda \in \mathbb{C} : \mathcal{L} - \lambda \text{ is injective and onto, and } \\
&\qquad (\mathcal{L} - \lambda)^{-1} \text{ is bounded}\}, \\
\sigma_{\text{pt}}(\mathcal{L}) &:= \{\lambda \in \mathbb{C} : \mathcal{L} - \lambda \text{ is Fredholm with index zero and } \\
&\qquad \text{has a non-trivial kernel}\}, \\
\sigma_{\text{ess}}(\mathcal{L}) &:= \{\lambda \in \mathbb{C} : \mathcal{L} - \lambda \text{ is either not Fredholm or } \\
&\qquad \text{has index different from zero}\}.
\end{aligned}$$

We define the **spectrum** of $\mathcal{L}$ as $\sigma(\mathcal{L}) = \sigma_{\text{ess}}(\mathcal{L}) \cup \sigma_{\text{pt}}(\mathcal{L})$.

Observations:

- For our purposes, $\sigma(\mathcal{L})$ will denote the L^2 spectrum
- Since $\mathcal{L}$ is closed, then $\rho(\mathcal{L}) = \mathbb{C}\backslash\sigma(\mathcal{L})$
- There are many definitions of essential spectra. The one presented here is due to Weyl (1910)
- Although it makes σ_{ess} a large set, it has the advantage that the remaining point spectrum σ_{pt} is a discrete set of isolated eigenvalues (Kapitula, Promislow, 2013)
- For solitary waves or fronts, σ_{ess} is easy to compute
- Reminder: an operator $\mathcal{L}$ is said to be Fredholm if its range $\mathcal{R}(\mathcal{L})$ is closed and both its nullity, $\text{nul}\,(\mathcal{L}) = \dim\ker\mathcal{L}$, and its deficiency, $\text{def}\,(\mathcal{L}) = \text{codim}\mathcal{R}(\mathcal{L})$, are finite. In such a case its index is defined as $\text{ind}(\mathcal{L}) = \text{nul}\,(\mathcal{L}) - \text{def}\,(\mathcal{L})$

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First order formulation

Alexander, Gardner, Jones (1990): The spectral problem can be recast as a first order system of the form

$$\mathbf{w}_z = \mathbf{A}(z,\lambda)\mathbf{w},$$

$$\mathbf{w} := \begin{pmatrix} w \\ \partial_z w \\ \vdots \\ \partial_z^{n-1} w \end{pmatrix} \in H^1(\mathbb{R};\mathbb{C}^n),$$

where the periodic coefficient matrices $\mathbf{A}(z,\lambda)$ are defined by

$$\mathbf{A}(z,\lambda) = \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ \lambda - a_0(z) & -a_1(z) & \cdots & -a_{n-2}(z) & -a_{n-1}(z) \end{pmatrix}$$

The coefficients are T-periodic, $\mathbf{A}(z+T,\lambda) = \mathbf{A}(z,\lambda)$ for all $z \in \mathbb{R}$, $\lambda \in \mathbb{C}$; analytic in $\lambda \in \mathbb{C}$ and bounded for all $z \in \mathbb{R}$.

Alternative definition of spectrum

Consider the family of closed, densely defined operators:

$$\mathcal{T}(\lambda) : D \subset X \to X$$

$$\mathcal{T}(\lambda)\mathbf{w} := \mathbf{w}_z - \mathbf{A}(z,\lambda)\mathbf{w}.$$

This family is indexed by $\lambda \in \mathbb{C}$. Once again we consider $X = L^2(\mathbb{R};\mathbb{C}^n)$, $D = H^1(\mathbb{R};\mathbb{C}^n)$.

Definition (cf. Sandstede (2002))

The **resolvent** ρ, the **point spectrum** σ_{pt} and the **essential spectrum** are defined as

$$\begin{aligned}\rho &:= \{\lambda \in \mathbb{C} : \mathcal{T}(\lambda) \text{ is one-to-one and onto, and} \\ &\qquad \mathcal{T}(\lambda)^{-1} \text{ is bounded}\}, \\ \sigma_{\text{pt}} &:= \{\lambda \in \mathbb{C} : \mathcal{T}(\lambda) \text{ is Fredholm with zero index} \\ &\qquad \text{and has a non-trivial kernel}\}, \\ \sigma_{\text{ess}} &:= \{\lambda \in \mathbb{C} : \mathcal{T}(\lambda) \text{ is either not Fredholm or} \\ &\qquad \text{has index different from zero}\}.\end{aligned}$$

The **spectrum** is $\sigma = \sigma_{\text{ess}} \cup \sigma_{\text{pt}}$. ($\mathcal{T}(\lambda)$ closed $\Rightarrow \rho = \mathbb{C} \backslash \sigma$.)

