Universidad Nacional Autónoma de México
Programa de Maestría y Doctorado en Ciencias Matemáticas y de la Especialización en Estadística Aplicada

Bifurcation and Dynamics in Hyperbolic Burgers-Fisher Equation

T E S I N A

QUE PARA OPTAR POR EL GRADO DE:
MAESTRO EN CIENCIAS

PRESENTA:
Ricardo Yadel Murillo Pérez

DIRECTOR DE LA TESINA:
Dr. Ramón Gabriel Plaza Villegas
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas (IIMAS-UNAM)

Ciudad de México. Abril 2021

Murillo Pérez Ricardo Yadel

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To my parents
to María Elena Murillo

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Acknowledgments

The work present here is the result of the efforts of many people who with their support contributed to its development and conclusion. Without them, this work would not have been possible. The successes that may be in it are the result of their comments and contributions. However, any errors that may exist are my own.

My first thanks are for my parents, Alma Rosa Pérez Pérez and Rubén Murillo Loza for their unconditional support. Next I must thank Lizbeth Ramírez for her advises and support. I thank my colleagues and friends who gave me their support and words of encouragement.

Introduction

In recent years there has been increasing interest in the convection-reaction-diffusion equations due mainly in the great amount to applications in which they naturally emerge. For instance, Burgers-Fisher equation has many applications a lot of branches of physics, biology, medicine, etc.

The Burgers-Fisher equation can be considered as a viscous balance law and models convective transport that includes a diffusive phenomenon. In first place we will show that the Fisher equation combined with Burgers equation give us a viscous balance law. The importance of Burgers-Fisher equation arises mainly in the context of population dynamical, particularly in chemotaxis phenomenona. Viscous conservation laws are parabolic type with the deffective of infinite propagation velocity of its solutions. That is why has been proposed before in the literature coupling the Burgers-Fisher equation with the delay Cattaneo-Maxwell equation. The resulting system is hyperbolic one with finite propagation velocity of its solutions.

The main topic in this work is to investigate the dynamical behavior of a scalar variable u subject to a transport mechanism of hyperbolic type coupled with a reaction process. The qualitative behaviour of the parabolic Burgers-Fisher equation has been studied in [16]. In this work, we will show that the hyperbolic Burgers-Fisher equation presents in part a similar qualitative behaviour in its travelling wave solutions. We will prove the existence of a family of bounded periodic travelling wave solutions with finite fundamental period from suitable assumptions. The existence emerge from Hopf bifurcation around a critical value of the wave speed.

Chapter 1

Hyperbolic Burgers-Fisher Equation

In this first chapter we are concerned with a well understanding of the Hyperbolic Burgers-Fisher model. The chapter has two principal parts. In the first part, is observed that Burgers-Fisher model is a standard viscous balance law. Each component in the standard viscous balance law has a meaning which is explained. Further, we are concerned with a correct understanding of the very particular features in the Burgers-Fisher model as a viscous balance law. In the second part, it is observed a defect in the model. Burguers-Fisher equation is a parabolic one and its classical solutions have infinite speed of propagation. In order to avoid this feature, the system is coupled with Maxwell-Cattaneo transfer law and we obtain a new hyperbolic model called the hyperbolic Burgers-Fisher equation.

1.1 Burgers-Fisher Equation

Nonlinear Partial Differential Equations are a powerful tools in mathematical modelling of real processes arising in physics, chemistry, biology, ecology, economy, medicine, etc. In recent years, Burgers-Fisher equation (and in general all the called *nonlinear-reaction-diffusion equations*) has received increasing attention due to its real world application [2]. The next equation is known as the Burgers-Fisher equation,

$$u_t + uu_x = u_{xx} + u(1-u)$$

The term uu_x is the *convective term*, the term u_{xx} is a *diffusive term* and $u(u-1)$ is a *reaction term*.

