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EXISTENCE OF NON-NEGATIVE WEAK SOLUTIONS TO A REACTION-DIFFUSION-CHEMOTAXIS SYSTEM WITH A CROSS-DIFFUSION TERM

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Chapter 1

Introduction

In ecology, the precept of adaptability of the organisms is crucial in order to understand the mechanisms of survival in complex populations. One of these adaptations is such as basic as the concept of movement: the navigation within a complex environment through the detection, integration and processing of a variety of internal and external signals. These movement mechanisms can include many behaviors, among which we find for instance, location of food sources, avoidance of predators and of course attracting mates (see [17]).

Take for instance, the notable example of bacteria populations. When those populations face hostile environment conditions, such as low levels of nutrients or high concentrations of agar (the most commonly used support medium for bacterial and fungal culture), bacterial colonies may show various morphological aggregation patterns, among them we find: a fractal morphogenesis for nutrient-poor solid agar; a dense-branching morphology on semi-solid agar; and finally on a softness-nutrient-agar environment the bacteria can show a simple homogeneously circular pattern (see [11], [14], [18] [23]).

In order to explain the experiments showing those patterns in bacterial colonies in laboratory, lots of mathematical models have been proposed. One of the classical models corresponds to the deterministic reactions-diffusion approach in which bacterial density, $b = b(x,t)$, and the nutrient concentrate, $n = n(x,t)$, are described with continous time evolution systems of partial differential equations of the form

$$\left\{ \begin{array}{ll} b_t & = D_b \, \nabla^2 b + F(b,n), \\ n_t & = D_n \, \nabla^2 n + G(b,n), \end{array} \right. \tag{1.1}$$

where $(x,t) \in \mathbb{R}^n \times [0,+\infty)$, the model assumes that the bacteria and the nutrients diffuse, with diffusion coefficients D_b and D_n, respectively. There are kinetic functions, $F = F(b,n)$ and $G = G(b,n)$, which represents the growth rate of bacteria cells and the consumption rate of the nutrient, respectively (see [16]).

However, there exist other models explaining phenomena which need to be taken into consideration. A key property of many bacteria is that in the presence of certain chemicals they move preferentially towards higher concentration of the chemical, when it is a chemoattractant, or towards a lower concentration when it is a repellent. This yields to the concept of *chemotaxis*: the directed movement of cells and organisms in response to chemical gradients (see [10]).

1.1 Mathematical models for chemotaxis

The first mathematical models for chemotaxis phenomena date from the 1950s by the Patlak's works and the nowadays Keller and Segel's prevailing methods in the 1970s. The original Keller-Seller (KS) model consists of four coupled reaction-advection-diffusion equations, which it can be reduced under quasi-steady-state assumptions to a model for two unknown functions. The general form of the model is

$$\begin{cases} b_t &= \nabla \cdot (k_1(b,n)\nabla b - k_2(b,n)\nabla n) + F(b,n), \\ n_t &= D_n \nabla^2 n + G(b,n), \end{cases} \tag{1.2}$$

where b represents the bacteria, cell or any organism density, n is the chemical or nutrient concentration; both within a domain $\Omega \subset \mathbb{R}^n$. Also, $k_1(b,n)$ describes the diffusivity of the cells (sometimes also called motility) while $k_2(b,n)$ is the chemotactic sensitivity; both functions may depend on the levels of b and n. The functions $F(b,n)$ and $G(b,n)$ again are kinetic functions describing the growth rate of bacteria cells and the consumption or degradation rate of the nutrient, respectively.

In contrast to the reaction diffusion system (1.1), these models come under the class of strongly coupled parabolic systems and are also known in the literature as cross-diffusion systems, as they naturally incorporate inter-species and intra-species competition, namely in the system (1.2), the interaction between the chemical and the bacteria.

Later, Mimura and Kawasaki [15] are who brought the concept of cross-diffusion to describe numerically the segregation effects for large time of a two-component (inter-species) competitive system. Cross-diffusion expresses the population fluxes of one species due to the presence of the other species. Positive cross-diffusion term denotes one species tends to move in the direction of lower concentration of another species.

After these groundbreaking works for chemotaxis, the literature has speedily grown in the last decades thanks to the work and analysis of some special and remarkable cases. The main focus on these models consists on the study of their solvability.

The great obstacle when these cross-diffusion models are studied is the lack of structure a that allows to use directly standard theorems in parabolic partial differential equations, as the classic results from [1], [2], [3] and [12].

However, in the last two decades, there have been some other techniques developed to obtain results thereon. Galiano and et al. [9] introduced a nonlinear population model with cross-diffusion terms for two competing species, which is studied analytically and numerically. Due to the cross diffusion terms, the problem is strongly nonlinear and so, no maximum principle generally applies.

Most recently, Anaya *et al.* [4] worked with a model of the indirect transmission of an epidemic disease between two spatially distributed host populations having non-coincident spatial domains with nonlocal and cross-diffusion, they proved the existence of weak solutions and, by regularity, theory they showed the existence and uniqueness of classical solutions.

Subsequently, Arumugam *et al.* [6] established the existence of weak solutions for a Keller–Segel chemotaxis system with an additional cross–diffusion term in the second equation.

In a recent contribution, Plaza [19] made the justification of the macroscopic, mean-field nutrient taxis system with doubly degenerate cross-diffusion (1.3) proposed by Leyva et al. [13], which models the complex spatio-temporal dynamics exhibited by the bacterium *Bacillus subtilis* during experiments run in vitro (Ohgiwari et al. [18]). This system is established as follows

$$\partial_t \begin{pmatrix} b \\ n \end{pmatrix} = \nabla \cdot \left(\begin{pmatrix} \sigma bn & -\sigma b^2 n \chi(n) \\ 0 & 1 \end{pmatrix} \nabla \begin{pmatrix} b \\ n \end{pmatrix} \right) + \begin{pmatrix} bn \\ -bn \end{pmatrix}, \tag{1.3}$$

where $\sigma > 0$ a constant measuring the hardness of the agar medium (large σ for low agar concentration) and $\chi = \chi(n)$ the receptor law (see [13]).

In the present work, we study the Cauchy problem for a variation to the model (1.3). The structure of this study is the following.

In the next sections, we introduce a chemotaxis model, we present the main hypotheses and parameters for the Cauchy problem, we give the definition of weak solution and a theorem asserting its existence is established.

In chapter two, the regularized systems are introduced in order to present the approximated solutions as a result of Schauder's fixed point theorem. To achieve this goal some estimates are demonstrated. We prove the non negativity of approximated solutions and some more estimations which lead to a limit solution.

Finally, in chapter three, we conclude the proof of the existence of a weak solution to the Cauchy problem (1.4)-(1.8) as a limit of approximated solutions. We also give a short conclusion.

1.2 Existence framework

In this work we consider a chemotaxis model for a bacterial nutrient-taxis system with doubly degenerate cross-diffusion, which reads as follows:

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \nabla \cdot \left(A_\delta(u,v) \nabla \begin{pmatrix} u \\ v \end{pmatrix} \right) + F(u,v) \qquad \text{on } Q_T := \Omega \times (0,T), \tag{1.4}$$

where $T > 0$ is a fixed time and Ω is a bounded domain in $\mathbb{R}^d, d \in \mathbb{N}$, with smooth boundary $\partial\,\Omega$. For $\delta > 0$ fixed we consider $A_\delta(u,v)$, the difussion matrix, defined as

$$A_\delta(u,v) = A(u,v) + \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix} := \begin{pmatrix} \delta + \sigma u^{\frac{1}{2}} v^{\frac{1}{2}} & -\sigma u^2 v \chi(u,v) \\ 0 & \delta + 1 \end{pmatrix} \tag{1.5}$$

with $\sigma > 0$ and $F(u,v) = \begin{pmatrix} F_1(u,v) \\ F_2(u,v) \end{pmatrix}$, where F_1 and F_2 are smooth functions for $(u,v) \in \mathbb{R}^2$, and $\chi = \chi(u,v)$ is a bounded non-negative continuous function on $\mathbb{R}^2$. Additionally, the system (1.4) is supplemented by non-flux boundary conditions

$$\Big((\delta + \sigma u^{\frac{1}{2}} v^{\frac{1}{2}}) \nabla u - \sigma u^2 v \chi(u,v) \nabla v \Big) \cdot \eta = 0, \tag{1.6}$$

$$\nabla v \cdot \eta = 0, \quad \text{on } \Gamma_T := \partial\Omega \times (0,T), \tag{1.7}$$

where η is the exterior unit normal to $\partial\,\Omega$. A no-flux boundary condition is imposed on $\partial\Omega$ such that the ecosystem is closed to the exterior environment.