Definition

Assume $\lambda \in \sigma_{\rm pt}$. Its geometric multiplicity (*g.m.*) is the maximal number of linearly independent elements in $\ker \mathcal{T}(\lambda)$. Suppose $\lambda \in \sigma_{\rm pt}$ has $g.m. = 1$, so that $\ker \mathcal{T}(\lambda) =$ span $\{\mathbf{w}_0\}$. We say λ has algebraic multiplicity (*a.m.*) equal to m if we can solve $\mathcal{T}(\lambda)\mathbf{w}_j = (\partial_\lambda \mathbf{A}(z,\lambda))\mathbf{w}_{j-1}$, for each $j = 1, \ldots, m-1$, with $\mathbf{w}_j \in H^n$, but there is no bounded H^n solution $\mathbf{w}$ to $\mathcal{T}(\lambda)\mathbf{w} = (\partial_\lambda \mathbf{A}(z,\lambda))\mathbf{w}_{m-1}$. For an arbitrary eigenvalue $\lambda \in \sigma_{\rm pt}$ with $g.m. = l$, the algebraic multiplicity is defined as the sum of the multiplicities $\sum_k^l m_k$ of a maximal set of linearly independent elements in $\ker \mathcal{T}(\lambda) =$ span $\{\mathbf{w}_1, \ldots, \mathbf{w}_l\}$.

Lemma

For any Banach space X and any operator $\mathcal{L}$ as considered before,

$$\sigma_{pt} = \sigma_{pt}(\mathcal{L}), \quad \sigma_{ess} = \sigma_{ess}(\mathcal{L}), \quad \rho = \rho(\mathcal{L}).$$

Moreover, σ_{pt} and $\sigma_{pt}(\mathcal{L})$ coincide with same algebraic and geometric multiplicities.

Proof: Follows form the fact that $\mathcal{T}(\lambda)$ and $\mathcal{L} - \lambda$ have the same Fredholm properties for each $\lambda \in \mathbb{C}$, and there is a one-to-one correspondence between $\ker \mathcal{T}(\lambda)$ and $\ker(\mathcal{L} - \lambda)$ and between associated Jordan chains with the same length.

$\square$

All L^2 spectrum is "continuous"

It is well-known that the $L^2(\mathbb{R})$ spectrum of a differential operator with periodic coefficients contains no isolated eigenvalues. This fact can be verified easily with the new formulation of spectra in terms of the operators $\mathcal{T}(\lambda)$ and persists, for example, for more general spectral problems (e.g. the quadratic pencils), no necessarily in standard form.

Lemma

All L^2 spectrum in the periodic case is purely "continuous", that is, $\sigma = \sigma_{ess}$ and σ_{pt} is empty.

Proof: Let $\lambda \in \sigma_{\text{pt}}$. Then by definition, $\mathcal{T}(\lambda)$ is Fredholm with zero index and has a non-trivial kernel. This implies that $N := \ker \mathcal{T}(\lambda) \subset H^1(\mathbb{R};\mathbb{C}^n)$ is a finite dimensional Hilbert space. Let us denote $\mathcal{S} : L^2(\mathbb{R};\mathbb{C}^n) \to L^2(\mathbb{R};\mathbb{C}^n)$ as the (unitary) shift operator with period T, defined as $\mathcal{S}\mathbf{w}(z) := \mathbf{w}(z+T)$. Since the coefficient matrix $\mathbf{A}(z,\lambda)$ is periodic with period T, there holds $\mathcal{S}\mathcal{T}(\lambda) = \mathcal{T}(\lambda)\mathcal{S}$ in $L^2(\mathbb{R};\mathbb{C}^n)$, making N an invariant subspace of $\mathcal{S}$. Let us define $\hat{\mathcal{S}}$ as the restriction of $\mathcal{S}$ to N. Then $\hat{\mathcal{S}} : N \to N$ is a unitary map in a finite-dimensional Hilbert space. Therefore, $\hat{\mathcal{S}}$ must have an eigenvalue $\alpha \in \mathbb{C}$ such that $\mathcal{S}\mathbf{w}^0 = \alpha \mathbf{w}^0$ for some $\mathbf{w}^0 \in N \subset L^2(\mathbb{R};\mathbb{C}^n)$, $\mathbf{w}^0 \neq 0$.