Burgers-Fisher equation is a combination of two famous models, the *Fisher-KPP equation* and the *Burgers equation*. Both of them have a general form of a *standard balance law*. A standard balance law describes how a conserved quantity diffuses obeying a balance function. Every standard balance law has the following form

$$u_t + f(u)_x = g(u)$$

with $(x,t) \in \mathbb{R} \times [0,\infty)$, $u(x,t) \in \Omega \subseteq \mathbb{R}$, where Ω is an open and connected region. A standard balance law asserts that a quantity u diffuses according a flux function $f \in C^2(\mathbb{R})$ and grows or decays like the balance function $g \in C^2(\mathbb{R})$.

Including the diffusive term u_{xx}, the standard balance law receives the name of a *viscous balance law*. The viscous balance law includes *viscous effects* and from the physical point of view is more complete. The inclusion of the viscous effects is obtained adding the viscous term (diffusive term) ϵu_{xx} in the balance law as follows

$$u_t + f(u)_x = \epsilon u_{xx} + g(u).$$

The meaning of the viscous term can be interpreted depending of the context. In a fluid, the viscosity term represents the resistance to deformation. In general, the inclusion of the viscous term permet the model represent the appearance of a diffusive phenomenon. The viscous term makes the system parabolic and it thus possesses only smooth solutions [15]

1.1.1 Burgers and Fisher models

On one hand, we have the Fisher-KPP equation. The Fisher-KPP equation can be seen as an example of a viscous balance law as follows. Let us consider the general form of the scalar viscous balance law

$$u_t + f(u)_x = u_{xx} + g(u)$$

where u is the conserved quantity. We take the particular flux $f \equiv 0$ and then the equation turns into the semilinear parabolic equation

$$u_t = u_{xx} + g(u).$$

Now the balance term $g(u)$ is called the *reaction term*. Taking the particular reaction term $g(u) := u(1-u)$, the resultant equation is the Fisher-KPP equation:

$$u_t = u_{xx} + u(1-u)$$

The Fisher-KPP equation was originally proposed like a model for gene spread [11]. Fisher-KPP equation models diffusion in a closed environment, it represents diffusive dispersal and logistic population growth. $g(0) = 0$ represents not spread at all and $g(1) = 0$ a complete spread in a population.

On the other hand, we have the Burgers equation. Let us take in the general balance law the flux function $f(u) = \frac{1}{2}u^2$ with no balance term $g(u) = 0$ (there is no flux of conserved quantity u in or out the region) and we obtain the *inviscid Burguers equation*:

$$u_t + uu_x = 0,$$

where the term uu_x is called *convective term.* The inviscid Burgers equation can be interpreted as a simplification of the Navier-Stokes equations that describes the dynamical behaviour of an ideal fluid. Taking into account viscous effects in a fluid we add the term ϵu_{xx} and we obtain the *viscous Burgers equation*

$$u_t + uu_x = \epsilon u_{xx},$$

As it is pointed out in [15], the solutions to Burgers equation without viscous term usually produce discontinuities called shocks waves. However, when the viscosity is present, all the discontinuities that appears turn out into smooth transition layers.

Now, adding a balance term like Fisher-KPP reaction term, we obtain the Burgers-Fisher equation:

$$u_t + uu_x = u_{xx} + u(1 - u).$$

The Burgers-Fisher equation represents the non-linear convective movement in closed environments for a density population including diffusive effects. The equation models the non-linear convective movement in which individuals move to neighbouring regions with lower densities more rapidly as the population gets more crowded. Such equation arises in a variety of contexts like mathematical biology modelling individual populations by transport phenomena, mainly in *chemotaxis phenomena.*

1.1.2 Chemotaxis

Chemotaxis refers to several mechanisms through which individuals in a population can move in response to an external chemical signal. For example, insects and animals usually produce chemical signals like *pheromones* as a sex attractant, etc. Bacterial or unicellular populations can move only in response to this signal. These organisms have several biological devices for moving. For example [12], the bacteria *Escherichia coli* is are composed of flagella, *Proteus mirabilis* can swim, creating an adequate substance, *Amebae Dictyostelium discoıdeum* crawl by sending an internal arm, etc.

The understanding of Chemotaxis is of fundamental importance in medical research. Chemotactic activity can be altered due to pharmaceutical agents. For example, when a bacterial infection invades the body it can be attacked by "moving" leuckocyte cells towards the region of bacterial inflammation or the infection can be decreased or inhibited.