Finally we implement the initial conditions

$$u(x,0) = u_0(x) \in L^2(\Omega), \;\; v(x,0) = v_0(t) \in L^2(\Omega) \quad \text{for all } \; x \in \Omega. \tag{1.8}$$

1.3 Hypotheses and main result

From now on we fix $\delta > 0$. We want the matrix $A_\delta(u,v)$ to be *uniformly definite positive* whenever $u, v \geq 0$, then we ask for some conditions on the function $\chi(u,v)$. We assume that

$$0 < \chi(u,v)^4 \leq \frac{16}{\sigma^2 u^7 v^3} \quad \text{for any } u, v > 0. \tag{1.9}$$

Notice that the last conditions assure that χ is bounded. For instance, we accomplish the conditions (1.9) with

$$\chi(u,v)^4 = \frac{16}{\sigma^2 (K_d + u^2)^{\frac{7}{2}} (K_d + v^2)^{\frac{3}{2}}},$$

where K_d is a positive constant (in some models, K_d represents the receptor-ligand binding dissociation constant, which has nutrient concentration units, and represents the nutrient level needed for half of receptor to be occupied). So it turns out that the diffusion matrix is uniformly definite positive.

In the following, we denote by I the identity matrix in $M_{2\times 2}(\mathbb{R})$.

Proposition 1.1. Assuming (1.9) it holds that $A(u,v) \geq \delta I$, for any $u, v \geq 0$.

Proof. Let be $u, v \geq 0$. From the hypothesis, we have that $2\sqrt{\sigma}(uv)^{\frac{1}{4}} - \sigma u^2 v \chi(u,v) \geq 0$ and then

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix}^{\intercal} A_\delta(u,v) \begin{pmatrix} x \\ y \end{pmatrix} &= \delta(x^2+y^2) + (\sigma(uv)^{\frac{1}{2}} x^2 + y^2) - \sigma u^2 v \chi(u,v) xy \\ &\geq \delta(x^2+y^2) + 2\sqrt{\sigma}(uv)^{\frac{1}{4}}\,|xy| - \sigma u^2 v \chi(u,v)|xy| \\ &\geq \delta(x^2+y^2) = \begin{pmatrix} u \\ v \end{pmatrix}^{\intercal} (\delta I) \begin{pmatrix} u \\ v \end{pmatrix},\end{aligned}$$

where it was used $ax^2 + by^2 \geq 2\sqrt{ab}\,|xy|$ for $a, b \geq 0$. □

If condition (1.9) does not hold, the ellipticity condition established in Proposition 1.1 could not be true, which leads to unbounded solutions in finite time that is called blow-up. Therefore, the assumption (1.9) allows us to actually prove the existence of weak solutions, via suitable estimations.

Another important ingredient to have in mind is the form of the kinetic functions. In order to have our model realistic and congruent to the biological phenomena, we need some kinetic functions appearing in the literature in chemotaxis systems (see [4], [6], [9]):

i) **Lotka-Volterra terms.** Assuming $R_i, \gamma_{ij} > 0$, for $i, j \in \{1, 2\}$,

$$F_1(u,v) = (R_1 - \gamma_{11}u - \gamma_{12}v)u, \tag{1.10}$$
$$F_2(u,v) = (R_2 - \gamma_{21}u - \gamma_{22}v)v, \tag{1.11}$$

herein R_1 and R_2 are the intrinsic growth rates of the first and second species or chemicals (in our case the bacteria and the nutrient), respectively, γ_{11} and γ_{22} are the coefficients of intra-specific competition, and γ_{12} and γ_{21} are those of inter-specific competitions.

ii) **Creation-degradation terms.**

$$F_1(u,v) = \beta_1 v - \alpha_1 u, \tag{1.12}$$
$$F_2(u,v) = \alpha_2 u - \beta_2 v, \tag{1.13}$$

where $\beta_1 > 0$ and $\alpha_2 > 0$ are the creation rates for bacteria and nutrient, respectively, and $\alpha_2, \beta_2 > 0$ are the degradation rates.

iii) **Logistic growth terms.**

$$F_1(u,v) = u(1-u), \tag{1.14}$$
$$F_2(u,v) = v(1-v). \tag{1.15}$$

We can also choose any combination of the kinetic terms F_1 and F_2 stated before.

Now, we introduce the definition of a weak solution to our problem (1.4)-(1.8).

Definition 1.2. We say that (u,v) is a weak solution to the problem (1.4)-(1.8) if

a) $u, v \geq 0$ satisfy the regularity properties

$$u, v \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \cap H^1(0,T; \big(W^{1,\infty}(\Omega)\big)^*),$$

b) the equations (1.4)-(1.7) are satisfied in the following sense

$$\int_0^T \langle u_t, \varphi\rangle \, dt + \iint_{Q_T} \Big((\delta + \sigma(uv)^{\frac{1}{2}})\nabla u - (\sigma u^2 v \chi(u,v))\nabla v \Big) \cdot \nabla\varphi \, dx\, dt, \\ = \iint_{Q_T} F_1(u,v)\varphi \, dx\, dt \tag{1.16}$$

$$\int_0^T \langle v_t, \psi\rangle \, dt + (\delta+1) \iint_{Q_T} \nabla v \cdot \nabla\psi \, dx\, dt = \iint_{Q_T} F_2(u,v)\psi \, dx\, dt, \tag{1.17}$$

for all $\varphi, \psi \in L^2(0,T; W^{1,\infty}(\Omega))$ test functions, where $\langle \cdot,\cdot\rangle$ represents the duality product in $(W^{1,\infty}(\Omega))^* \times W^{1,\infty}(\Omega)$,

c) and the initial conditions (1.8) are satisfied in the sense

$$\lim_{t\longrightarrow 0} \|u(\cdot,t) - u_0(\cdot)\|_{(W^{1,\infty}(\Omega))^*} = 0, \tag{1.18}$$

$$\lim_{t\longrightarrow 0} \|v(\cdot,t) - v_0(\cdot)\|_{(W^{1,\infty}(\Omega))^*} = 0. \tag{1.19}$$

We can now establish the main-existence result.

Theorem 1.3. *Assume the condition (1.9), if $u_0, v_0 \in L^2(\Omega)$ are such that $u_0, v_0 \geq 0$ a.e. in Ω and F_1, F_2 satisfy*

$$|F_1(p,q)|, |F_2(p,q)| \leq C_F(p^2 + q^2 + 1), \tag{1.20}$$

$$F_1(0,q) \geq 0, \tag{1.21}$$

$$F_2(p,0) \geq 0, \tag{1.22}$$

$$F_1(p,q) \leq C_F\, p, \tag{1.23}$$

$$F_2(p,q) \leq C_F\, q, \qquad \textit{for all} \;\; p, q \geq 0, \tag{1.24}$$

for some $C_F > 0$ uniform constant, then the problem (1.4)-(1.8) has a weak solution (u,v) in the sense of Definition 1.2.

Notice that the kinetic functions described in (1.10)-(1.15) satisfy the hypotheses of Theorem 1.3. The proof of Theorem 1.3 can be found in Section 3.1.

Chapter 2

Regularized systems and approximated solutions

In this chapter we introduce the approximated problems to (1.4)-(1.8) which we will call *regularized systems*. These systems help us to develop fixed point technique to prove the existence of approximated solutions, since we have established before, that the matrix diffusion $A(u,v)$ does not have uniformly upper bounds.

In the following we assume the hypothesis of Theorem 1.3. For $\varepsilon > 0$ arbitrary, we start by defining

$$f_\varepsilon(s) := \frac{s^+}{1+\varepsilon\, s^+}, \quad \text{where } \; s^+ := \max\{0, s\}.$$

Then it follows that

$$\begin{aligned} 0 \le f_\varepsilon(s) &\le \frac{1}{\varepsilon}, \\ \lim_{\varepsilon \longrightarrow 0} f_\varepsilon(s) &= s^+, \quad \text{for any } \; s \in \mathbb{R}. \end{aligned} \tag{2.1}$$

We consider now the approximated diffusion matrices

$$A_{\delta,\varepsilon}(\overline{u},\overline{v}) := A_\delta(f_\varepsilon(\overline{u}), f_\varepsilon(\overline{v})) = \begin{pmatrix} \delta + \sigma\,[f_\varepsilon(\overline{u}) f_\varepsilon(\overline{v})]^{\frac{1}{2}} & -\sigma f_\varepsilon(\overline{u})^2 f_\varepsilon(\overline{v}) \chi(f_\varepsilon(\overline{u}), f_\varepsilon(\overline{v})) \\ 0 & \delta + 1 \end{pmatrix}, \tag{2.2}$$

for all $(\overline{u},\overline{v}) \in \mathbb{R}^2$. Note that by Proposition 1.1, we have that $A_{\delta,\varepsilon}$ is uniformly definite positive in $\mathbb{R}^2$, namely,

$$A_{\delta,\varepsilon}(\overline{u},\overline{v}) \ge \delta I \quad \text{for any } \; (\overline{u},\overline{v}) \in \mathbb{R}^2. \tag{2.3}$$

In the same way, we define

$$F_\varepsilon(\overline{u},\overline{v}) = \begin{pmatrix} F_{1,\varepsilon}(\overline{u},\overline{v}) \\ F_{2,\varepsilon}(\overline{u},\overline{v}) \end{pmatrix} := \begin{pmatrix} F_1(f_\varepsilon(\overline{u}), f_\varepsilon(\overline{v})) \\ F_2(f_\varepsilon(\overline{u}), f_\varepsilon(\overline{v})) \end{pmatrix}. \tag{2.4}$$

Since F_i is continuous and $0 \le f_\varepsilon(\overline{u}), f_\varepsilon(\overline{v}) \le \frac{1}{\varepsilon}$, we obtain that

$$|F_{i,\varepsilon}(\overline{u},\overline{v})| \le M_\varepsilon, \tag{2.5}$$

where M_ε is a constant depending on $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ and ε, and for $i = 1, 2$.