Since $\hat{\mathcal{S}}$ is unitary, we have that $|\alpha| = 1$, whence

$$|\mathbf{w}^0(z+T)| = |(\hat{\mathcal{S}}\mathbf{w}^0)(z)| = |\alpha \mathbf{w}^0(z)| = |\mathbf{w}^0(z)|$$

(here $|\cdot|$ means the Euclidean norm in $\mathbb{C}^n$), that is, $|\mathbf{w}^0(z)|^2$ is T-periodic. But since $\mathbf{w}^0 \neq 0$, this is a contradiction with $\mathbf{w}^0 \in L^2(\mathbb{R};\mathbb{C}^n)$. Thus, σ_{pt} is empty and $\sigma = \sigma_{\text{ess}}$.

□

Floquet characterization of the spectrum

We are going to exploit the periodic nature of the spectral problem to characterize the spectrum in terms of **Floquet theory**.

Let $\mathbf{F}(z,\lambda)$ denote the fundamental solution matrix for the first order system,

$$\mathbf{F}_z(z,\lambda) = \mathbf{A}(z,\lambda)\mathbf{F}(z,\lambda),$$

with initial condition $\mathbf{F}(0,\lambda) = \mathbf{I}$, $\forall\lambda \in \mathbb{C}$.

The T-periodicity in z of the coefficient matrix $\mathbf{A}$ then implies that

$$\mathbf{F}(z+T,\lambda)=\mathbf{F}(z,\lambda)\mathbf{M}(\lambda), \quad \forall z\in\mathbb{R}, \quad \text{where} \quad \mathbf{M}(\lambda):=\mathbf{F}(T,\lambda)$$

(Indeed, by differentiation
$\partial_z\mathbf{F}(z+T,\lambda)=\mathbf{A}(x+T,\lambda)\mathbf{F}(z+T,\lambda)$ is also a solution.
By uniqueness theorem,
$\mathbf{F}(z+T,\lambda)=\mathbf{F}(z,\lambda)\mathbf{F}(T,\lambda)=\mathbf{F}(z,\lambda)\mathbf{M}(\lambda)$.)

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Properties:

- The monodromy matrix is really a representation of the linear mapping taking a given solution $\mathbf{w}(z,\lambda)$ evaluated for $z = 0 \pmod{T}$ to its value one period later
- Since the elements of the coefficient matrix $\mathbf{A}$ are entire functions of λ, and since the Picard iterates for $\mathbf{F}(z,\lambda)$ converge uniformly for bounded z, the elements of the monodromy matrix $\mathbf{M}(\lambda)$ are also entire functions of $\lambda \in \mathbb{C}$

Floquet multipliers

Let $\mu(\lambda)$ denote an eigenvalue of $\mathbf{M}(\lambda)$, and let $\mathbf{w}^0(\lambda) \in \mathbb{C}^n$ denote a corresponding (nonzero) eigenvector:

$$\mathbf{M}(\lambda)\mathbf{w}^0(\lambda) = \mu(\lambda)\mathbf{w}^0(\lambda)$$

Then $\mathbf{w}(z,\lambda) := \mathbf{F}(z,\lambda)\mathbf{w}^0(\lambda)$ is a nontrivial solution of the first-order system that satisfies

$$\begin{aligned}\mathbf{w}(z+T,\lambda) &= \mathbf{F}(z+T,\lambda)\mathbf{w}^0(\lambda) = \mathbf{F}(z,\lambda)\mathbf{M}(\lambda)\mathbf{w}^0(\lambda)\\ &= \mu(\lambda)\mathbf{F}(z,\lambda)\mathbf{w}^0(\lambda) = \mu(\lambda)\mathbf{w}(z,\lambda), \quad \forall z \in \mathbb{R}.\end{aligned}$$

for all $z \in \mathbb{R}$.