The Burgers-Fisher equation arises naturally in the study of pattern formation by bacterial colonies. Bacteria have several biological devices for moving themselves. Bacterial movement is a random process which can be approximated by diffusion. The movement of bacterial obeys a convective transport in a spatial homogeneous situation with limited availability of nutrients that affects the reproduction of bacteria.

Chemotaxis phenomenon is a rather complex one. However, the movement in a cell population can be modelled taking into account the interaction between chemical concentration and the density attracted cells. Let us denote by $\rho(x,t)$ the density of bacteria chemotactic to a single chemical element of concentration $s(x,t)$ with $(x,t) \in \mathbb{R} \times [0,\infty)$. The density ρ, with logistic growth, evolves according to,

$$\rho_t = (D\rho_x - \rho\chi s_x)_x + \rho(1-\rho)$$

where $D > 0$ is a diffusive constant and $\chi > 0$ is chemotactic factor.

In cases where the rate of chemical consumption is due mainly to the ability of the bacteria to consume it we have

$$s_t = -k\rho.$$

where $k > 0$ is a diffusive constant. If we now look for traveling wave solutions $s = s(x-ct)$ and $\rho = \rho(x-ct)$, then $s_t = -cs_x$, so $s_x = \frac{k\rho}{c}$. The problem reduces to following partial differential equation for ρ,

$$\rho_t = D\rho_{xx} - \frac{\chi k}{c}(\rho^2)_x + \rho(1-\rho).$$

The last equation is Burgers-Fisher equation. In next section is considered a refinement of Burgers-Fisher equation. Coupling Burgers-Fisher equation with Cattaneo-Maxwell equation we have a better model as we will show.

1.2 Hyperbolic Model

In the previous section we mentioned the interpretation of Burgers-Fisher equation. Nevertheless in spite of its usefulness, the model is defective and unrealistic. Burgers-Fisher model is classified like a non-linear parabolic equation and have the same criticism that the standard linear diffusion equation and Fisher-KPP equation concerning the infinite speed of propagation of disturbances. Even for the general linear parabolic equation it is proved [4] that if the coefficients are bounded and uniformly Hölder continuous, the equation possesses a positive fundamental equation. It follows that under the same assumptions on the coeficients [5], the general reaction-diffusion-convection equation has classical solutions with infinite speed of propagation.

To avoid this difficulty, in [6] a coupled Burgers-Fisher equation with Maxwell-Cattaneo transfer law was proposed. In Burgers-Fisher equation, we take $v = u_x$ giving the equation a conservative form with flux function v

$$u_t + uu_x = v_x + u(1-u).$$

To circumvent the problem of infinite speed of propagation in the solutions, let us assume the flux does not depend instantaneously at a point. We replace the equation,

$$v(x,t) = u_x(x,t)$$

by a delay equation,

$$v(x,t+\tau) = u_x(x,t).$$

This way, there is a short time before the effect is felt. Let us write the Taylor's series expansión,

$$v(x,t+\tau) = v(x,t) + \tau\frac{\partial v}{\partial t}(x,t) + O(\tau^2).$$

The terms $O(\tau^2)$ are neglected and substitute in order to obtain the Maxwell-Cattaneo transfer law:

$$\tau v_t + v = u_x.$$

The equation states that the flux v relaxes toward u_x in a time-scale $\tau > 0$. By its form, Maxwell-Cattaneo equation is of hyperbolic type. We now observe that this is the classical form of an hyperbolic system with unknowns density u and flux v,

$$\begin{cases} u_t + uu_x = v_x + u(1-u) & x \in \mathbb{R}, t \geq 0. \\ \tau v_t + v = u_x \end{cases}$$

with initial conditions

$$\begin{aligned} u(x,0) &= u_0(x) \qquad x \in \mathbb{R} \\ v(x,0) &= v_0(x) \end{aligned}$$

Let us note that, by the *Kac's trick* we can eliminate the variable v. Differentiate the first equation with respect to t obtaining,

$$v_{xt} = u_{tt} + (uu_x)_t - (u(1-u))_t.$$

Differentiate the second equation with respect to x,

$$\tau v_{tx} + v_x = u_{xx}.$$

Substituing v_{xt} and adding $u_t + uu_x$ we obtain

$$u_t + uu_x + \tau(u_{tt} + (uu_x)_t - (u(1-u))_t) + v_x = u_t + uu_x + u_{xx}.$$