2.1 Fixed point technique for regularized systems

Our next aim is to prove the existence of a non negative solution $(u_\varepsilon, v_\varepsilon)$ to the regularized system

$$\begin{cases} \partial_t \begin{pmatrix} w \\ z \end{pmatrix} = \nabla \cdot \left(A_{\delta,\varepsilon}(w,z) \nabla \begin{pmatrix} w \\ z \end{pmatrix} \right) + F_\varepsilon(w,z) & \text{in } Q_T := \Omega \times (0,T), \quad (2.6a) \\ (a_\varepsilon(w,z)\nabla w - b_\varepsilon(w,z)\nabla z) \cdot \eta = 0 & \quad (2.6b) \\ \nabla z \cdot \eta = 0 & \text{on } \Gamma_T := \partial\Omega \times (0,T), \quad (2.6c) \\ w(x,0) = u_0(x) \in L^2(\Omega), \;\; z(x,0) = v_0(x) \in L^2(\Omega) & \text{for all } x \in \Omega, \quad (2.6d) \end{cases}$$

where

$$\begin{aligned} a_\varepsilon(w,z) &= \delta + \sigma \left[f_\varepsilon(w) f_\varepsilon(z) \right]^{\frac{1}{2}}, \\ b_\varepsilon(w,z) &= \sigma f_\varepsilon(w)^2 f_\varepsilon(z)\, \chi(f_\varepsilon(w), f_\varepsilon(z)). \end{aligned}$$

We demonstrate the existence of a non negative solution $(u_\varepsilon, v_\varepsilon)$ by using Schauder's fixed point theorem (see [21]).

So, in order to apply Shauder's fixed point theorem, we consider fist the Banach space

$$\mathcal{X}_T := \left\{ u \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^1(\Omega)) \; : \; u_t \in L^2(0,T; \left(H^1(\Omega)\right)^*) \right\}$$

whose norm is defined by

$$\|u\|_{\mathcal{X}_T} := \max \left\{ \|u\|_{L^\infty(0,T;L^2(\Omega))}, \|u\|_{L^2(0,T;H^1(\Omega))}, \|u_t\|_{L^2(0,T;(H^1(\Omega))^*)} \right\},$$

and the following subspace

$$K := \{\overline{u} \in L^2(Q_T) \cap \mathcal{X}_T \,:\, \|\overline{u}\|_{\mathcal{X}_T} \leq \rho\}$$

where $\rho > 0$ is a constant that we need to choose in a convenient way and where K is thought of as a subspace of $L^2(Q_T)$, i.e., K is endowed with the norm $\|\cdot\|_{L^2(Q_T)}$.

It is straightforward to verify that K is a closed convex subspace of $L^2(Q_T)$.

Additionally, if we fix $(\overline{u}, \overline{v}) \in K$, we can apply the general results for linear systems of equations associated to uniformly parabolic operators in [12, Chapter VII], to show that there exists a unique solution $(u, v) \in \mathcal{X}_T \times \mathcal{X}_T$ for the problem

$$\begin{cases} \partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \nabla \cdot \left(A_{\delta,\varepsilon}(\overline{u}, \overline{v}) \nabla \begin{pmatrix} u \\ v \end{pmatrix} \right) + F_\varepsilon(\overline{u}, \overline{v}) & \text{in } Q_T := \Omega \times (0, T), \quad (2.7a) \\ (a_\varepsilon(\overline{u}, \overline{v}) \nabla u - b_\varepsilon(\overline{u}, \overline{v}) \nabla v) \cdot \eta = 0 & \quad (2.7b) \\ \nabla v \cdot \eta = 0 & \text{on } \Gamma_T := \partial\Omega \times (0, T), \quad (2.7c) \\ u(x, 0) = u_0(x) \in L^2(\Omega), \;\; v(x, 0) = v_0(t) \in L^2(\Omega) & \text{for all } x \in \Omega. \quad (2.7d) \end{cases}$$

For this, we notice that $(A_{\delta,\varepsilon}(\overline{u}, \overline{v}))_{ij} \in L^\infty(Q_T)$ due to (2.1),(2.2) and the boundedness of χ; and that $F_{i,\varepsilon}(\overline{u}, \overline{v}) \in L^2(Q_T)$ by (2.5). Finally, we recall the uniform definite positiveness of $A_{\delta,\varepsilon}(\overline{u}, \overline{v})$. Then the classical results from [12, Chapter VII] leads to the existence and uniqueness of the problem (2.7a)-(2.7d).

The following step is to define the map $\mathcal{T} : K \times K \longrightarrow \mathcal{X}_T \times \mathcal{X}_T$ by $T(\overline{u}, \overline{v}) = (u, v)$ where $(u, v) \in \mathcal{X}_T \times \mathcal{X}_T$ is the unique solution to (2.7a)-(2.7d). We want now to apply the Schauder's fixed point theorem to $\mathcal{T}$. First, we shall prove that $\mathcal{T}$ is invariant, which means that we need to find ρ such that $\mathcal{T}(K \times K) \subseteq K \times K$, for this purpose we establish the next proposition:

Proposition 2.1. For each $\varepsilon > 0$, there exists a uniform constant $R_\varepsilon > 0$, independent on ρ, such that for any $(\overline{u}, \overline{v}) \in (L^2(Q_T) \cap \mathcal{X}_T)^2$ the estimate $\|u\|_{\mathcal{X}_T}, \|v\|_{\mathcal{X}_T} \leq R_\varepsilon$ holds.

Proof. Let $(\overline{u}, \overline{v}) \in (L^2(Q_T) \cap \mathcal{X}_T)^2$ and (u, v) be the unique solution to (2.7a)-(2.7d). Due to the regularity of (u, v) we can use them as test functions in the equations (2.7a)-(2.7c) to obtain, for almost every $t \in [0, T]$,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (u^2 + v^2)\, dx + \delta \int_\Omega \left(|\nabla u|^2 + |\nabla v|^2 \right) \, dx \qquad (2.8) \\ \leq &\frac{1}{2}\frac{d}{dt}\int_\Omega (u^2 + v^2)\, dx + \int_\Omega \left(A_{\delta,\varepsilon}(\overline{u}, \overline{v}) \nabla \begin{pmatrix} u \\ v \end{pmatrix} \right) \cdot \nabla \begin{pmatrix} u \\ v \end{pmatrix} \, dx \\ = &\int_\Omega \left(F_{1,\varepsilon}(\overline{u}, \overline{v}) u + F_{2,\varepsilon}(\overline{u}, \overline{v}) v \right) \, dx \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2} \int_\Omega \left(u^2 + v^2 + F_{1,\varepsilon}^2(\overline{u}, \overline{v}) + F_{2,\varepsilon}^2(\overline{u}, \overline{v})\right) \, dx \\
&\leq M_\varepsilon^2 \, |\Omega| + \frac{1}{2} \int_\Omega (u^2 + v^2) \, dx,
\end{aligned} \tag{2.9}
$$

where it was used (2.3) and (2.5). From (2.8) and (2.9) we deduce

$$
\frac{d}{dt} \int_\Omega (u^2 + v^2) \, dx \leq 2 \, M_\varepsilon^2 \, |\Omega| + \int_\Omega (u^2 + v^2) \, dx
$$

and, applying the Gronwall's lemma to the function $E(t) = \displaystyle\int_\Omega (u^2 + v^2) \, dx$, it follows that

$$
\int_\Omega (u^2 + v^2) \, dx \leq \left(2M^2 \, |\Omega| + \int_\Omega (u_0^2 + v_0^2) \, dx \right) e^t. \tag{2.10}
$$