Thus $\mathbf{w}(z,\lambda)$ is a particular solution that goes into a multiple of itself upon translation by a period in z. These solutions are the classical **Floquet solutions**, and the eigenvalue $\mu(\lambda)$ of the monodromy matrix $\mathbf{M}(\lambda)$ is called a **Floquet multiplier**.

Denote by $b(\lambda)$ modulo 2π,

$$e^{b(\lambda)} = \mu(\lambda).$$

Then, it is evident that $e^{-b(\lambda)z/T}\mathbf{w}(z,\lambda)$ is a T-periodic function of z, or, equivalently (Floquet Theorem) $\mathbf{w}(z,\lambda)$ can be written in the form

$$\mathbf{w}(z,\lambda) = e^{b(\lambda)z/T}\mathbf{p}(z,\lambda),$$

where $\mathbf{p}(z+T,\lambda) = \mathbf{p}(z,\lambda)$, $\forall z \in \mathbb{R}$. The quantity $b(\lambda)$ is a **Floquet exponent**.

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Lemma

$\lambda \in \sigma$ if and only if there is at least one Floquet exponent with Re $b(\lambda) = 0$

Proof: From standard Floquet theory, if the first-order system has a nontrivial solution in $L^\infty(\mathbb{R};\mathbb{C}^n)$, it is necessarily a linear combination of solutions having the form $\mathbf{w}(z,\lambda) = e^{b(\lambda)z/T}\mathbf{p}(z,\lambda)$ with $b(\lambda)$ purely imaginary, that is, it is a superposition of Floquet solutions corresponding to Floquet multipliers $\mu(\lambda)$ with $|\mu(\lambda)| = 1$, yielding Re $b(\lambda) = 0$. Conversely, if there is at least one Floquet exponent with Re $b(\lambda) = 0$, the same procedure yields at least one non-trivial L^2 solution to the first order system, and $\lambda \in \sigma$.

□

As a consequence, the L^2 spectrum can be characterized in terms of the monodromy matrix as follows:

Lemma

$\lambda \in \sigma$ if and only if there exists $\mu \in \mathbb{C}$ with $|\mu| = 1$ such that

$$\det(\mathbf{M}(\lambda) - \mu \mathbf{I}) = 0,$$

that is, at least one of the Floquet multipliers lies on the unit circle.

Proof sketch: σ consists entirely of essential spectrum. Moreover, $\lambda \in \sigma_{\text{ess}}$ if and only if the system admits a non-trivial, uniformly bounded solution in L^2. Any such solution is necessarily a linear combination of Floquet solutions with multipliers μ satisfying $|\mu| = 1$.

It is not difficult to verify that $\mathcal{T}(\lambda)$ has a bounded inverse provided all Floquet exponents have non-zero real part (cf. Gardner, 1993, Proposition 2.1). Hence $\lambda \in \sigma = \sigma_{\text{ess}}$ if and only if there exists a eigenvalue of $\mathbf{M}(\lambda)$ of the form $\mu = e^{i\theta}$ with $\theta \in \mathbb{R}$.

□

Floquet spectrum

It is possible to parametrize the spectrum in terms of the Floquet multipliers $\mu = e^{i\theta} \in S^1$, or equivalently $\theta \in \mathbb{R} \pmod{2\pi}$. Let us define the set σ_θ as the set of complex numbers λ for which there exists a nontrivial solution of the boundary-value problem

$$\mathbf{w}_z = \mathbf{A}(z,\lambda)\mathbf{w}, \quad \mathbf{w}(T) = e^{i\theta}\mathbf{w}(0), \qquad \theta \in \mathbb{R}$$

Clearly $\sigma_\theta = \sigma_{\theta+2\pi k}$, for all $k \in \mathbb{Z}$. We thus define the **Floquet spectrum** σ_F as:

$$\sigma_F := \bigcup_{-\pi<\theta\leq\pi} \sigma_\theta$$

From previous arguments its clear that:

- $\sigma = \sigma_F$ (see Lemma above)
- Each set σ_θ is **discrete**: it is the zero set of the analytic function $\det(\mathbf{M}(\lambda) - e^{i\theta}\mathbf{I}) = 0$
- When $\theta = 0$, then the boundary conditions become periodic and σ_0 detects perturbations which are co-periodic
- By a symmetric argument: σ_π detects anti-periodic perturbations