Thus

$$u_t + uu_x + \tau(u_{tt} + (uu_x)_t - (u(1-u))_t) = u_{xx} + u(1-u).$$

with initial conditions

$$\begin{aligned} u(x,0) &= u_0(x) \qquad x \in \mathbb{R} \\ u_t(x,0) &= -u_0(x)u_0'(x) + v_0'(x) + u_0(x)(1-u_0(x)) \end{aligned}$$

Now, if we take limit $\tau \downarrow 0$, then we recover Burgers-Fisher equation.

The hyperbolic coupled system can be interpreted as a model for a reaction-diffusion-convection process. By hyperbolicity, the new model has the realistic feature of finite speed of propagation of disturbances, correcting in this way the parabolic defective feature for reaction-diffusión-convection model established in Burgers-Fisher equation. Even more, it has been showed [3] that coupling

Burgers-Fisher equation with Cattaneo-Maxwell equation produces a better adjustment to experimental research [3].

Finally, Burgers-Fisher equation models convective and diffusive movement in a limited space. That is why the model has particular importance in modelling population phenomena like chemotaxis in which bacteria are dispersed in a closed environment attracted by a particular chemical stimulus.

The next chapter is devoted to the qualitative analysis of the travelling wave solutions of the hyperbolic Burguers-Fisher equation. The Hopf bifurcation theorem for planar systems is the main tool for proving the conditions of existence of a bounded periodic travelling wave solution of small amplitud. Those wave solutions emerge from a Hopf bifurcation around a critical value of the speed.

Chapter 2

Periodic Waves

This chapter is devoted to the qualitative analysis of the hyperbolic Burgers-Fisher equation. The Hopf bifurcation theorem is the main tool to prove the existence of a bounden periodic travelling wave solution. The first section shows the Hopf bifurcation theorem for planar systems and explains its meaning. The second section presents the main result and is proof, with the aid of Hopf bifurcation theorem. The main result asserts the existence and conditions of existence of a bounded periodic travelling wave solution of small amplitude to hyperbolic Burgers-Fisher equation.

2.1 Hopf Bifurcation Theorem

In general way, bifurcation theory works on systems of differential equations that depend on parameters and detects changes in the structure of solutions. The qualitative behavoir of solutions may change when the parameters change. The parameter values in which the structure of the solutions change is called bifurcation values.

There are many kinds of bifurcation. A Hopf Bifurcation occurs when a periodic solution or limit cycle, surrounding an equilibrium point arises as a parameter in a system of differential equations varies. When a stable limit cycle surrounds an unstable equilibrium point, the bifurcation is called a *supercritical Hopf bifurcation*. If the limit cycle is unstable and surrounds a stable equilibrium point, then the bifurcation is called a *subcritical Hopf bifurcation* [10].

Next, we present the version of the Hopf Bifurcation Theorem in two dimensions. The following version was read in [10]. The Theorem reads as follows,

Theorem 1 (Hopf Bifurcation). *Let us consider the planar system,*

$$\begin{aligned} \dot{x} &= f_\mu(x, y) \\ \dot{y} &= g_\mu(x, y) \end{aligned} \tag{2.1}$$

where μ is the parameter. Let us suppose there is a fixed point. Without loss of generality we may assume that the fixed point is located at the origin, $(x, y) =$

$(0,0)$. *Let the eigenvalues of the linearised system about the fixed point be given by,*

$$\lambda(\mu) = \alpha(\mu) + \beta(\mu)$$
$$\bar{\lambda}(\mu) = \alpha(\mu) - \beta(\mu) \tag{2.2}$$

Suppose further that for a certain value of μ, which we may assume to be 0, the following conditions are satisfied:

1. *(non-hyperbolicity condition: conjugate pair of imaginary eigenvalues)*

$$\alpha(0) = 0$$
$$\beta(0) = \omega \neq 0 \tag{2.3}$$

where $sgn(\omega) = sgn\left(\frac{\partial g_\mu}{\partial x}\right)|_{\mu=0}(0,0)$

