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"DISSIPATIVE STRUCTURE OF A ONE-DIMENSIONAL QUANTUM HYDRODYNAMICS SYSTEM WITH NONLINEAR VISCOCITY THROUGH THE GENUINE COUPLING CONDITION"

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A todas las personas, seres y situaciones que han hecho esto posible.

Contents

Introduction

Consider the one dimensional quantum hydrodynamics (QHD) system with nonlinear viscosity in Eulerian coordinates:

$$\begin{aligned} \rho_t + m_x &= 0, \\ m_t + \left(\frac{m^2}{\rho} + p(\rho)\right)_x &= \varepsilon\mu\rho\left(\frac{m_x}{\rho}\right)_x + \varepsilon^2 k^2 \rho \left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x, \end{aligned} \tag{1.1}$$

with $x \in \mathbb{R}$ and $t > 0$ being the spatial and time variables respectly. The scalar functions $\rho(x,t) > 0$ and $m(x,t) = \rho(x,t)u(x,t)$ denotes the density and the momentum fields and $u(x,t)$ is the velocity. Here $p(\rho)$ denotes the pressure and it's an equation of state. It will be assumed that $p(\rho) = \rho^\gamma$, with $\gamma \geqslant 1$ a constant. The constants $0 \leqslant \varepsilon \ll 1$, $\mu > 0$ and $k > 0$ determine the viscosity $\varepsilon\mu$, and the dispersion (or capilarity) $\varepsilon^2 k^2$, respectly.

The function $\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)$ is known as the (normalized) quantum Bohm potential [1, 2], providing the model with a nonlinear third order dispersive term. It can be interpreted as a quantum correction to the classical pressure (stress tensor). On the otherside, the nonlinear viscosity term is motivated by the theory of superfluidity (see for example [3]).

Models in QHD represent an equivalent alternative formulation of the Schrödinger equation, written in terms of hydrodynamical variables, and structurally similar to the Navier–Stokes equations of fluid dynamics.

The first derivation of the QHD equations is due to Madelung [4], during the early times of quantum mechanics and it was a precursor of the de Broglie–Bohm causal interpretation of quantum theory [5]. Since then, quantum fluid models have been applied to describe many physical phenomena, such as the modeling of quantum semiconductors, the dynamics of Bose–Einstein [6, 7] condensates and the mathematical description of superfluidity [3, 8], among others.
The analysis of QHD models has recently attracted the attention of many mathematicians, specially researchers working in analysis and PDE because of the relevance in physics and the underlying mathematical challenges; look for example [9, 10, 11].

Our investigation focuses on showing that system (1.1) of partial differential equations satisfies the strict dissipativity property, or, in other words, it tell us that solutions to the linearized problem around equilibrium states satisfy some decay structure. In physical terms, this property tell us that the dissipation terms do not allow solutions of traveling wave type to be, simultaneously, solutions to the associated hyperbolic system without dissipation.

This characterization of strict dissipativity, known as *genuine coupling* , has been extensively studied by Kawashima and Shizuta, see for example [12, 13, 14, 15, 16]. The genuine coupling condition tell us that no eigenvectors of the hyperbolic part of our system, lies in the kernel of the viscous terms, Kawashima and Shizuta, prove that for symmetric systems, the strict dissipativity and the genuine coupling condition are equivalent.

Jeffrey Humphreys in [17] proposed and extension of Kawashima and Shizuta theory, by considering higher order derivatives. In our case, we will follow Humphreys, method, if we notice, the quantum Bohm potential $\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)$ will turn into a third order dispersive term, so it will be necessary to consider higher order derivatives.

The structure of the work is as follows. Section 2, is devoted to write our system in conservative form, we use the concept of *enthalpy* (see for instance [18]) to transform system (1.1) into a one that their conserved variables are the mass density ρ and the velocity u. Once with that, we write our system in quasi-linear form, to see which structure will have our system while linearizing around an equilibrium state. After that, we proceed to prove that the obtained system satisfies the strictly hyperbolic condition, which ensure us that the local Cauchy problem have an unique solution.

In section 3, the Kawashima Shizuta theory is developed, and once with that, we extended to the Humphreys theory for higher order derivatives, this section concludes by proving that our system of interest satisfies the genuine coupling condition (which is equivalent to the dissipative structure).

Finally, we end with a discussion of the obtained results and some proposals to extend the results for the non-linear cases.

QHD system and Strictly Hyperbolic Condition.

This section is devoted to write our system in conservative form, thanks to that, system (1.1) is transform into a system where preserved variables are the density and the velocity, after that, we write our system in quasi-linear form. To finish this section, we prove that our system satisfies the stricly hyperbolic conditon.

2.1 QHD system

First, we express the system (1.1) in conservative form, where the conserved variables are (ρ, u) (see [19, 20]), to obtain this, we recall the definition of *entalphy*:

$$h(\rho) := \begin{cases} \ln \rho, & \gamma = 1, \\ \frac{\gamma}{\gamma-1}\rho^{\gamma-1}, & \gamma > 1, \end{cases} \tag{2.1}$$

so, it will be satisfied the equality:

$$p(\rho)_x = \rho h(\rho)_x. \tag{2.2}$$

Applying equation (2.2) to system (1.1), we obtain the system:

$$\begin{aligned} \rho_t + (\rho u)_x &= 0, \\ u_t + \left(\frac{1}{2}u^2 + h(\rho)\right)_x &= \varepsilon\mu\left(\frac{(\rho u)_x}{\rho}\right)_x + \varepsilon^2 k^2\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x. \end{aligned} \tag{2.3}$$

Remark. In contrast to its counter part in the classical fluid mechanics, with nonlinear viscosity and capillarity (see e.g., [21, 22, 23, 24]), it is to be noticed that the conserved variables in the QHD system (2.3) are the mass density and the velocity (not the momentum m), the conserved quantities are different for both systems (the quantum and classical dynamics), changing the structure of the equations. Indeed, the viscosity term for standard compressible fluids has the general form $(\hat{\mu}(\rho)u_x)_x$, with $\hat{\mu}(\rho)$ is a general, nonlinear viscosity term. In our case, the superfluidity case, the viscosity term depends on $\Big((\rho_x u)/\rho\Big)_x$, which comes from the second equation of (2.3). On the other hand, the dispersive term was already different due to the quantum Bohm potential $\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)$, which contrasts with the capillarity terms of Korteweg type, where the capillarity is assumed to be constant and have the form $k(\rho\rho_{xx} - \frac{1}{2}\rho_x^2)_x$ (see [23, 24] for details).