Bloch wave decomposition

It is convenient to remove the θ-dependence associated with the boundary conditions. The change of variables $\mathbf{y} = e^{-i\theta z/T}\mathbf{w}$ transforms the first order boundary value problem into a system with periodic and θ-independent boundary conditions:

$$\mathbf{y}_z = \widetilde{\mathbf{A}}(z, \lambda, \theta)\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{y}(T),$$

$$\widetilde{\mathbf{A}}(z, \lambda, \theta) := \mathbf{A}(z, \lambda) - (i\theta/T)\mathbf{I},$$

Reminder: periodic Sobolev spaces are defined as

$$H^s_{\mathrm{per}}([0,T];\mathbb{C}^n) = \{u \in H^s_{\mathrm{loc}}(\mathbb{R};\mathbb{C}^n) : u(z+T) = u(z) \text{a.e. in } z\}$$

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Lemma

$\lambda \in \sigma = \sigma_F$ if and only if for some $-\pi < \theta \leq \pi$ there exists a non-trivial solution $\mathbf{y} \in H^1_{\mathrm{per}}([0,T];\mathbb{C}^n)$ to the boundary value problem

$$\mathbf{y}_z = \widetilde{\mathbf{A}}(z,\lambda,\theta)\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{y}(T).$$

Proof: The above system possesses a nontrivial solution $\mathbf{y} \in H^1_{\mathrm{per}}([0,T];\mathbb{C}^n)$ for some $\lambda \in \mathbb{C}$ and $-\pi < \theta \leq \pi$ if and only if $\mathbf{w} = e^{iz/T}\mathbf{y}$ solves the original system and $\lambda \in \sigma_F = \sigma$.

□

Denote by $\tilde{\mathbf{F}} = \tilde{\mathbf{F}}(z, \lambda, \theta)$ the identity-normalized fundamental matrix of the system $\mathbf{y}_z = \widetilde{\mathbf{A}}(z, \lambda, \theta)$. We define the **Bloch monodromy matrix** as

$$\widetilde{\mathbf{M}}(\lambda, \theta) := \tilde{\mathbf{F}}(T, \lambda, \theta).$$

Therefore, an immediate corollary is the following:

Corollary

$\lambda \in \sigma$ *if and only if* $\widetilde{\mathbf{M}}(\lambda, \theta)$ *has an eigenvalue equal to one, for some* $\theta \in (-\pi, \pi]$.

One of the advantages of the “periodic” formulation is that we may pose the system in a proper periodic space (such as $H^1_{\text{per}}([0,T];\mathbb{C}^n)$), independently of (but indexed by) $\theta \in (-\pi,\pi]$. It is strongly related to the so-called **Bloch-wave decomposition**, which involves a different transformation:

Consider the original spectral problem in terms of the periodic coefficient operator $\mathcal{L}w = \lambda w$. Make the transformation,

$$w(z) =: e^{i\theta z/T} v(z) \;\Rightarrow\; \partial_z^j w = e^{i\theta z/T} (\partial_z + i\theta/T)^j v(z),$$

for all $j = 0, 1, \ldots, n-1$.

Then the operator $\mathcal{L}$ transforms into the θ-dependent operator

$$\mathcal{L}_\theta v := (\partial_z + i\theta/T)^n v + a_{n-1}(z)(\partial_z + i\theta/T)^{n-1} v + \ldots$$
$$\ldots + a_1(z)(\partial_z + i\theta/T)v + a_0(z)v,$$

and the boundary conditions of form $\mathbf{w}(T) = e^{i\theta}\mathbf{w}(0)$ (namely, $(\partial_z^j w)(T) = e^{i\theta}(\partial_z^j w)(0)$) transform into

$$\partial_z^j v(T) = \partial_z^j v(0), \quad j = 0, 1, \ldots, n-1.$$

We may thus define a **family of Bloch operators** in periodic spaces:

$$\mathcal{L}_\theta : D = H^n_{\text{per}}([0,T];\mathbb{C}^n) \to L^2_{\text{per}}([0,T];\mathbb{C}^n), \quad \theta \in (-\pi, \pi].$$

Equivalence

If we write the above spectral problem as a first order system via $\mathbf{v} = (v, \partial_z v, \ldots, \partial_z^{n-1} v)^\top$, then we may calculate

$$\mathbf{w}(z) = e^{i\theta z/T}\mathbf{T}(\theta/T)\mathbf{v}(z),$$

where

$$\mathbf{T}(\zeta) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ i\zeta & 1 & 0 & 0 & 0 & \cdots & 0 \\ (i\zeta)^2 & 2(i\zeta) & 1 & 0 & 0 & \cdots & 0 \\ (i\zeta)^3 & 3(i\zeta)^2 & 3(i\zeta) & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (i\zeta)^{n-1} & n(i\zeta)^{n-2} & \cdots & \cdots & \cdots & n(i\zeta) & 1 \end{pmatrix}$$

(non-singular matrix).