(transversality condition: the eigenvalues cross the imaginary axis with non-zero speed)

$$\frac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\mu}|_{\mu=0} = d \neq 0 \tag{2.4}$$

(genericity condition)

$$a \neq 0$$

where

$$a = \frac{1}{16}(f_{xxx}+f_{xyy}+g_{xxy}+g_{yyy})+\frac{1}{16\omega}(f_{xy}(f_{xx}+f_{yy})-g_{xy}(g_{xx}+g_{yy})-f_{xx}g_{xx}+f_{yy}g_{yy})$$

Then a unique curve of periodic solutions bifurcates from the origin into the region

The Hopf Bifurcation Theorem provides conditions of existence of a periodic solution from an equilibrium point as a parameter crosses a bifurcation value. It determines the existence of a periodic solution for a system of equations. In next section the existence of periodic travelling wave solutions in hyperbolic Burgers-Fisher is established.

2.2 Existence of periodic solutions

The coupled hyperbolic Burgers-Fisher equation reads,

$$\begin{cases} u_t + uu_x &= v_x + f(u), \\ \tau v_t + v &= u_x. \end{cases} \tag{2.5}$$

Here $f(u) = u(1-u)$ and τ is such that $0 < \tau < \tau_m := \frac{1}{\sup_{u\in[0,1]}|f'(u)|} = 1$. Let us substitute the travelling wave solutions to (2.5) and define $\xi = x - ct$. Using the traveling wave profile $(u,v)(x,t) = (U,V)(\xi) = (U,V)(x-ct)$, we obtain

$$u_t = -cU'$$
$$v_t = -cV'$$
$$u_x = U'$$
$$v_x = V'$$

The new system is,

$$\begin{cases} (U-c)U' - V' &= f(U) \\ U' + \tau c V' &= V \end{cases} \tag{2.6}$$

Let us verify that the system (2.6) satisfies Hopf Bifurcation Theorem hypothesis. In matrix form we express the system as,

$$\begin{pmatrix} U-c & -1 \\ 1 & \tau c \end{pmatrix} \begin{pmatrix} U' \\ V' \end{pmatrix} = \begin{pmatrix} U(1-U) \\ V \end{pmatrix}$$

We define the matrix

$$A := \begin{pmatrix} U-c & -1 \\ 1 & \tau c \end{pmatrix}$$

and calculate its determinant $\det(A) = 1 - \tau c^2 + c\tau U$. Like $U \in [0,1]$ and the critical value for the velocity of the travelling wave profile is $c = 0$, then $\det(A) \neq 0$ for all c such that $|c|(1+c^2) < \frac{1}{2\tau}$. Therefore the matrix A is a invertible with inverse,

$$A^{-1} := \frac{1}{1-\tau c^2 + c\tau U} \begin{pmatrix} \tau c & 1 \\ -1 & U-c \end{pmatrix}.$$

Thus

$$\begin{pmatrix} U' \\ V' \end{pmatrix} = \frac{1}{1-\tau c^2 + c\tau U} \begin{pmatrix} \tau c & 1 \\ -1 & U-c \end{pmatrix} \begin{pmatrix} f(U) \\ V \end{pmatrix} \tag{2.7}$$

We define,

$$f_c(U,V) := \frac{f(U)\tau c + V}{1-\tau c^2 + c\tau U}$$

and

$$g_c(U,V) := \frac{UV - f(U) - cV}{1-\tau c^2 + c\tau U}$$

Calculating Jacobian matrix we obtain

$$J(U,V) := \begin{pmatrix} \frac{\partial f_c(U,V)}{\partial U} & \frac{\partial g_c(U,V)}{\partial U} \\ \frac{\partial f_c(U,V)}{\partial V} & \frac{\partial g_c(U,V)}{\partial V} \end{pmatrix}$$

where

$$\begin{aligned} \frac{\partial f_c(U,V)}{\partial U} &= \frac{\tau c f'(U)}{1-\tau c^2 + c\tau U} - \frac{(c\tau)(f(U)\tau c - V)}{(1-\tau c^2 + c\tau U)^2} \\ \frac{\partial f_c(U,V)}{\partial V} &= \frac{1}{1-\tau c^2 + c\tau U} \\ \frac{\partial g_c(U,V)}{\partial U} &= \frac{V - f'(U)}{1-\tau c^2 + c\tau U} - \frac{(UV - f(U) - cV)(c\tau)}{(1-\tau c^2 + c\tau U)^2} \\ \frac{\partial g_c(U,V)}{\partial V} &= \frac{U-c}{1-\tau c^2 + c\tau U} \end{aligned}$$