To write our system (2.3) in quasi-linear form, let's notice that:

$$\left(\frac{(\rho u)_x}{\rho}\right)_x = \left(\frac{\rho u_x + u\rho_x}{\rho}\right)_x = u_{xx} + \left(\frac{u\rho_x}{\rho}\right)_x = u_{xx} + \frac{(u_x\rho_x + u\rho_{xx})\rho - u\rho_x^2}{\rho^2}.$$

On the otherside, we have that

$$(\sqrt{\rho})_{xx} = \left(\frac{\rho_x}{2\sqrt{\rho}}\right)_x = \frac{1}{2}\left(\frac{\rho_{xx}\sqrt{\rho} - \frac{\rho_x^2}{2\sqrt{\rho}}}{\rho}\right) = \frac{1}{2}\left(\frac{\frac{2\rho_{xx}\rho - \rho_x^2}{2\sqrt{\rho}}}{\rho}\right) = \frac{1}{2}\left(\frac{2\rho_{xx}\rho - \rho_x^2}{2\sqrt{\rho}\rho}\right).$$

So, making the quotient with $\sqrt{\rho}$ and taking the spatial derivative, we have

$$\begin{aligned}
\left(\frac{(\sqrt{\rho})_{xx}}{\sqrt{\rho}}\right)_x &= \frac{1}{4}\left(\frac{2\rho_{xx}\rho - \rho_x^2}{\rho^2}\right)_x, \\
&= \frac{1}{2}\left(\frac{\rho_{xx}}{\rho}\right)_x - \frac{1}{4}\left(\frac{\rho_x^2}{\rho^2}\right)_x, \\
&= \frac{1}{2}\left(\frac{\rho_{xxx}}{\rho} - \frac{\rho_{xx}\rho_x}{\rho^2}\right) - \frac{1}{4}\left(\frac{2\rho_x\rho_{xx}\rho^2 - 2\rho\rho_x\rho_{xx}}{\rho^4}\right).
\end{aligned}$$

We will consider a vector of the form $U = (\rho, u)^\top \in \mathcal{U} \subset \mathbb{R}^2$ being the vector of state variables, where:

$$\mathcal{U} := \{(\rho, u)^\top \in \mathbb{R}^2 : 0 < \rho\},$$

is known as the state space.
In this way, for $U \in \mathcal{U}$ the system (2.3) can be written in quasi-linear form as:

$$U_t + A^1(U)U_x = B(U)U_{xx} + C(U)U_{xxx} + G(U, U_x, U_{xx}). \tag{2.4}$$

Where

$$A^1(U) := \begin{pmatrix} u & \rho \\ h'(\rho) & u \end{pmatrix}; \quad B(U) := \varepsilon\mu \begin{pmatrix} 0 & 0 \\ \frac{u}{\rho} & 1 \end{pmatrix}; \quad C(U) := \varepsilon^2 k^2 \begin{pmatrix} 0 & 0 \\ \frac{1}{2\rho} & 0 \end{pmatrix},$$

and $G(U, U_x, U_{xx})$ denotes the high order (fully nonlinear) terms, given by:

$$G(U, U_x, U_{xx}) := \begin{pmatrix} 0 \\ \varepsilon\mu\left(\frac{u_x\rho_x\rho - u\rho_x^2}{\rho^2}\right) + \frac{\varepsilon^2 k^2}{2}\left(\frac{\rho_x\rho_{xx} - \rho\rho_x\rho_{xx}}{\rho^3} - \frac{\rho_x\rho_{xx}}{\rho^2}\right) \end{pmatrix}.$$

Remark. Let us notice that $A^1, B, C \in C^\infty(\mathcal{U}; \mathbb{R}^{2\times 2})$, $G \in C^\infty(\mathcal{U} \times \mathbb{R}^4 \times \mathbb{R}^4, \mathbb{R}^4)$, this thanks to the fact that $0 < \rho$, and also notice that $B \geqslant 0$ is semi-positive definite for all state $U \in \mathcal{U}$.

Relaxation terms, are those of the form:

$$Q(U) = \begin{pmatrix} q_1(\rho, u) \\ q_2(\rho, u) \end{pmatrix},$$

with $q_i : \mathcal{U} \to \mathbb{R}$, for $i = \{1, 2\}$.
In our case, for system (2.4), we would not have any relaxation term, that is $Q(U) \equiv 0$, for all $U \in \mathcal{U}$.

In the study of systems of conservation laws with relaxation, the large time behavior of solutions is determined by a "relaxed" structure [25, 26], chosen so that the dynamics leads solutions towards an equilibrium manifold. In quasi-linear systems, the equilibrium manifold in two dimensions is defined as:

$$\mathcal{V} := \{U \in \mathcal{U} : Q(U) = 0\}.$$

A solution $U = U(x, t)$ to quasi-linear system, is said to be an *equilibrium solution* (or a *Maxwellian*) if it lies in the equilibrium manifold $\mathcal{V}$.
In our case, and thanks to the lack of relaxation any constant state $\bar{U} \in \mathcal{U}$ will be an equilibrium solution.