By taking the supremum over $(0, T)$, we get

$$
\sup_{t \in (0,T)} \left[\|u(\cdot, t)\|_{L^2(\Omega)}^2 + \|v(\cdot, t)\|_{L^2(\Omega)}^2 \right] \leq e^T \left(2M_\varepsilon^2 \, |\Omega| + \|u_0\|_{L^2(\Omega)}^2 + \|v_0\|_{L^2(\Omega)}^2 \right), \tag{2.11}
$$

which proves that

$$
\|u\|_{L^\infty(0,T;L^2(\Omega))} \, , \|v\|_{L^\infty(0,T;L^2(\Omega))} \leq R_1 \tag{2.12}
$$

where $R_1 > 0$ is a constant depending entirely on $u_0, v_0, \varepsilon, \Omega$ and T. Now, integrating over $(0, T)$ the two sides of the inequalities, (2.8) and (2.9), and using the estimate (2.10), it holds that

$$
\begin{aligned}
\delta \iint_{Q_T} \left(|\nabla u|^2 + |\nabla v|^2 \right) \, dx \, dt \leq M_\varepsilon^2 \, |\Omega| \, T + \left(M_\varepsilon^2 \, |\Omega| + \frac{1}{2} \int_\Omega (u_0^2 + v_0^2) \, dx \right) \left(e^T - 1 \right) \\
+ \frac{1}{2} \int_\Omega (u_0^2 + v_0^2) \, dx,
\end{aligned}
$$

which leads to

$$
\begin{aligned}
\|u\|_{L^2(0,T;H^1(\Omega))}^2 + \|v\|_{L^2(0,T;H^1(\Omega))}^2 &= \iint_{Q_T} \left(u^2 + v^2 + |\nabla u|^2 + |\nabla v|^2 \right) \, dx \, dt \\
&\leq \left(2 + \frac{1}{\delta} \right) \left(M_\varepsilon^2 \, |\Omega| + \frac{1}{2} \int_\Omega (u_0^2 + v_0^2) \, dx \right) \left(e^T - 1 \right) \\
&\quad + \frac{1}{\delta} \left(\frac{1}{2} \int_\Omega (u_0^2 + v_0^2) \, dx + M_\varepsilon^2 \, |\Omega| \, T \right),
\end{aligned}
$$

where we integrated again (2.10) over $(0, T)$. This gives us the estimate

$$
\|u\|_{L^2(0,T;H^1(\Omega))} \, , \|v\|_{L^2(0,T;H^1(\Omega))} \leq R_2, \tag{2.13}
$$

where $R_2 > 0$ is a constant depending entirely on $u_0, v_0, \varepsilon, \Omega$, δ and T.

Finally we prove that $\|u_t\|_{L^2(0,T;(H^1(\Omega))^*)}, \|v_t\|_{L^2(0,T;(H^1(\Omega))^*)} \leq R_3$, where $R_3 > 0$ is a constant (independent of ρ). For this, we fix $\varphi \in L^2(0,T;(H^1(\Omega)))$ as a test function and using the first equation of (2.7a) we obtain

$$\left|\int_0^T \langle u_t, \varphi\rangle \, dt\right| \leq \iint_{Q_T} |a_\varepsilon(\overline{u},\overline{v})||\nabla u \cdot \nabla\varphi| \, dx\, dt + \iint_{Q_T} |b_\varepsilon(\overline{u},\overline{v})||\nabla v \cdot \nabla\varphi| \, dx\, dt$$
$$+ \iint_{Q_T} |F_{1,\varepsilon}(\overline{u},\overline{v})| \, |\varphi| \, dx\, dt$$

$$\leq \left(\delta + \frac{\sigma}{\varepsilon}\right)\left(\iint_{Q_T} |\nabla u|^2 \, dx\, dt\right)^{\frac{1}{2}}\left(\iint_{Q_T} |\nabla\varphi|^2 \, dx\, dt\right)^{\frac{1}{2}}$$
$$+ \left(\frac{\sigma\chi_{max}}{\varepsilon^3}\right)\left(\iint_{Q_T} |\nabla u|^2 \, dx\, dt\right)^{\frac{1}{2}}\left(\iint_{Q_T} |\nabla\varphi|^2 \, dx\, dt\right)^{\frac{1}{2}}$$
$$+ \left(\iint_{Q_T} F_{1,\varepsilon}(\overline{u},\overline{v})^2 \, dx\, dt\right)^{\frac{1}{2}}\left(\iint_{Q_T} \varphi^2 \, dx\, dt\right)^{\frac{1}{2}},$$

where it was used the Hölder's inequality and (2.1). By using the estimate (2.13) and (2.5) it follows that

$$\left|\int_0^T \langle u_t, \varphi\rangle \, dt\right| \leq \left(\left(\delta + \frac{\sigma}{\varepsilon} + \frac{\sigma\chi_{max}}{\varepsilon^3}\right) R_2 + M_\varepsilon \sqrt{T\,|\Omega|}\right) \|\varphi\|_{L^2(0,T;H^1(\Omega))}. \tag{2.14}$$

Analogously, it is proved that

$$\left|\int_0^T \langle v_t, \varphi\rangle \, dt\right| \leq \left(R_2 + M_\varepsilon \sqrt{T\,|\Omega|}\right) \|\varphi\|_{L^2(0,T;H^1(\Omega))}. \tag{2.15}$$

The estimates (2.14) and (2.15) yield

$$\|u_t\|_{L^2(0,T;(H^1(\Omega))^*)}, \|v_t\|_{L^2(0,T;(H^1(\Omega))^*)} \leq R_3 \tag{2.16}$$

for some constant R_3 and finishing the proof by taking $R_\varepsilon = \max\{R_1, R_2, R_3\}$. □

Proposition 2.1 assure us that if we take $0 < R_\varepsilon < \rho$ then $\mathcal{T} : K \times K \longrightarrow K \times K$ is well defined.

Now, we are in shape to prove the continuity of $\mathcal{T} : K \times K \longrightarrow K \times K$, for this purpose we need some classical results of compact embeddings for Sobolev spaces and for Banach-space-valued functions.

We adopt, at this point, the notation $X \subset\subset Y$ meaning that X is compactly (linearly) embedded in Y, and $X \hookrightarrow Y$ for a continuous (linear) embedding, where X and Y are

Banach spaces.

Theorem 2.2 ([8], Section 5.7). *For any U bounded open subset of $\mathbb{R}^d, d \in \mathbb{N}$, with $\partial U \in C^1$ and for all $1 \leq p < \infty$, it satisfies*

$$W^{1,p}(U) \subset\subset L^p(U).$$

In particular $H^1(U) \subset\subset L^2(U)$.

The second result establishes the compactness embedding for Banach-space-valued functions, first proved by Aubin in 1963 for reflexive Banach spaces [7] and subsequently Simon removed this condition in 1986 [22].

Theorem 2.3 (Aubin-Lions-Simmon lemma, [20], [22]). *Let X_0, X and X_1 be Banach spaces satisfying $X_0 \subset\subset X \hookrightarrow X_1$ with continuous embeddings. For $0 < T < \infty$ and $1 \leq p, q \leq \infty$ let*

$$W = \{u \in L^p(0,T;X_0) \,:\, u_t \in L^q(0,T;X_1)\}$$

Banach space whose norm is defined by $\|u\|_W = \max\left\{\|u\|_{L^p(0,T;X_0)}, \|u_t\|_{L^q(0,T;X_1)}\right\}$. Then

i) if $p < \infty$, $W \subset\subset L^p(0,T;X)$;

ii) if $p = \infty$ and $q > 1$, $W \subset\subset C([0,T];X)$.

Next, we apply Theorem 2.2 to obtain $H^1(\Omega) \subset\subset L^2(\Omega) \hookrightarrow (H^1(\Omega))^*$ continuous embedding and then, by Theorem 2.3, we conclude that

$$\left\{u \in L^2(0,T;H^1(\Omega)) \,:\, u_t \in L^2(0,T;(H^1(\Omega))^*)\right\} \subset\subset L^2(0,T;L^2(\Omega)) = L^2(Q_T),$$

thus, by the definition of $\mathcal{X}_T$, we have the compact embedding

$$\mathcal{X}_T \subset\subset L^2(Q_T). \tag{2.17}$$

The next proposition proves the continuity of $\mathcal{T} : K \times K \longrightarrow K \times K$.

Proposition 2.4. $\mathcal{T}$ is a compact continuous function in $K \times K$.