Note $f(0) = 0$. Thus, $f_c(0,0) = 0$ and $g_c(0,0) = 0$, so the point $(0,0)$ is an equilibrium point of (2.7). Evaluating the Jacobian matrix we obtain,

$$J(0,0) := \begin{pmatrix} \frac{\partial f_c(0,0)}{\partial U} & \frac{\partial g_c(0,0)}{\partial U} \\ \frac{\partial f_c(0,0)}{\partial V} & \frac{\partial g_c(0,0)}{\partial V} \end{pmatrix} = \frac{1}{1-\tau c^2} \begin{pmatrix} \tau c f'(0) & -f'(0) \\ 1 & -c \end{pmatrix} \tag{2.8}$$

The following theorem establishes the existence of small amplitude periodic traveling waves corresponding the small amplitude limit cycles around the equilibrium $(0,0)$ arising from Hopf bifurcation

Theorem 2. *Exists a unique periodic solution to* (2.5) *wich bifurcates form the equilibrium point if $c > 0$. Even more, the equilibrium $(0,0)$ is stable and the periodic solution is unstable (from the dynamical systems viewpoint).*

Proof. Let us consider the linearized system (2.8). The matrix $J(0,0)$ has eigenvales $\lambda_{1,2} = \alpha(c) \pm i\beta(c)$, where

$$\alpha(c) = \frac{-c(1-\tau f'(0))}{2}$$
$$\beta(c) = \frac{\sqrt{-c^2(1-\tau f'(0))^2 + 4f'(0)(1-\tau c^2)}}{2}$$

We observe that if $c = 0$ then $\alpha(0) = 0$ and $\beta(0) = \sqrt{f'(0)} \neq 0$. The eigenvalues are imaginary and complex conjugates. We shall see that the Hopf's bifurcation theorem conditions are satisfaced.

1. Non-Hiperbolicity condition.
$\beta(0) = \sqrt{f'(0)} \neq 0$. Let us calculate,

$$g_c(U,V) = \frac{UV - f(U) - cV}{1-\tau c^2 + c\tau U}$$
$$\frac{\partial g_c(U,V)}{\partial U} = \frac{V - f'(U)}{1-\tau c^2 + c\tau U} - \frac{(UV - f(U) - cV)(c\tau)}{(1-\tau c^2 + c\tau U)^2}$$
$$\left.\frac{\partial g_c(0,0)}{\partial U}\right|_{c=0} = -f'(0) < 0.$$

Then,

$$sgn(\sqrt{f'(0)}) = sgn\left(\left.\frac{\partial g_c(0,0)}{\partial U}\right|_{c=0}\right) = sgn(-f'(0)) < 0.$$

2. Transversality condition.
Take $\alpha(c) = \frac{-c(1-\tau f'(0))}{2}$ and derive. If $0 < \tau \leq \sup_{U\in[0,1]} \frac{1}{|f'(U)|} = \tau_m$ then

$$\frac{d\alpha(c)}{dc} = \frac{-1+\tau f'(0)}{2} := d$$
$$\left.\frac{d\alpha(c)}{dc}\right|_{c=0} = \frac{-1+\tau f'(0)}{2} \neq 0$$