2.2 Hyperbolicity

Considering the hyperbolic part of our system (2.4)

$$U_t + A^1(U)U_x = 0. \tag{2.5}$$

Which results from neglacting the viscous, dispersive and non-linear terms in system (2.4). For any state $U \in \mathcal{U}$, (2.5) is a quasi-linear, strictly hyperbolic first order system, to prove this, let us obtain the characteristics speeds.

For any $U \in \mathcal{U}$ let us consider the polynomial given by:

$$\pi(\zeta) = \det\Big(A^1(U) - \zeta \mathrm{I}\Big).$$

Where I denotes the identity matrix in $\mathbb{M}_{2\times 2}(\mathbb{R})$.
The roots of the polynomial $\pi(\zeta) = 0$ are known as characteristic speeds of the system (2.5). If those roots are all real and distinct, we will say that the system (2.5) is strictly hyperbolic for $U \in \mathcal{U}$.
Let us remember, that the notion of hyperbolicity is motivated by the existence of

traveling wave solutions to system (2.5) of the form $U(x,t) = \varphi(x - st)$, for some real propagating speed $s \in \mathbb{R}$ and a profile vector function φ.
This reduce our problem to a spectral one, of the form:

$$(A^1(\varphi) - s\mathrm{I})\varphi' = 0,$$

with eigenvalue $s \in \mathbb{R}$ and eigenfunction φ'.

After a straightforward computation we see that

$$\begin{aligned}\pi(\zeta) &= \det\left| \begin{pmatrix} u-\zeta & \rho \\ h'(\rho) & u-\zeta \end{pmatrix} \right| , \\ &= (u-\zeta)^2 - h'(\rho)\rho \ .\end{aligned}$$

So $\pi(\zeta) = 0$ if only if

$$\zeta = u \pm \sqrt{h'(\rho)\rho} \ . \tag{2.6}$$

Notice that

$$h'(\rho)\rho = \gamma\rho^{\gamma-1},$$

and thanks to the fact that $\rho > 0$, then the roots of $\sqrt{h'(\rho)\rho}$ are both real, so the characteristic speeds will be given by:

$$\zeta_1 = u - \sqrt{h'(\rho)\rho} < u + \sqrt{h'(\rho)\rho} = \zeta_2 \ .$$

Due to the fact that both roots are real and dinstinct, we can ensure that our system is strictly hyperbolic.
We can summary the previous results as the following:

Lemma. *For each $U = (\rho, u)^\top \in \mathcal{U} \subset \mathbb{R}^2$, the first order system (2.5) is strictly hyperbolic at $U \in \mathcal{U}$, and the characteristic speeds are given by:*

$$\begin{aligned}\zeta_1(U) &= u - \sqrt{h'(\rho)\rho}, \\ \zeta_2(U) &= u + \sqrt{h'(\rho)\rho} \ .\end{aligned} \tag{2.7}$$

In [27], Plaza and collaborators, studied the existance and structure of dispersive shock profiles for system (1.1), with nonlinear viscosity, the study of the case with linear viscosity was mainly developed by Lattanzio and Zhelyazov in [19] (for related results, see [28]).

Kawashima and Shizuta Theory

3.1 Genuine coupling condition

Kawashima's theory [12, 13, 14, 15, 16, 29, 30] considers the second-order constant coefficient systems of the form:

$$A^0 U_t + A^1 U_x = B U_{xx}. \tag{3.1}$$

Where A^0, A^1 and B are symmetric matrices with B positive semi-definite .
Since it is a system with constant coefficients, the solution can be determined by its Fourier transform in the spatial variables $x \in \mathbb{R}$. The resulting equation is

$$A^0 \hat{U}_t + i\xi A^1 \hat{U} + \xi^2 B \hat{U} = 0. \tag{3.2}$$

Where $\hat{U} = \hat{U}(\xi, t)$ denotes the Fourier transform of the state variable U. The fact that $A^0, B > 0$ is not enough to ensure the decay of solutions to the linear. We resort to the following sufficient condition for the essential spectrum of the linear constant coefficient differential operator to be stable. For $\xi \in \mathbb{R}$, $\xi \neq 0$, let $\lambda = \lambda(\xi) \in \mathbb{C}$ denote the eigenvalues of the corresponding characteristic equation, namely, the roots of the following dispersion relation,

$$\det(\lambda A^0 + i\xi A^1 + \xi^2 B) = 0. \tag{3.3}$$

Definition. (Strict dissipativity). System (3.1) is said to be *strictly dissipative* if $\mathrm{Re}\lambda(\xi) < 0$ for all $\xi \in \mathbb{R}$, $\xi \neq 0$.

Closely related to the dissipativity condition is the following.

Definition. (Genuine coupling). System (3.1) satisfies *genuine coupling condition* at any state $\bar{U} \in \mathcal{U}$ if for any $V \in \mathbb{R}^2$, $V \neq 0$ with $BV = 0$ then we have that $(\lambda A^0 + A^1)V = 0$, for all $\lambda \in \mathbb{R}$.

This condition basically expresses that no eigenvector of the hyperbolic part of the operator lies in the kernel of the dissipative terms. Such property is physically relevant. We remark that a loss of coupling in these instances means that a purely hyperbolic direction exists whereby discontinuous "shock wave" solutions can persist.

It tell us that traveling wave solutions to system (2.5) are not dissipated by the viscous and relaxation terms.

Definition. A matrix K is a *compansating function* for system (3.1) provided that:

- KA^0 is skrew symmetric.
- $\frac{1}{2}(KA^1 + (KA^1)^\top) + B$ is positive definite.