Thus, the spectral problem takes the **Bloch wave form**

$$\mathbf{v}_z = \mathbf{T}(\theta/T)\big(\mathbf{A}(z,\lambda) - (i\theta/T)\mathbf{I}\big)\mathbf{T}(\theta/T)^{-1}\mathbf{v}, \qquad \mathbf{v}(T) = \mathbf{v}(0).$$

By making

$$\mathbf{y}(z) := \mathbf{T}(\theta/T)^{-1}\mathbf{v}(z),$$

we recover the system

$$\mathbf{y}_z = \widetilde{\mathbf{A}}(z,\lambda,\theta)\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{y}(T).$$

As an immediate consequence we have

Lemma

The (continuous) L^2 spectrum of the periodic operator $\mathcal{L}$ is the union of the (discrete) spectra, $\cup_{\theta \in (-\pi,\pi]} \sigma(\mathcal{L}_\theta)$, of the family of associated Bloch operators acting on $L^2_{\text{per}}([0,T];\mathbb{C})$

Spectral stability

Definition

We say that a spatially periodic traveling wave solution is **spectrally stable** if the spectrum of its linearization σ (or equivalently, $\sigma = \sigma(\mathcal{L}) = \cup_{\theta \in (-\pi,\pi]} \sigma_\theta$) is contained in $\{\lambda \in \mathbb{C} : \text{Re}\, \lambda \leq 0\}$.

The Bloch transform

Generic differential operator

$$\mathcal{L}u := \partial_x^n u + a_{n-1}(x)\partial_x^{n-1} u + \ldots + a_1(x)\partial_x u + a_0(x)u,$$

with 2π-periodic coefficients, $a_j(x+2\pi) = a_j(x)$, $\forall x \in \mathbb{R}$. Suppose $a_j(\cdot) \in C_0^\infty(\mathbb{R})$.

Fourier series expansion:

$$a_j(x) = \sum_{k\in\mathbb{Z}} a_j^k e^{ikx}$$

$$a_j^k = \frac{1}{2\pi}\int_{-\pi}^{\pi} a_j(x)e^{-ikx}\,dx$$

Motivation: Suppose $u \in \mathcal{S}(\mathbb{R})$ (Schwarz class). Then taking the Fourier transform,

$$\begin{aligned}
(\widehat{a_j(x)\partial_x^j u(x)})(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} a_j(x) \partial_x^j u(x) e^{-i\xi x}\, dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} a_j^k e^{ikx} \partial_x^j u(x) e^{-i\xi x}\, dx \\
&= \sum_{k \in \mathbb{Z}} \frac{a_j^k}{\sqrt{2\pi}} \int_{\mathbb{R}} \partial_x^j u(x) e^{-i(x-k)}\, dx \\
&= \sum_{k \in \mathbb{Z}} a_j^k \widehat{(\partial_x^j u)}(\xi - k) \\
&= \sum_{k \in \mathbb{Z}} a_j^k (i(\xi - k))^j \hat{u}(\xi - k).
\end{aligned}$$

Thus, $\widehat{\mathcal{L}u}(\xi)$ depends only on the values $\hat{u}(\xi - k)$ for $k \in \mathbb{Z}$.

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Thus, $\widehat{\mathcal{L}u}(\xi)$ depends only on the values $\hat{u}(\xi - k)$ for $k \in \mathbb{Z}$.

The Bloch transform will diagonalize periodic coefficient operators in the same way the Fourier transform diagonalizes constant coefficient operators.

Bloch periodic functions

Definition

Let $T > 0$ be a reference fundamental period. For each $\eta \in \mathbb{R}$, a function $f : \mathbb{R} \to \mathbb{R}$ is said to be (η, T)-**Bloch periodic** if

$$f(x + kT) = e^{i2\pi k\eta} f(x),$$

for all $x \in \mathbb{R}$, and all $k \in \mathbb{Z}$.