Proof. Recall that K is a subspace of $L^2(Q_T)$. In order to demonstrate the continuity of $\mathcal{T}$, let $(\overline{u}_n, \overline{v}_n) \in K \times K$ be a subsequence and $(\overline{u}, \overline{v}) \in K \times K$ such that

$$(\overline{u}_n, \overline{v}_n) \longrightarrow (\overline{u}, \overline{v}) \quad \text{a.e. in } Q_T \text{ and strongly in } \left(L^2(Q_T)\right)^2. \tag{2.18}$$

We need to prove that

$$(u_n, v_n) := \mathcal{T}(\overline{u}_n, \overline{v}_n) \longrightarrow (u, v) := \mathcal{T}(\overline{u}, \overline{v}) \quad \text{a.e. in } Q_T \text{ strongly in } \left(L^2(Q_T)\right)^2. \tag{2.19}$$

To this end, by a characterization of convergence, it is sufficient to prove that any subsequence of (u_n, v_n) has a further subsequence which convergences to (u, v) strongly in $\left(L^2(Q_T)\right)^2$.

Let (u_{n_k}, v_{n_k}) be a subsequence of (u_n, v_n). By Proposition 2.1, by (2.17) and since $L^2(0,T;H^1(\Omega))$ and $L^2(0,T;(H^1(\Omega))^*)$ are Hilbert spaces, we can assure the existence of a further subsequence $(u_{n_{k_j}}, v_{n_{k_j}})$ such that

$$
\begin{cases}
(u_{n_{k_j}}, v_{n_{k_j}}) \to (u', v') & \text{a.e. in } Q_T \text{ and strongly in } \left(L^2(Q_T)\right)^2 \qquad (2.20a)\\
(u_{n_{k_j}}, v_{n_{k_j}}) \rightharpoonup (u', v') & \text{weakly in } \left(L^2(0,T;H^1(\Omega))\right)^2 \qquad (2.20b)\\
(\partial_t u_{n_{k_j}}, \partial_t v_{n_{k_j}}) \rightharpoonup (u'_t, v'_t) & \text{weakly in } \left(L^2(0,T;(H^1(\Omega))^*)\right)^2 \qquad (2.20c)
\end{cases}
$$

for some $(u', v') \in L^2(Q_T)$.

Since $(u_{n_{k_j}}, v_{n_{k_j}})$ are classical solutions of (2.7a)-(2.7d) for $(\overline{u}_{n_{k_j}}, \overline{v}_{n_{k_j}})$ respectively, in particular they are weak solutions in the sense of distributions, which means that

$$
\begin{aligned}
\int_\Omega \left(\partial_t u_{n_{k_j}}\right) \varphi\, dx + \int_\Omega \left(a_\varepsilon(\overline{u}_{n_{k_j}}, \overline{v}_{n_{k_j}})\nabla u_{n_{k_j}} - b_\varepsilon(\overline{u}_{n_{k_j}}, \overline{v}_{n_{k_j}})\nabla v_{n_{k_j}}\right) \cdot \nabla\varphi\, dx \\
= \int_\Omega F_{1,\varepsilon}(\overline{u}_{n_{k_j}}, \overline{v}_{n_{k_j}})\varphi\, dx, \qquad (2.21)
\end{aligned}
$$

$$
\int_\Omega \left(\partial_t v_{n_{k_j}}\right) \psi\, dt + \int_\Omega \left((\delta+1)\nabla v_{n_{k_j}}\right) \cdot \nabla\psi\, dx = \int_\Omega F_{2,\varepsilon}(\overline{u}_{n_{k_j}}, \overline{v}_{n_{k_j}})\psi\, dx; \qquad (2.22)
$$

for any $\varphi, \psi \in C^\infty(\Omega)$ and for a.e. $t \in [0,T]$.

Using (2.18), (2.20a)-(2.20c), the fact that $F_{1,\varepsilon}(\overline{u}_{n_{k_j}}, \overline{v}_{n_{k_j}}), F_{2,\varepsilon}(\overline{u}_{n_{k_j}}, \overline{v}_{n_{k_j}}) \in L^\infty(\Omega)$ (see (2.5)), and the continuity of $a_\varepsilon, b_\varepsilon, F_1$ and F_2, we deduce, tending $n_{k_j} \longrightarrow \infty$, that

$$
\begin{aligned}
\int_\Omega u'_t\, \varphi\, dx + \int_\Omega (a_\varepsilon(\overline{u}, \overline{v})\nabla u' - b_\varepsilon(\overline{u}, \overline{v})\nabla v') \cdot \nabla\varphi\, dx &= \int_\Omega F_{1,\varepsilon}(\overline{u}, \overline{v})\varphi\, dx, \\
\int_\Omega v'_t\, \psi\, dx + \int_\Omega ((\delta+1)\nabla v') \cdot \nabla\psi\, dx &= \int_\Omega F_{2,\varepsilon}(\overline{u}, \overline{v})\psi\, dx;
\end{aligned}
$$

for any $\varphi, \psi \in C^\infty(\Omega)$ and for a.e. $t \in [0,T]$.

This shows, by the uniqueness of weak solutions in distribution sense (see again [12]) and the definition of $\mathcal{T}$, that $(u', v') = (u, v)$ a.e. in Q_T, and which proves in turn that

$$
(u_{n_{k_j}}, v_{n_{k_j}}) \to (u, v) \quad \text{a.e. in } Q_T \text{ and strongly in } \left(L^2(Q_T)\right)^2.
$$

Therefore (2.19) holds, demonstrating the continuity of $\mathcal{T}$.

By the same arguments, the compactness of $\mathcal{T}$ is proved using again the compact embedding (2.17). $\square$

Finally from Proposition 2.4 we are able to use Schauder's fixed point theorem to operator $\mathcal{T}$; then, for each $\varepsilon > 0$, there exists a solution $(u_\varepsilon, v_\varepsilon) \in K \times K$ to the Cauchy problem (2.6a)-(2.6d). In particular, the solutions $(u_\varepsilon, v_\varepsilon)$ satisfy the following weak conditions in a distribution sense:

$$\int_\Omega (\partial_t u_\varepsilon)\, \varphi \, dx + \int_\Omega (a_\varepsilon(u_\varepsilon, v_\varepsilon)\nabla u_\varepsilon - b_\varepsilon(u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon) \cdot \nabla \varphi \, dx = \int_\Omega F_{1,\varepsilon}(u_\varepsilon, v_\varepsilon)\varphi \, dx, \tag{2.23}$$

$$\int_\Omega (\partial_t v_\varepsilon)\, \psi \, dx + (\delta + 1) \int_\Omega \nabla v_\varepsilon \cdot \nabla \psi \, dx \, dt = \int_\Omega F_{2,\varepsilon}(u_\varepsilon, v_\varepsilon)\psi \, dx, \tag{2.24}$$

for all $\varphi, \psi \in L^2(0, T; W^{1,\infty}(\Omega))$.

2.2 Estimates for approximated solutions

Proposition 2.5. Assuming the hypothesis of Theorem 1.3. Then the solution $(u_\varepsilon, v_\varepsilon)$ is non negative. Moreover, there exist uniform constants $C_1, C_2, C_3 > 0$ not depending on ε such that

$$\|u_\varepsilon\|_{L^\infty(0,T;\, L^2(\Omega))}, \|v_\varepsilon\|_{L^\infty(0,T;\, L^2(\Omega))} \le C_1, \tag{2.25}$$

$$\|u_\varepsilon\|_{L^2(0,T;\, H^1(\Omega))}, \|v_\varepsilon\|_{L^2(0,T;\, H^1(\Omega))} \le C_2, \tag{2.26}$$

$$\|\partial_t u_\varepsilon\|_{L^2(0,T;\, (W^{1,\infty}(\Omega))^*)}, \|\partial_t v_\varepsilon\|_{L^2(0,T;\, (W^{1,\infty}(\Omega))^*)} \le C_3. \tag{2.27}$$