Theorem. *(Shizuta-Kawashima) [12] Assume A^j, B $j = 0, 1$ are real symmetric matrices, with $A^0, B > 0$. Then the following statements are equivalent:*

1. *System (3.1) is stricly dissipative.*
2. *System (3.1) satisfies genuine coupling condition at $\bar{U} \in \mathcal{U}$.*
3. *There exists a compensation function K for system (3.1).*
4. *There exists a positive constant $k > 0$ such that for any $\xi \in \mathbb{R}$, $\xi \neq 0$,and any root $\lambda = \lambda(\xi)$ of the characteristic equation (3.3) there holds*

$$\mathrm{Re}\lambda(\xi) \leqslant -\frac{k\xi^2}{1+\xi^2}. \tag{3.4}$$

The last theorem was generalized by Humpherys by considering the general linear system of the form:

$$A^0 U_t = -\sum_{k=0}^{n} D^k \partial_x^k U; \quad U \in \mathbb{R}^m. \tag{3.5}$$

Where each $m \times m$ matrix D^k is a constant matrix.
Likewise, by taking the Fourier transform, the evolution of (3.5) reduces to the eigenvalue problem

$$\lambda A^0 \hat{U}_t + \sum_{k=1}^{n} (i\xi)^k D^k U = 0. \tag{3.6}$$

Simplifying by separating the odd and even terms in (3.6), we get

$$\big(\lambda A^0 + i\xi\mathbb{A}(\xi) + \mathbb{B}(\xi)\big) = 0, \tag{3.7}$$

where

$$\mathbb{A}(\xi) := \sum_{k\ \ odd} D^k (i\xi)^{k-1}; \quad \mathbb{B}(\xi) := \sum_{k\ \ even} (-1)^{k/2} D^k \xi^k. \tag{3.8}$$

The matrix $\mathbb{A}(\xi)$ and $\mathbb{B}(\xi)$ are referred as the generalized flux and the generalized viscosity, respectively.

Then we have the following definitions

Definition.

1. System (3.5) is called strictly dissipative if for each $\xi \neq 0$ we have $\mathrm{Re}(\lambda(\xi)) < 0$.

2. System (3.5) is said to be genuinely coupled if no eigenvalue of $\mathbb{A}(\xi)$ is in the kernel of $\mathbb{B}(\xi)$.

Mathematically, it is easy to see that genuine coupling is a necessary condition for strict dissipativity. The main result in Humpherys paper shows that for symmetric systems, the properties of strict dissipativity, genuine coupling for systems of the form like system (3.5), and the existence of a skew-Hermitian compensating matrix K are equivalent.
The following assumptions are made

(H1) $\mathbb{A}(\xi)$ is symmetric and of constant multiplicity in ξ.

(H2) $\mathbb{B}(\xi) \geqslant 0$ (symmetric and positive semi-definite).

Next, we state the main result without proof and after that we make some important remarks.

Theorem. *(Humpherys) [17]. Given (H1) and (H2) above, the following statements are equivalent:*

1. *System (3.5) is strictly dissipative.*

2. *System (3.5) is genuinely coupled.*

3. *There exists a real-analytic skew-Hermitian matrix-values $K(\xi)$ such that $[K(\xi), \mathbb{A}(\xi)] + \mathbb{B}(\xi) > 0$ for all $\xi \neq 0$.*

Where $[K(\xi), \mathbb{A}(\xi)] = K(\xi)\mathbb{A}(\xi) - \mathbb{A}(\xi)K(\xi)$.

Remarks. The compensation matrix K is of the form:

$$K(\xi) = \sum_{i \neq j} \frac{\pi_i \mathbb{B}(\xi) \pi_j}{\mu_i - \mu_j}, \tag{3.9}$$

that is the Drazin inverse or reduced resolvent of the commutator operator, where $\{\mu_k\}_{k=1}^r$ denote the distinct eigenvalues of $\mathbb{A}(\xi)$ with corresponding eigenprojections $\{\pi_k\}_{k=1}^r$.

The above theorem can be extended with the following definition:

Definition. System (3.5) is called symmetrizable in the sense of Humpherys if there exists a symmetric real-analytic matrix-valued $S(\xi) > 0$ such that both $S(\xi)\mathbb{A}(\xi)$ and $S(\xi)\mathbb{B}(\xi)$ are symmetric, and $S(\xi)\mathbb{B}(\xi) \geqslant 0$. We say that $S(\xi)$ is a symmetrizer of system (3.5).

We notice that this notion of symmetrizability differs from the Friedrichs' one. With this more general notion of symmetrizability, we can extend easily Theorem above to the following:

Theorem. *If $S(\xi)$ is a symmetrizer of system (3.5) then the following statements are equivalent:*

1. *System (3.5) is strictly dissipative.*

2. *System (3.5) is genuinely coupled.*

3. *There exists a real-analytic skew-Hermitian matrix-values $K(\xi)$ such that $[K(\xi), S\cdot \mathbb{A}(\xi)] + S \cdot \mathbb{B}(\xi) > 0$ for all $\xi \neq 0$.*

It is the last generalization of the standard definition of symmetrizability the one that we adopt here.