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Observations:

- The case of a $(0,T)$-Bloch periodic function coincides with the definition of a T-periodic function
- If each $\eta \in \mathbb{R}$ is replaced by $\eta + m$ with $m \in \mathbb{Z}$, the bondary condition remains unchanged

$$f(x+kT) = e^{i2\pi k(\eta+m)}f(x) = e^{i2\pi k\eta}f(x).$$

Thus it suffices to consider $\eta \in (-1/2, 1/2]$.

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Theorem (Bloch decomposition)

Given $f \in L^2(\mathbb{R})$ there exists a unique function, called its ***Bloch transform***, $\mathcal{B}f := f_b \in L^2([0,2\pi)) \times (-1/2, 1/2])$ *such that*

$$f(x) = \int_{-1/2}^{1/2} f_b(x,\eta) e^{i\eta x}\, dx.$$

In addition, for any $f, g \in L^2(\mathbb{R})$ the Plancherel theorem holds,

$$\int_{\mathbb{R}} f(x) g(x)^* \, dx = \int_0^{2\pi} \int_{-1/2}^{1/2} f_b(x,\eta) g_b(x\eta)^* e^{i\eta x}\, dx.$$

In particular, the Bloch transform $f \mapsto \mathcal{B}f = f_b$ is an isometry from $L^2(\mathbb{R})$ to $L^2([0,2\pi)) \times (-1/2, 1/2])$

Proof sketch:

For any $f \in C_0^\infty(\mathbb{R})$ and each $\eta \in (-1/2, 1/2]$ define

$$f_b(x,\eta) := \sum_{k\in\mathbb{Z}} f(x+2\pi k)e^{-i(x+2\pi k)\eta}, \quad \text{(Bloch transform).}$$

Clearly, $f_b(x,\eta)$ is 2π-periodic in x,

$$f_b(x+2\pi,\eta) = \sum_{k+1\in\mathbb{Z}} f(x+2\pi k)e^{-i(x+2\pi k)\eta} = f_b(x,\eta)$$

Similarly, $e^{ix\eta}f_b(x,\eta)$ is $(-1/2, 1/2]$-periodic in η (integer periodic):

$$\begin{aligned} e^{ix(\eta+m)} f_b(x,\eta+m) &= e^{ix(\eta+m)} \sum_{k\in\mathbb{Z}} f(x+2\pi k) e^{-i(x+2\pi k)(\eta+m)} \\ &= e^{ix(\eta+m)} e^{-ixm} \sum_{k\in\mathbb{Z}} f(x+2\pi k) e^{-i(x+2\pi k)\eta} e^{-i2\pi k m} \\ &= e^{ix\eta} \sum_{k\in\mathbb{Z}} f(x+2\pi k) e^{-i(x+2\pi k)\eta} \\ &= e^{ix\eta} f_b(x,\eta). \end{aligned}$$

Observe that

$$e^{ix\eta} f_b(x,\eta) = \sum_{k\in\mathbb{Z}} f(x+2\pi k) e^{-i2\pi k\eta}$$

Hence,

$$\begin{aligned}\int_{-1/2}^{1/2} e^{ix\eta} f_b(x,\eta) &= f(x) + \sum_{k\neq 1} f(x+2\pi k) \int_{-1/2}^{1/2} e^{-2\pi i k\eta}\, d\eta \\ &= f(x) - \sum_{k\neq 0} f(x+2\pi k) \left(\frac{e^{-i\pi k} - e^{i\pi k}}{2i\pi k} \right) \\ &= f(x)\end{aligned}$$

Similarly, Plancherel can be proved and, by density, the Bloch transform can be extended to L^2.

□

Observation:

From the Bloch transformation theorem we readily have that

$$\mathcal{B}(\mathcal{L}u)(x,\eta) = \mathcal{L}_\eta(f_b(\cdot,\eta))(x),$$

where $\mathcal{L}_\eta$ is the family of Bloch operators,

$$\mathcal{L}_\eta = (\partial_x + i\eta)^n + a_{n-1}(x)(\partial_x + i\eta)^{n-1} + \ldots + a_0(x),$$

hence the associated Bloch operators can be viewed as operator-valued symbols under $\mathcal{B}$ acting on $L^2_{\text{per}}([0,2\pi])$.

Thank you!