3. Lyapunov's exponent.

Now let us calcule the Lyapunov's exponent:

$$\begin{aligned} a = &\frac{1}{16}(f_{UUU} + f_{UVV} + g_{UUV} + g_{VVV}) \\ &+ \frac{1}{16\omega}(f_{UV}(f_{UU} - f_{VV}) - g_{UV}(g_{UU} + g_{VV}) - f_{UU}g_{UU} + f_{VV}g_{VV}) \end{aligned}$$

where $\omega = \sqrt{f'(0)} = 1$,

$$\begin{aligned}
f_{UU} &= \left.\frac{\partial^2 f_c(0,0)}{\partial U^2}\right|_{c=0} = 0 \\
f_{UV} &= \left.\frac{\partial^2 f_c(0,0)}{\partial U \partial V}\right|_{c=0} = 0 \\
f_{VV} &= \left.\frac{\partial^2 f_c(0,0)}{\partial V^2}\right|_{c=0} = 0 \\
g_{UU} &= \left.\frac{\partial^2 g_c(0,0)}{\partial U^2}\right|_{c=0} = f''(0) = -2 \\
g_{UV} &= \left.\frac{\partial^2 f_c(0,0)}{\partial U \partial V}\right|_{c=0} = 1 \\
g_{VV} &= \left.\frac{\partial^2 g_c(0,0)}{\partial V^2}\right|_{c=0} = 0 \\
f_{UUU} &= \left.\frac{\partial^3 f_c(0,0)}{\partial U^3}\right|_{c=0} = 0 \\
f_{UVV} &= \left.\frac{\partial^3 f_c(0,0)}{\partial U \partial V^2}\right|_{c=0} = 0 \\
g_{UUV} &= \left.\frac{\partial^3 g_c(0,0)}{\partial U^2 \partial V}\right|_{c=0} = 0 \\
g_{VVV} &= \left.\frac{\partial^3 g_c(0,0)}{\partial V^3}\right|_{c=0} = 0
\end{aligned}$$

This implies,

$$a = \frac{2}{16\omega} = \frac{1}{8}$$

Now

$$d = \frac{-1 + \tau f'(0)}{2}$$

Then,

$$ad = \frac{-1 + \tau}{16}$$

We need to see the sign of ad. Remember that $0 < \tau \leq \sup_{U \in [0,1]} \frac{1}{|f'(U)|} = \tau_m = 1$. This implies,

$$\tau - 1 < 0$$

So $ad < 0$. By the Hopf's bifurcation theorem, exists a unique curve of periodic solutions bifurcates from the equilibrium point into the region $c > 0$. Like $d < 0$ the point $(0, 0)$ is stable and the periodic solution is unstable.

□

Next let us examine the results of numerical simulations illustrating the periodic solution whose existence is proved in the Theorem. In the first figure we take $c = 0.01 > 0$ and $\tau = 0.1$. Te solution in red is taken with initial conditions $(0, 0.3)$ and we go back in time. The solution in blue is taken with initial conditions $(0, 0.17)$ and we go back in time too. Observe they approximate the periodic cycle.

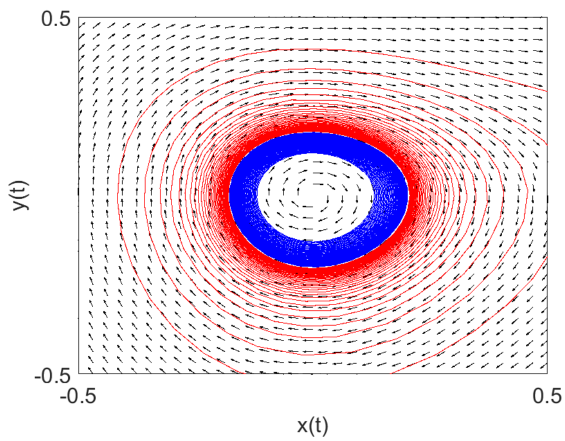

Figure 2.1: Graph of two solutions with $c > 0$

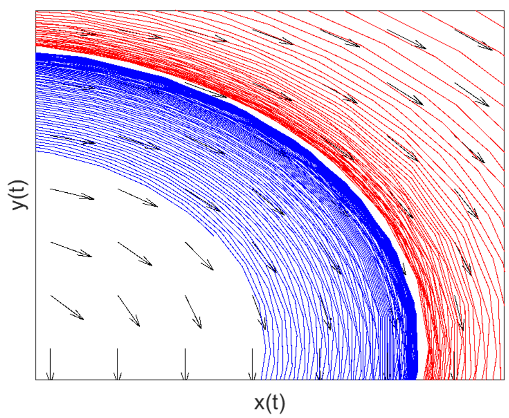

Figure 2.2: Near sight to the periodic solution

Conclusions

Burgers-Fisher equation is an example of a diffusion-convection-reaction equation and is a combination from two famous and well known equations namely Burgers equation and Fisher-KPP equation. The importance of Burgers-Fisher equation is due to its many applications mainly in population dynamics. The equation models non-linear convective movement in witch individuals move aside a diffusive effect in a closed environment.