3.2 Symmetrizability

Symmetrizability implies hyperbolicity of system (2.5). As we can see in Friedrichs [31] and Goudunov [32] works, symmetrizability has established itself as an important property. It plays a key role, for example, to perform energy estimates and to study existence and stability of solutions.
Let us notice that writting our QHD system as in (3.6) we have

$$\mathbb{A}(\xi) = A^1(U) + \xi^2 C(U) = \begin{pmatrix} u & \rho \\ h'(\rho) & u \end{pmatrix} + \varepsilon^2 k^2 \xi^2 \begin{pmatrix} 0 & 0 \\ \frac{1}{2\rho} & 0 \end{pmatrix} = \begin{pmatrix} u & \rho \\ h'(\rho) + \frac{\varepsilon^2 k^2 \xi^2}{2\rho} & u \end{pmatrix},$$

$$\mathbb{B}(\xi) = \xi^2 B(U) = \xi^2 \varepsilon \mu \begin{pmatrix} 0 & 0 \\ \frac{u}{\rho} & 1 \end{pmatrix}.$$

We define

$$\beta(\xi) := h'(\rho) + \frac{\varepsilon^2 k^2 \xi^2}{2\rho}.$$

In this way, we need to find a symmetric matrix $S = S(\xi)$ positive definite such that $S \cdot \mathbb{A}(\xi)$ and $S \cdot \mathbb{B}(\xi)$ are both symmetric.
Let us consider S being of the form:

$$S := \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}.$$

Making the product $S \cdot \mathbb{B}(\xi)$ we have:

$$S \cdot \mathbb{B}(\xi) = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix} \cdot \xi^2 \varepsilon \mu \begin{pmatrix} 0 & 0 \\ \frac{u}{\rho} & 1 \end{pmatrix}.$$

In order that the product $S \cdot \mathbb{B}(\xi)$ being symmetric, the next equality must satisfy

$$s_2 = s_3 \frac{u}{\rho}.$$

Following the same process for $\mathbb{A}(\xi)$, we obtain the equality

$$s_1 = s_3 \frac{\beta(\xi)}{\rho}.$$

With these, we have that S need to be of the form:

$$S = \begin{pmatrix} s_3 \frac{\beta}{\rho} & s_3 \frac{u}{\rho} \\ s_3 \frac{u}{\rho} & s_3 \end{pmatrix},$$

which is a symmetric matrix.

Let us remember the readers that our matrix S will be positive definite if only if all it's eigenvalues λ_i are both real and positive, thanks that our matrix S is symmetric, we can ensure that the eigenvalues are real, so we only have to establish enough conditions to make our eigenvalues positive.

We now show under which conditions our matrix S is positive definite. To this, we proceed to calculate the eigenvalues:

$$|S - \lambda I| = det \left| \begin{pmatrix} s_3 \frac{\beta(\xi)}{\rho} - \lambda & s_3 \frac{u}{\rho} \\ s_3 \frac{u}{\rho} & s_3 - \lambda \end{pmatrix} \right| .$$

So, we obtain the polynomial

$$\begin{aligned} \left(s_3 \frac{\beta(\xi)}{\rho} - \lambda \right) (s_3 - \lambda) - \left(s_3 \frac{u}{\rho} \right) \left(s_3 \frac{u}{\rho} \right) &= 0, \\ \lambda^2 - \left(s_3 \left[\frac{\beta(\xi)}{\rho} + 1 \right] \right) \lambda + s_3^2 \left(\frac{\beta(\xi)}{\rho} - \frac{u^2}{\rho^2} \right) &= 0. \end{aligned} \tag{3.10}$$

Using the quadratic formula, the roots of the polynomial (3.10) are given by:

$$\lambda_{1,2} = \frac{\left(s_3\left[\frac{\beta(\xi)}{\rho}+1\right]\right) \pm \sqrt{s_3^2\left[\frac{\beta(\xi)}{\rho}+1\right]^2 - 4\left(s_3^2\left(\frac{\beta(\xi)}{\rho}-\frac{u^2}{\rho^2}\right)\right)}}{2},$$
$$= s_3\left(\frac{\frac{\beta(\xi)}{\rho}+1 \pm \sqrt{\left[\frac{\beta(\xi)}{\rho}-1\right]^2+4\left(\frac{u^2}{\rho^2}\right)}}{2}\right).$$

So, we can ensure that the eigenvalues, will be positive, if the term

$$\left(\frac{\beta(\xi)}{\rho}+1\right) \pm \sqrt{\left[\frac{\beta(\xi)}{\rho}-1\right]^2+4\left(\frac{u^2}{\rho^2}\right)}, \tag{3.11}$$

do not suffers a change of sign, and have the same sign of s_3.
Let us notice that in the case that we consider the eigenvalue

$$\lambda_1 = s_3\left(\frac{\frac{\beta(\xi)}{\rho}+1+\sqrt{\left[\frac{\beta(\xi)}{\rho}-1\right]^2+4\left(\frac{u^2}{\rho^2}\right)}}{2}\right).$$

Then, (3.11) will have a positive sign and we must ask s_3 being positive, that is $s_3 > 0$, using these, we must ensure that

$$\sqrt{\left(\frac{\beta(\xi)}{\rho}-1\right)^2+4\left(\frac{u^2}{\rho^2}\right)} < \left(\frac{\beta(\xi)}{\rho}+1\right). \tag{3.12}$$

Via direct calculation, condition (3.12) is satisfy if only if

$$\left(\frac{\beta(\xi)}{\rho}-1\right)^2+4\left(\frac{u^2}{\rho^2}\right) < \left(\frac{\beta(\xi)}{\rho}+1\right)^2, \tag{3.13}$$

or equivalent

$$\left(\frac{u^2}{\rho}\right) < \beta(\xi). \tag{3.14}$$

Remark. Let us notice that condition (3.14) must be satisfy for all $\xi \in \mathbb{R}$, so condition (3.14) can be rewritten as:

$$u^2 < h'(\rho)\rho. \tag{3.15}$$

If we notice in our characteristics speeds given in (2.7), condition (3.15) is equivalent that our system satisfies a subsonic condition.

For systems in conservation form the symmetrizer is usually the Hessian of a convex entropy function. Even in the case of quasi-linear systems not in conservation form (where the coefficients A^j are not necessarily Jacobians of the flux functions f^j) it is possible to define a convex entropy, as shown by Kawashima and Yong [16]: if the symmetrizer is the Jacobian of a diffeomorphic change of variables $S(U) := D_U\Psi(U)$, then a convex entropy function can be introduced.