Proof. We start proving the non negativeness of $(u_\varepsilon, v_\varepsilon)$. Since we have the regularity properties for u_ε, we can consider

$$\varphi = u_\varepsilon^- = \min\{u_\varepsilon, 0\},$$
$$\psi = v_\varepsilon^- = \min\{v_\varepsilon, 0\},$$

as test functions in the equations (2.23) and (2.24), adding both of them and noticing that;

$$\begin{aligned}
&\nabla w \cdot \nabla w^- = \left|\nabla w^-\right|^2, \\
&(\partial_t w)\, w^- = \left(\partial_t w^-\right) w^-, \\
&a_\varepsilon(u, v) = a(0, f_\varepsilon(v)) = \delta && \text{in } \{u \le 0\}, \\
&b_\varepsilon(u, v) = b(0, f_\varepsilon(v)) = 0 && \text{in } \{u \le 0\};
\end{aligned}$$

we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \left[(u_\varepsilon^-)^2 + (v_\varepsilon^-)^2\right]\,dx + \delta\int_\Omega \left(|\nabla u_\varepsilon^-|^2 + |\nabla v_\varepsilon^-|^2\right)\,dx \\
&\le \int_\Omega \partial_t \begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}\cdot\begin{pmatrix} u_\varepsilon^- \\ v_\varepsilon^- \end{pmatrix}\,dx + \int_\Omega \left(A_{\delta,\varepsilon}(u_\varepsilon,v_\varepsilon)\nabla\begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}\right)\cdot\nabla\begin{pmatrix} u_\varepsilon^- \\ v_\varepsilon^- \end{pmatrix}\,dx \\
&= \int_\Omega \left[F_{1,\varepsilon}(u_\varepsilon,v_\varepsilon)\,u_\varepsilon^- + F_{2,\varepsilon}(u_\varepsilon,v_\varepsilon)\,v_\varepsilon^-\right]\,dx \\
&= \int_\Omega \left[F_1(0,f_\varepsilon(v_\varepsilon))\,u_\varepsilon^- + F_2(f_\varepsilon(u_\varepsilon),0)\,v_\varepsilon^-\right]\,dx \le 0,
\end{aligned}
$$

where it was used the hypothesis (1.21) and (1.22). Thus it follows that

$$
\int_\Omega \left[(u_\varepsilon^-)^2 + (v_\varepsilon^-)^2\right]\,dx \le 0,
$$

and therefore, since $u_0, v_0 \ge 0$,

$$
\int_\Omega \left[(u_\varepsilon^-)^2 + (v_\varepsilon^-)^2\right]\,dx \le \int_\Omega \left[(u_0^-)^2 + (v_0^-)^2\right]\,dx = 0.
$$

Which proves that $u_\varepsilon^-, v_\varepsilon^- = 0$ a.e. in Ω for a.e. $t \in [0,T]$. Showing that u_ε and v_ε are non negative.

Now, in order to demonstrate the estimates (2.25)-(2.27), we will follow the same arguments in Proposition 2.1 with some slightly differences, since we need to obtain uniformly bounds not depending on ε. Indeed, taking this time $\varphi = u_\varepsilon$ and $\psi = v_\varepsilon$ as test functions in (2.23) and (2.24), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (u_\varepsilon^2 + v_\varepsilon^2)\,dx + \delta\int_\Omega \left(|\nabla u_\varepsilon|^2 + |\nabla v_\varepsilon|^2\right)\,dx \\
&\le \int_\Omega \partial_t \begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}\cdot\begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}\,dx + \int_\Omega \left(A_{\delta,\varepsilon}(u_\varepsilon,v_\varepsilon)\nabla\begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}\right)\cdot\nabla\begin{pmatrix} u_\varepsilon \\ v_\varepsilon \end{pmatrix}\,dx \\
&= \int_\Omega \left[F_{1,\varepsilon}(u_\varepsilon,v_\varepsilon)\,u_\varepsilon + F_{2,\varepsilon}(u_\varepsilon,v_\varepsilon)\,v_\varepsilon\right]\,dx \\
&= \int_\Omega \left[F_1(f_\varepsilon(u_\varepsilon),f_\varepsilon(v_\varepsilon))\,f_\varepsilon(u_\varepsilon)(1+\varepsilon\,u_\varepsilon) + F_2(f_\varepsilon(u_\varepsilon),f_\varepsilon(v_\varepsilon))\,f_\varepsilon(v_\varepsilon)(1+\varepsilon\,v_\varepsilon)\right]\,dx \\
&\le \int_\Omega \left[f_\varepsilon(u_\varepsilon)^2(1+\varepsilon\,u_\varepsilon) + f_\varepsilon(v_\varepsilon)^2(1+\varepsilon\,v_\varepsilon)\right]\,dx \\
&\le \int_\Omega \left(u_\varepsilon^2 + v_\varepsilon^2\right)\,dx,
\end{aligned}
$$

where we have used (1.23), (1.24), (2.3) and the definition of F_ε (see (2.4)). Summing up the latter inequalities we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u_\varepsilon^2+v_\varepsilon^2)\,dx+\delta\int_\Omega \left(|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\right)\,dx\le\int_\Omega \left(u_\varepsilon^2+v_\varepsilon^2\right)\,dx \tag{2.28}$$

and then, applying Gronwall's inequality, we can deduce

$$\int_\Omega (u_\varepsilon^2+v_\varepsilon^2)\,dx\le\left(\|u_0\|^2_{L^2(\Omega)}+\|v_0\|^2_{L^2(\Omega)}\right)e^{2t}. \tag{2.29}$$

Therefore and taking supremum over $[0,T]$,

$$\|u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))},\|v_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}\le\left(\|u_0\|^2_{L^2(\Omega)}+\|v_0\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}e^T=:C_1.$$

For the estimate (2.26), we integrate (2.28) and (2.29) over $[0,T]$ so we can have

$$\begin{aligned}\|u_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))}+\|v_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))}&=\iint_{Q_T}\left[u_\varepsilon^2+v_\varepsilon^2+|\nabla u_\varepsilon|^2+|\nabla v_\varepsilon|^2\right]\,dx\,dt\\&\le\frac{1}{2}\left(1+\frac{1}{\delta}\right)\left(\|u_0\|^2_{L^2(\Omega)}+\|v_0\|^2_{L^2(\Omega)}\right)(e^{2T}-1)\\&\quad+\frac{1}{2\delta}\left(\|u_0\|^2_{L^2(\Omega)}+\|v_0\|^2_{L^2(\Omega)}\right)=:C_2^2.\end{aligned}$$

Hence $\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega))},\|v_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\le C_2$.

Finally, we prove the estimate (2.27). For this purpose, we notice first that $a_\varepsilon(u_\varepsilon,v_\varepsilon)$ and $b_\varepsilon(u_\varepsilon,v_\varepsilon)$ are uniformly bounded in $L^\infty(0,\,T;\,L^2(\Omega))$. Indeed, from hypothesis (1.9) and the definition of a_ε and b_ε,

$$0\le a_\varepsilon(u_\varepsilon,v_\varepsilon)\le\delta+u_\varepsilon^{\frac{1}{2}}v_\varepsilon^{\frac{1}{2}}\le\delta+\frac{1}{2}(u_\varepsilon+v_\varepsilon), \tag{2.30}$$

$$b_\varepsilon(u_\varepsilon,v_\varepsilon)\le 2\sqrt{\sigma}u_\varepsilon^{\frac{1}{4}}v_\varepsilon^{\frac{1}{4}}\le\sqrt{\sigma}(u_\varepsilon^{\frac{1}{2}}+v_\varepsilon^{\frac{1}{2}})\le\sqrt{\sigma}\left(\frac{1}{2}u_\varepsilon+\frac{1}{2}v_\varepsilon+1\right), \tag{2.31}$$

where Young's inequality was applied. These bounds together with the estimate (2.25) lead to

$$\|a_\varepsilon(u_\varepsilon,v_\varepsilon)\|_{L^\infty(0,\,T;\,L^2(\Omega))},\|b_\varepsilon(u_\varepsilon,v_\varepsilon)\|_{L^\infty(0,\,T;\,L^2(\Omega))}\le C,$$

where C is a positive constant independent on ε.

Analogously, by the hypothesis (1.20), we have that $F_{1,\varepsilon}(u_\varepsilon,v_\varepsilon),F_{2,\varepsilon}(u_\varepsilon,v_\varepsilon)$ are uniformly bounded in $L^\infty(0,\,T;\,L^1(\Omega))$, namely

$$\|F_{1,\varepsilon}(u_\varepsilon,v_\varepsilon)\|_{L^\infty(0,\,T;\,L^1(\Omega))},\|F_{2,\varepsilon}(u_\varepsilon,v_\varepsilon)\|_{L^\infty(0,\,T;\,L^1(\Omega))}\le C'$$

and C' is again a positive constant independent on ε.