An example in which Burgers-Fisher equation arises in a natural way is in pattern formation of bacterial model. Bacterial are attracted by chemical signals due nutrients. with a diffusive effect obeying a convective transport.

Burgers-Fisher equation is of parabolic type and can be proved that its classical solution have infinite speed of propagation. Nevertheless, the defective characteristic in its solutions can be corrected coupling with Cattaneo-Maxwell transfer law. The resulting system is an hyperbolic one and its classical solutions have finite speed of propagation. The hyperbolic model have a similar qualitative behaviour that the parabolic one. In chapter two we proved the existence of bounded periodic travelling wave solutions with small amplitude and bounded period which emerge which emerge from a Hopf bifurcation around a critical value of the wave speed.

Bibliography

[1] Alvarez E, Plaza R. 2021, "Existence and Spectral Instability of Bounded Spatially Periodic Travelling Waves for Scalar Viscous Balance Laws" *Quarterly Applied Mathematics*, in press.

[2] Cherniha Roman, Serov Mykola, PliukhinDolak Oleksii, T. 2018, "Nonlinear reaction-diffusion-convection equations: Lie and conditional symmetry, exact solutions, and their applications", Boca Raton, FL: CRC Press, 2018. 1.1

[3] Dolak, Y. Hillen, T. 2003, "Cattaneo models for chemosensitive movement. Numerical solution and pattern formation" *Journal of Mathematical Biology*, Vol. 46, No. 5, 461-478. 1.2

[4] Friedman, A. 2008, "Partial differential equations of parabolic type", Mineola, New York : Dover, 2008 1.2

[5] Gilding B. H. 1996, "The Characterization of Reaction-Convection-Diffusion Processes by Travelling Waves" *Journal of Differential Equations*, Vol. 124, No. 0002, 27-79. 1.2

[6] Hillen T, Stevens A. 2000, "Hyperbolic models for chemotaxis in 1-D" *Nonlinear Analysis: Real World Applications*, Vol. 1, No. 3, 409–433. 1.2

[7] Kuznetsov, Y. 2004, "Elements of applied bifurcation theory", New York : Springer Verlag, c2004.

[8] Lattanzio, C., Mascia C., Plaza R.G., Simeoni C. 2016, "Analytical and numerical investigation of travelling waves for the Allen-Cahn model with relaxation" *Mathematical Models and Methods in Applied Sciences*, Vol.26, No. 5, 931-985.

[9] Lynch L. 2014, "Dynamical systems with applications using MATLAB", Cham : Birkhauser, 2014.

[10] Muñoz, R. 2011, "Introduction to Bifurcations and The Hopf Bifurcation Theorem for Planar Systems", URL=http://www.math.colostate.edu/~shipman/47/volume3b2011/M640_MunozAlicea.pdf. 2.1

[11] Murray J.D. 2002, "Mathematical biology I: an introduction", Berlin : Springer Verlag, 2002. 1.1.1

[12] Perthame, B. 2007, "Transport equations in biology", Basel : Birkhäuser Verlag, 2007 1.1.2

[13] Perthame, B. 2015, "Parabolic equations in biology : growth, reaction, movement and diffusion", Cham : Springer, 2015

[14] Straughan, B. 2011, "Heat Waves", New York : Springer, c2011.

[15] Tai-Ping, L. 2000 "Hyperbolic and viscous conservation laws", Philadelphia, Society for Industrial and Applied Mathematics, c2000 1.1, 1.1.1

[16] Zhou, Y. Quian, L. Weinian, Z. 2013 "Bounded Travelling Waves of the Generalized Burguers-Fisher Equation", *International Journal of Bifurcation and Chaos*, Vol.23, No. 3. (document)