As a way of conclusion, if we consider $s_3 = \rho$ we have the following result.

Lemma. *If our flux satisfies the subsonicity condition (3.15), then, the linearized system associated to system (2.4) is symmetrizable in the sense of Humpherys and the symmetric matrix $S \in C^\infty(\mathcal{U}, \mathbb{R}^{2\times 2})$ is given by*

$$S(\xi) := \begin{pmatrix} \beta(\xi) & u \\ u & \rho \end{pmatrix}. \tag{3.16}$$

Proof: Clearly S is smooth in the convex open set $\mathcal{U}$. Moreover, S is symmetric, and thaks to condition (3.15), will be positive definite.
That S symmetrizes system (2.4) follows from straighforward computations that yield

$$\begin{aligned} \hat{A}(\xi) &: = S \cdot \mathbb{A}(\xi) = \begin{pmatrix} 2\beta(\xi)u & \beta\rho + u^2 \\ \beta\rho + u^2 & 2\beta(\xi)u \end{pmatrix}, \\ \hat{B}(\xi) &: = S \cdot \mathbb{B}(\xi) = \xi^2 \varepsilon\mu \begin{pmatrix} \frac{u^2}{\rho} & u \\ u & \rho \end{pmatrix}, \end{aligned}$$

which are smooth symmetric matrix functions of $U \in \mathcal{U}$.

Once we have prove that our system can put in symmetric form, we are in the right way to follow Humpherys theory and study the strict dissipativity of our system.

3.3 Strict Dissipativity and Genuine Coupling Condition for QHD system

In order to define the strict dissipativity of the system, let us consider solutions around a constant equilibrium state

$$\bar{U} = (\bar{\rho}, \bar{u})^\top \in \mathcal{U} \subset \mathbb{R}^2.$$

If $U + \bar{U}$ is a solution to our system, then we can recast the system as:

$$U_t + A^1(\bar{U})U_x = B(\bar{U})U_{xx} + C(\bar{U})U_{xxx} + \mathcal{N}(U, U_x, U_{xx}, U_{xxx}), \tag{3.17}$$

where $\mathcal{N}$ comprises the nonlinear terms. Let us consider the linear part of system (3.17) that is, the linear system:

$$U_t + A^1(\bar{U})U_x = B(\bar{U})U_{xx} + C(\bar{U})U_{xxx}. \tag{3.18}$$

If we consider our matrix $\mathbb{A}(\xi)$ and $\mathbb{B}(\xi)$ as in (3.6), we have

$$\mathbb{A}(\xi) = A^1(\bar{U}) + \xi^2 C(\bar{U}) = \begin{pmatrix} \bar{u} & \bar{\rho} \\ h'(\bar{\rho}) & \bar{u} \end{pmatrix} + \varepsilon^2 k^2 \xi^2 \begin{pmatrix} 0 & 0 \\ \frac{1}{2\bar{\rho}} & 0 \end{pmatrix} = \begin{pmatrix} \bar{u} & \bar{\rho} \\ h'(\bar{\rho}) + \frac{\varepsilon^2 k^2 \xi^2}{2\bar{\rho}} & \bar{u} \end{pmatrix},$$

$$\mathbb{B}(\xi) = \xi^2 B(\bar{U}) = \xi^2 \varepsilon\mu \begin{pmatrix} 0 & 0 \\ \bar{u}/\bar{\rho} & 1 \end{pmatrix}.$$

And we define, the term $\beta(\xi)$ as

$$\beta(\xi) = h'(\bar{\rho}) + \frac{\varepsilon^2 k^2 \xi^2}{2\bar{\rho}}.$$

We are now in a suitable position to show that for any fixed state $\bar{U} \in \mathcal{U}$ system (3.17) satisfies the genuine coupling condition at $\bar{U} = (\bar{\rho}, \bar{u})^\top \in \mathcal{U}$.

Theorem. *The system (3.18) satisfies the genuine coupling condition at any state* $\bar{U} = (\bar{\rho}, \bar{u})^\top \in \mathcal{U}$.

Proof. The eigenvalues of the matrix $\mathbb{A}(\xi)$ will be given by:

$$\lambda_{1,2} = u \pm \sqrt{\beta(\xi)\bar{\rho}}. \tag{3.19}$$

So the eigenvectors will be of the form:

$$V_1 := \begin{pmatrix} \frac{\sqrt{\bar{\rho}}}{\sqrt{\beta(\xi)}} \\ 1 \end{pmatrix}; \quad V_2 := \begin{pmatrix} -\frac{\sqrt{\bar{\rho}}}{\sqrt{\beta(\xi)}} \\ 1 \end{pmatrix}. \tag{3.20}$$

In this way, we will have that for $i \in \{1, 2\}$, $V_i \in \mathrm{Ker}\mathbb{B}(\xi)$, if

$$\mathbb{B}(\xi) \cdot V_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{3.21}$$

or equivalent, if we consider the eigenvalue V_1, then we have:

$$\frac{\bar{u}}{\bar{\rho}} \frac{\sqrt{\bar{\rho}}}{\sqrt{\beta(\xi)}} + 1 = 0,$$

$$\frac{\bar{u}}{\sqrt{\bar{\rho}\beta(\xi)}} + 1 = 0.$$

These is satisfy if only if :

$$\bar{u} = -\sqrt{\bar{\rho}\beta(\xi)},$$

so, if we define ξ as:

$$\xi = \sqrt{2\left(\frac{\bar{u}^2 - h'(\bar{\rho})\bar{\rho}}{\varepsilon^2 k^2}\right)} \ . \tag{3.22}$$