Now, it is deduced from (2.23) and repeated occasions of Hölder inequality that, for every $\varphi \in L^2(0,T;W^{1,\infty}(\Omega))$,

$$\begin{aligned}
\left|\int_0^T \langle \partial_t u_\varepsilon, \varphi\rangle \; dt\right| &\le \iint_{Q_T} |a_\varepsilon(u_\varepsilon, v_\varepsilon)||\nabla u_\varepsilon \cdot \nabla\varphi| \, dx\, dt + \iint_{Q_T} |b_\varepsilon(u_\varepsilon, v_\varepsilon)||\nabla v_\varepsilon \cdot \nabla\varphi| \, dx\, dt \\
&\quad + \iint_{Q_T} |F_{1,\varepsilon}(u_\varepsilon, v_\varepsilon)||\varphi| \, dx\, dt \\
&\le \int_0^T \|\varphi(t)\|_{W^{1,\infty}(\Omega)} \, \|a_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))\|_{L^2(\Omega)} \, \|\nabla u_\varepsilon(t)\|_{L^2(\Omega)} \; dt \\
&\quad + \int_0^T \|\varphi(t)\|_{W^{1,\infty}(\Omega)} \, \|b_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))\|_{L^2(\Omega)} \, \|\nabla v_\varepsilon(t)\|_{L^2(\Omega)} \; dt \\
&\quad + \int_0^T \|F_{1,\varepsilon}(u_\varepsilon(t), v_\varepsilon(t))\|_{L^1(\Omega)} \, \|\varphi\|_{W^{1,\infty}(\Omega)} \; dt \\
&\le \int_0^T \|\varphi(t)\|_{W^{1,\infty}(\Omega)} \, \|a_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))\|_{L^2(\Omega)} \, \|u_\varepsilon(t)\|_{H^1(\Omega)} \; dt \\
&\quad + \int_0^T \|\varphi(t)\|_{W^{1,\infty}(\Omega)} \, \|b_\varepsilon(u_\varepsilon(t), v_\varepsilon(t))\|_{L^2(\Omega)} \, \|v_\varepsilon(t)\|_{H^1(\Omega)} \; dt \\
&\quad + \int_0^T \|F_{1,\varepsilon}(u_\varepsilon(t), v_\varepsilon(t))\|_{L^1(\Omega)} \, \|\varphi\|_{W^{1,\infty}(\Omega)} \; dt \\
&\le \|a_\varepsilon(u_\varepsilon, v_\varepsilon)\|_{L^\infty(0,T;\, L^2(\Omega))} \, \|u_\varepsilon\|_{L^2(0,T;\, H^1(\Omega))} \, \|\varphi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))} \\
&\quad + \|b_\varepsilon(u_\varepsilon, v_\varepsilon)\|_{L^\infty(0,T;\, L^2(\Omega))} \, \|v_\varepsilon\|_{L^2(0,T;\, H^1(\Omega))} \, \|\varphi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))} \\
&\quad + \|F_{1,\varepsilon}(u_\varepsilon, v_\varepsilon)\|_{L^2(0,T;\, L^1(\Omega))} \, \|\varphi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))} \\
&\le \|a_\varepsilon(u_\varepsilon, v_\varepsilon)\|_{L^\infty(0,T;\, L^2(\Omega))} \, \|u_\varepsilon\|_{L^2(0,T;\, H^1(\Omega))} \, \|\varphi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))} \\
&\quad + \|b_\varepsilon(u_\varepsilon, v_\varepsilon)\|_{L^\infty(0,T;\, L^2(\Omega))} \, \|v_\varepsilon\|_{L^2(0,T;\, H^1(\Omega))} \, \|\varphi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))} \\
&\quad + \sqrt{T}\, \|F_{1,\varepsilon}(u_\varepsilon, v_\varepsilon)\|_{L^\infty(0,T;\, L^1(\Omega))} \, \|\varphi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))} \\
&\le (2\, C\, C_2 + \sqrt{T}\, C') \; \|\varphi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))} \, ,
\end{aligned}$$

which gives the estimate $\|\partial_t u_\varepsilon\|_{L^2(0,T;\,(W^{1,\infty}(\Omega))^*)} \le \; 2\,C\,C_2 + \sqrt{T}\,C'$.

By the same arguments, we can derive that, for any $\psi \in L^2(0,\, T;\, W^{1,\infty}(\Omega))$,

$$
\begin{aligned}
\left|\int_0^T \langle \partial_t v_\varepsilon, \varphi\rangle \, dt\right| &\le (\delta+1)\iint_{Q_T} |\nabla v_\varepsilon \cdot \nabla\psi| \, dx\, dt + \iint_{Q_T} |F_{2,\varepsilon}(u_\varepsilon, v_\varepsilon)||\psi| \, dx\, dt \\
&\le (\delta+1)\int_0^T \|\nabla v_\varepsilon(t)\|_{L^1(\Omega)} \|\nabla\psi(t)\|_{L^\infty(\Omega)} \, dt \\
&\quad + \int_0^T \|F_{2,\varepsilon}(u_\varepsilon(t), v_\varepsilon(t))\|_{L^1(\Omega)} \|\psi\|_{L^\infty(\Omega} \, dt \\
&\le (\delta+1)\sqrt{\Omega}\int_0^T \|v_\varepsilon(t)\|_{H^1(\Omega)} \|\psi(t)\|_{W^{1,\infty}(\Omega)} \, dt \\
&\quad + \int_0^T \|F_{2,\varepsilon}(u_\varepsilon(t), v_\varepsilon(t))\|_{L^1(\Omega)} \|\psi\|_{L^\infty(\Omega} \, dt \\
&\le (\delta+1)\sqrt{\Omega}\, \|v\|_{L^2(0,T;\, H^1(\Omega))} \|\psi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))} \\
&\quad + \|F_{2,\varepsilon}(u_\varepsilon, v_\varepsilon)\|_{L^2(0,T;\, L^1(\Omega))} \|\psi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))} \\
&\le (\delta+1)\sqrt{\Omega}\, \|v\|_{L^2(0,T;\, H^1(\Omega))} \|\psi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))} \\
&\quad + \sqrt{T}\, \|F_{2,\varepsilon}(u_\varepsilon, v_\varepsilon)\|_{L^\infty(0,T;\, L^1(\Omega))} \|\psi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))} \\
&\le ((\delta+1)\sqrt{\Omega}\, C_2 + \sqrt{T}\, C')\, \|\psi\|_{L^2(0,T;\, W^{1,\infty}(\Omega))}
\end{aligned}
$$

and thus

$$
\|\partial_t u_\varepsilon\|_{L^2(0,T;\, (W^{1,\infty}(\Omega))^*)} \le (\delta+1)\sqrt{\Omega}\, C_2 + \sqrt{T}\, C'.
$$

Then we define

$$
C_3 := \max\{2\, C\, C_2 + \sqrt{T}\, C', (\delta+1)\sqrt{\Omega}\, C_2 + \sqrt{T}\, C'\}.
$$

This completes the proof of the proposition. □

Chapter 3

Existence of weak solutions

From Aubin's lemma (see Theorem 2.3) and Proposition 2.5, we deduce that $\{(u_\varepsilon, v_\varepsilon)\}_{\varepsilon>0}$ forms a bounded subset of

$$\left\{u \in L^2(0,T;H^1(\Omega)) \,:\, u_t \in L^2(0,T;(W^{1,\infty}(\Omega))^*)\right\} \subset\subset L^2(Q_T)$$

(we have used also that $H^1(\Omega) \subset\subset L^2(\Omega) \hookrightarrow (W^{1,\infty}(\Omega))^*$).

Using the previous fact together with the fact that $L^2(0,\, T;\, H^1(\Omega))$ and $L^2(0,\, T;\, (W^{1,\infty}(\Omega))^*)$ are Hilbert spaces and (2.17), we obtain the existence of an element $(u, v) \in \left(L^2(0,T;H^1(\Omega))\right)^2$ with $(u_t, v_t) \in \left(L^2(0,T;(W^{1,\infty}(\Omega))^*)\right)^2$ such that, as $\varepsilon \longrightarrow 0$, the following convergences (up to a subsequence) hold:

$$\begin{cases} (u_\varepsilon, v_\varepsilon) \to (u,v) & \text{a.e. in } Q_T \text{ and strongly in } \left(L^2(Q_T)\right)^2, & \text{(3.1a)} \\ (u_\varepsilon, v_\varepsilon) \rightharpoonup (u,v) & \text{weakly in } \left(L^2(0,T;H^1(\Omega))\right)^2, & \text{(3.1b)} \\ (\partial_t u_\varepsilon, \partial_t v_\varepsilon) \rightharpoonup (u_t, v_t) & \text{weakly in } \left(L^2(0,T;(W^{1,\infty}(\Omega))^*)\right)^2. & \text{(3.1c)} \end{cases}$$

Note that, in particular, (3.1a) and (3.1b) imply that

$$(\nabla u_\varepsilon, \nabla v_\varepsilon) \rightharpoonup (\nabla u, \nabla v) \qquad \text{weakly in } (L^2(Q_T))^2. \tag{3.2}$$