In this case, we will have

$$\begin{aligned}
\beta(\xi) &= h'(\bar{\rho}) + \frac{\varepsilon^2 k^2}{2\bar{\rho}} \cdot 2\left(\frac{\bar{u}^2 - h'(\bar{\rho})\bar{\rho}}{\varepsilon^2 k^2}\right), \\
\beta(\xi) &= h'(\bar{\rho}) + \frac{\bar{u}^2}{\bar{\rho}} - h'(\bar{\rho}), \\
\beta(\xi) &= \frac{\bar{u}^2}{\bar{\rho}}.
\end{aligned}$$

So, it must be satisfied that

$$-\sqrt{\bar{\rho}\beta(\xi)} = \sqrt{\frac{\bar{\rho}\bar{u}^2}{\bar{\rho}}} = \sqrt{\bar{u}^2}.$$

Taking the negative root, we will have that:

$$-\sqrt{\bar{\rho}\beta(\xi)} = \bar{u},$$

so V_1 will be in the kernel of $\mathbb{B}(\xi)$.
The same result arises in the case of V_2, that is, V_2 will be in Ker$\mathbb{B}(\xi)$ if

$$\bar{u} = \sqrt{\bar{\rho}\beta(\xi)}.$$

So, V_1 and V_2 will be in Ker$\mathbb{B}(\xi)$ if only if ξ is define as in (3.22), but if we define ξ in this way, then thanks to the subsonic condition (3.15), we will have that

$$\left(\frac{\bar{u}^2 - h'(\bar{\rho})\bar{\rho}}{\varepsilon^2 k^2}\right) < 0,$$

using this inequality, we arises that

$$\xi = \sqrt{2\left(\frac{\bar{u}^2 - h'(\bar{\rho})\bar{\rho}}{\varepsilon^2 k^2}\right)} \in \mathbb{C}.$$

Which is a contradiction because ξ must be a real number. So, it can not be any eigenvector of $\mathbb{A}(\xi)$ in the kernel of $\mathbb{B}(\xi)$, that means that genuine coupling condition is satisfy for our linearized system (3.18).

Although genuine coupling readily implies the existence of a compensating function, it is often possible to provide an explicit formula for it, via the Humphery's formula given in (3.9).

First, let us observe that for a constant state $\bar{U} = (\bar{\rho}, \bar{u})^\top$ the eigenvalues of the matrix symmetric matrix $\hat{A}(\xi) = S \cdot \mathbb{A}(\xi)$ are given by

$$\mu_1 = 2\beta(\xi)\bar{u} - (\bar{u}^2 + \beta(\xi)\bar{\rho}),$$
$$\mu_2 = 2\beta(\xi)\bar{u} + (\bar{u}^2 + \beta(\xi)\bar{\rho}).$$

The eigenvectors are respectively given by:

$$v_1 = (-1, 1)^\top; \quad v_2 = (1, 1)^\top.$$

So the canonical basis in 2 dimensions, can be written as:

$$(1, 0)^\top = -\frac{1}{2}(-1, 1)^\top + \frac{1}{2}(1, 1)^\top,$$
$$(0, 1)^\top = \frac{1}{2}(-1, 1)^\top + \frac{1}{2}(1, 1)^\top.$$

With these, the eigenprojections, will be given by:

$$\pi_1(\xi) = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}; \quad \pi_2(\xi) = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Applying formula (3.9), we have:

$$\frac{\pi_1 \hat{B}(\xi)\pi_2}{\mu_1 - \mu_2} = -\kappa(\xi)\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\bar{u}^2}{\bar{\rho}} & \bar{u} \\ \bar{u} & \bar{\rho} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = -\kappa(\xi)\begin{pmatrix} \frac{\bar{u}^2}{\bar{\rho}} - \bar{\rho} & \frac{\bar{u}^2}{\bar{\rho}} - \bar{\rho} \\ \bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}} & \bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}} \end{pmatrix},$$

$$\frac{\pi_2 \hat{B}(\xi)\pi_1}{\mu_2 - \mu_1} = \kappa(\xi)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{\bar{u}^2}{\bar{\rho}} & \bar{u} \\ \bar{u} & \bar{\rho} \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \kappa(\xi)\begin{pmatrix} \frac{\bar{u}^2}{\bar{\rho}} - \bar{\rho} & \bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}} \\ \frac{\bar{u}^2}{\bar{\rho}} - \bar{\rho} & \bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}} \end{pmatrix},$$

where

$$\kappa(\xi) = \frac{\xi^2 \varepsilon \mu}{8(\bar{u}^2 + \beta(\xi)\bar{\rho})}.$$

In this way, we obtain:

$$K(\xi) = \frac{\xi^2 \varepsilon \mu}{4(\bar{u}^2 + \beta(\xi)\bar{\rho})}\begin{pmatrix} 0 & \bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}} \\ \frac{\bar{u}^2}{\bar{\rho}} - \bar{\rho} & 0 \end{pmatrix}. \tag{3.23}$$

Clearly $K(\xi)$ is skew-hermitian, and we have

$$\begin{aligned} K(\xi) \cdot \hat{A}(\xi) - \hat{A}(\xi) \cdot K(\xi) =& 2\kappa(\xi)\begin{pmatrix} (\beta(\xi)\bar{\rho} + \bar{u}^2)\left(\bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}}\right) & 2\beta(\xi)\bar{u}\left(\bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}}\right) \\ -2\beta(\xi)\bar{u}\left(\bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}}\right) & (\beta(\xi)\bar{\rho} + \bar{u}^2)\left(\frac{\bar{u}^2}{\bar{\rho}} - \bar{\rho}\right) \end{pmatrix} \\ & -2\kappa(\xi)\begin{pmatrix} -(\beta(\xi)\bar{\rho} + \bar{u}^2)\left(\bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}}\right) & 2\beta(\xi)\bar{u}\left(\bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}}\right) \\ -2\beta(\xi)\bar{u}\left(\bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}}\right) & -(\beta(\xi)\bar{\rho} + \bar{u}^2)\left(\frac{\bar{u}^2}{\bar{\rho}} - \bar{\rho}\right) \end{pmatrix}, \\ K(\xi) \cdot \hat{A}(\xi) - \hat{A}(\xi) \cdot K(\xi) =& \frac{\xi^2 \varepsilon \mu}{2}\begin{pmatrix} \left(\bar{\rho} - \frac{\bar{u}^2}{\bar{\rho}}\right) & 0 \\ 0 & \left(\frac{\bar{u}^2}{\bar{\rho}} - \bar{\rho}\right) \end{pmatrix}. \end{aligned}$$

Adding the matrix $\hat{B}(\xi)$ we have:

$$\begin{aligned} K(\xi)\cdot\hat{A}(\xi)-\hat{A}(\xi)\cdot K(\xi)+\hat{B}(\xi) =& \frac{\xi^2\varepsilon\mu}{2}\begin{pmatrix} \bar{\rho}-\frac{\bar{u}^2}{\bar{\rho}} & 0 \\ 0 & \frac{\bar{u}^2}{\bar{\rho}}-\bar{\rho} \end{pmatrix} + \frac{\xi^2\varepsilon\mu}{2}\begin{pmatrix} 2\frac{\bar{u}^2}{\bar{\rho}} & 2\bar{u} \\ 2\bar{u} & 2\bar{\rho} \end{pmatrix}, \\ =& \frac{\xi^2\varepsilon\mu}{2}\begin{pmatrix} \bar{\rho}+\frac{\bar{u}^2}{\bar{\rho}} & 2\bar{u} \\ 2\bar{u} & \frac{\bar{u}^2}{\bar{\rho}}+\bar{\rho} \end{pmatrix}. \end{aligned}$$

To verify that the matrix $K(\xi)\cdot\hat{A}(\xi)-\hat{A}(\xi)\cdot K(\xi)+\hat{B}(\xi)$ is positive definite, let us obtain the eigenvalues of the matrix M, where M is defined as:

$$M := \begin{pmatrix} \bar{\rho}+\frac{\bar{u}^2}{\bar{\rho}} & 2\bar{u} \\ 2\bar{u} & \frac{\bar{u}^2}{\bar{\rho}}+\bar{\rho} \end{pmatrix}.$$

Let us notice, that its eigenvalues will be determined by the characteristic polynomial:

$$\begin{aligned} \left(\frac{\bar{\rho}^2+\bar{u}^2}{\bar{\rho}}-\lambda\right)^2 &= 4u^2, \\ \frac{\bar{\rho}^2+\bar{u}^2-\bar{\rho}\lambda}{\bar{\rho}} &= \pm 2\bar{u}. \end{aligned}$$

From these, we have that the eigenvalues of the matrix M are given by:

$$\begin{aligned} \lambda_1 &= \frac{(\bar{\rho}-\bar{u})^2}{\bar{\rho}}, \\ \lambda_2 &= \frac{(\bar{\rho}+\bar{u})^2}{\bar{\rho}}, \end{aligned}$$

and thanks to the fact that $\bar{\rho}>0$, then we can ensure that both eigenvalues λ_1, λ_2 are positive. So we can conclude that the matrix $K(\xi)\cdot\hat{A}(\xi)-\hat{A}(\xi)\cdot K(\xi)+\hat{B}(\xi)$ is positive definite.

Thus the matrix $K(\xi)$ defined in (3.23) is the required compensating matrix. It is worth to note that the matrix $K(\xi)$ is compensating function in the sense of Humpherys.

Conclusions

During the present work we have mainly developed the study of the dissipative structure for the one-dimensional quantum hydrodynamics system given by equation (1.1). Due to the presence of the third order dispersive term, given by the quantum Bohm potential, this study, follows the Humpherys method.

Although, the study of PDE models in quantum hydrodynamics have been a growing area in recent years, the study of the dissipative structure for our model has not been reported in the literature, and this is the main contribution of the present work.

We have proved strict dissipativity for these systems by verifying the genuine coupling condition in Humpherys sense, as well as by providing explicit forms for the compensating function using the Drazin inverse given in Humpherys paper [17].

Genuine coupling condition tell us that traveling wave solutions to the hyperbolic system are not dissipated by the viscous and relaxation terms. This implies deep consequences on the time asymptotic smoothing behavior of solutions.

As we have pointed out in the second chapter, symmetrizability is a fundamental property in the theory. Which enable us to ensure the hyperbolicity of the system. Let us outstand that in our case, in order to ensure the symmetrizability of our system, it must be satisfy to subsonic condition imposed in equation (3.15).

Let us noticed that the study of the dissipative structure is the first step to give a nonlinear result, once we have seen that our model of interest is well-posed and we can find a compensation matrix, we are in the right way to give decay rates of the solutions to the linearized system around a constant state. Once with that, we can extend the result into the nonlinear case.

So, the following step of the present work is to obtain the linear decay rates of our system, in the direction followed by Plaza, Angeles and Valdovinos in [21, 33], the big difference between our case and the classic Korteweg model presented in [21] is that the viscous contribution for the second equation of our system will depend of the two variables of state ρ, u, this small difference will be the one which will complicate the obtaining of the linear decay rates, which nowadays still being an open problem.

Another natural line of work is to ask whether multi-dimensional quantum hydrodynamics systems are strictly dissipative. With respect to this problem, let us remark, however, that not even the existence of a symmetrizer in several space dimensions is yet clear.

In this sense the present work represents a contribution in quantum hydrodynamics systems, by ensuring the dissipative structure of the presented problem, but not only that, it leads us to new open problems in the area.

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