Our goal is to show that (u, v) is the weak solution established in Definition 1.2. In order to achieve this aim, since $(u_\varepsilon, v_\varepsilon)$ verifies in particular the weak formulation

$$\int_0^T \langle \partial_t u_\varepsilon, \varphi \rangle \, dt + \iint_{Q_T} (a_\varepsilon(u_\varepsilon, v_\varepsilon)\nabla u_\varepsilon - b_\varepsilon(u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon) \cdot \nabla \varphi \, dx \, dt = \iint_{Q_T} F_{1,\varepsilon}(u_\varepsilon, v_\varepsilon)\varphi \, dx \, dt, \tag{3.3}$$

$$\int_0^T \langle \partial_t v_\varepsilon, \psi \rangle \, dt + (\delta + 1) \iint_{Q_T} \nabla v_\varepsilon \cdot \nabla \psi \, dx \, dt = \iint_{Q_T} F_{2,\varepsilon}(u_\varepsilon, v_\varepsilon)\psi \, dx \, dt; \tag{3.4}$$

for any $\varphi, \psi \in L^2(0,\, T;\, W^{1,\infty}(\Omega))$, we want to make $\varepsilon \longrightarrow 0$ to obtain

$$\int_0^T \langle u_t, \varphi \rangle \, dt + \iint_{Q_T} (a(u,v)\nabla u - b(u,v)\nabla v) \cdot \nabla \varphi \, dx \, dt = \iint_{Q_T} F_1(u,v) \, \varphi \, dx \, dt, \tag{3.5}$$

$$\int_0^T \langle v_t, \psi \rangle \, dt + (\delta + 1) \iint_{Q_T} \nabla v \cdot \nabla \psi \, dx \, dt = \iint_{Q_T} F_2(u,v) \, \psi \, dx \, dt; \tag{3.6}$$

where $a(u,v) = \delta + \sigma \, u^{\frac{1}{2}} v^{\frac{1}{2}}$ and $b(u,v) = \sigma \, u^2 \, v \, \chi(u,v)$. We verify this fact by proving the convergence term by term, namely, we show, as $\varepsilon \longrightarrow 0$, that

$$\int_0^T \langle \partial_t u_\varepsilon, \varphi \rangle \, dt \longrightarrow \int_0^T \langle u_t, \varphi \rangle \, dt, \tag{3.7}$$

$$\iint_{Q_T} a_\varepsilon(u_\varepsilon, v_\varepsilon)\nabla u_\varepsilon \cdot \nabla \varphi \, dx \, dt \longrightarrow \iint_{Q_T} a(u,v)\nabla u \cdot \nabla \varphi \, dx \, dt, \tag{3.8}$$

$$\iint_{Q_T} b_\varepsilon(u_\varepsilon, v_\varepsilon)\nabla v_\varepsilon \cdot \nabla \varphi \, dx \, dt \longrightarrow \iint_{Q_T} b(u,v)\nabla v \cdot \nabla \varphi \, dx \, dt, \tag{3.9}$$

$$\iint_{Q_T} F_{1,\varepsilon}(u_\varepsilon, v_\varepsilon)\varphi \, dx \, dt \longrightarrow \iint_{Q_T} F_1(u,v)\varphi \, dx \, dt, \tag{3.10}$$

$$\int_0^T \langle \partial_t v_\varepsilon, \psi \rangle \, dt \longrightarrow \int_0^T \langle v_t, \psi \rangle \, dt, \tag{3.11}$$

$$\iint_{Q_T} \nabla v_\varepsilon \cdot \nabla \psi \, dx \, dt \longrightarrow \iint_{Q_T} \nabla v \cdot \nabla \psi \, dx \, dt, \tag{3.12}$$

$$\iint_{Q_T} F_{2,\varepsilon}(u_\varepsilon, v_\varepsilon)\psi \, dx \, dt \longrightarrow \iint_{Q_T} F_2(u,v)\psi \, dx \, dt, \tag{3.13}$$

for all $\varphi, \psi \in L^2(0,\, T;\, W^{1,\infty}(\Omega))$.

Note that (3.7), (3.11) and (3.12) are deduced directly from (3.1c) and (3.2). Now, we show the other convergences, first we give the next lemma.

Lemma 3.1. For (u, v) satisfying (3.1a)-(3.1c) it follows, as $\varepsilon \longrightarrow, 0$ that

$$(f_\varepsilon(u_\varepsilon), f_\varepsilon(v_\varepsilon)) \longrightarrow (u, v) \qquad \text{a.e. in } Q_T \text{ and strongly in } \left(L^2(Q_T)\right)^2.$$

Proof. Recall that $\Omega \subseteq \mathbb{R}^d$. If $d \geq 3$ we take $p := p^* = \frac{2d}{d-2}$; if $d = 1$ or 2, then take any $d' \geq 3$ and consider again $p = \frac{2d'}{d'-2}$. In this way, by Gagliardo-Nirenberg-Sobolev inequalities, we know that $H^1(\Omega) \hookrightarrow L^p(\Omega)$ and thus

$$\|u\|_{L^p(\Omega)} \leq M \, \|u\|_{H^1(\Omega)}$$

for some $M > 0$ constant depending only on Ω and d. Therefore, using Hölder inequality, it is deduced:

$$\begin{aligned}
\iint_{Q_T} (u - f_\varepsilon(u_\varepsilon))^2 \, dx \, dt &= \iint_{Q_T} \left(\frac{u - u_\varepsilon + \varepsilon \, u \, u_\varepsilon}{1 + \varepsilon \, u_\varepsilon} \right)^2 dx \, dt \\
&\leq 2 \iint_{Q_T} (u - u_\varepsilon)^2 \, dx \, dt + 2 \iint_{Q_T} \left(\frac{\varepsilon \, u \, u_\varepsilon}{1 + \varepsilon \, u_\varepsilon} \right)^2 dx \, dt \\
&\leq 2 \, \|u - u_\varepsilon\|^2_{L^2(Q_T)} + 2 \iint_{Q_T} \frac{(\varepsilon \, u \, u_\varepsilon)^2}{(\varepsilon \, u_\varepsilon)^{2-\frac{4}{d}} (1 + \varepsilon \, u_\varepsilon)^{\frac{4}{d}}} \, dx \, dt \\
&\leq 2 \, \|u - u_\varepsilon\|^2_{L^2(Q_T)} + 2 \, \varepsilon^{\frac{4}{d}} \iint_{Q_T} u^2 \, u_\varepsilon^{\frac{4}{d}} \, dx \, dt \\
&\leq 2 \, \|u - u_\varepsilon\|^2_{L^2(Q_T)} \\
&\quad + 2 \, \varepsilon^{\frac{4}{d}} \int_0^T \left(\int_\Omega u^p(t) \, dx \right)^{\frac{2}{p}} \left(\int_\Omega u_\varepsilon^{\frac{4p}{d(p-2)}}(t) \, dx \right)^{\frac{p-2}{p}} dt \\
&\leq 2 \, \|u - u_\varepsilon\|^2_{L^2(Q_T)} + 2 \, \varepsilon^{\frac{4}{d}} \int_0^T \|u(t)\|^2_{L^p(\Omega)} \, \|u_\varepsilon(t)\|^{\frac{4}{d}}_{L^2(\Omega)} \, dt \\
&\leq 2 \, \|u - u_\varepsilon\|^2_{L^2(Q_T)} + 2 \, \varepsilon^{\frac{4}{d}} \, M^2 \, \|u_\varepsilon\|^{\frac{4}{d}}_{L^\infty(0,T;\, L^2(\Omega))} \int_0^T \|u(t)\|^2_{H^1(\Omega)} \, dt \\
&\leq 2 \, \|u - u_\varepsilon\|^2_{L^2(Q_T)} + 2 \, \varepsilon^{\frac{4}{d}} \, M^2 \, \|u_\varepsilon\|^{\frac{4}{d}}_{L^\infty(0,T;\, L^2(\Omega))} \, \|u\|^2_{L^2(0,T;\, H^1(\Omega))} \\
&\quad \longrightarrow 0 \qquad \text{as } \varepsilon \longrightarrow 0,
\end{aligned}$$

where it was used that (u_ε) is uniformly bounded in $L^\infty(0, T; \, L^2(\Omega))$. This proves the convergence $f_\varepsilon(u_\varepsilon) \longrightarrow u$ in $L^2(Q_T)$. Analogously, $f_\varepsilon(v_\varepsilon) \longrightarrow v$ in $L^2(Q_T)$ is deduced. $\square$

As a direct consequence of Lemma 3.1, we have the following convergences.

Lemma 3.2. The following convergences hold as $\varepsilon \longrightarrow 0$:

a) $a_\varepsilon(u_\varepsilon, v_\varepsilon) \longrightarrow a(u, v)$ and $b_\varepsilon(u_\varepsilon, v_\varepsilon) \longrightarrow b(u, v)$ in $L^2(Q_T)$;

b) $F_{i,\varepsilon}(u_\varepsilon, v_\varepsilon) \longrightarrow F_i(u, v)$ in $L^2(0, T; \, L^1(\Omega))$ for $i = 1, 2$.

Proof. a) From Lemma 3.1 we have in particular that $(f_\varepsilon(u_\varepsilon), f_\varepsilon(v_\varepsilon)) \longrightarrow (u, v)$ a.e. in Q_T^2 and, by the continuity of $a(\cdot, \cdot)$,

$$a_\varepsilon(u_\varepsilon, v_\varepsilon) \longrightarrow a(u,v) \qquad \text{a.e. in } \; Q_T.$$

Now, since $(f_\varepsilon(u_\varepsilon), f_\varepsilon(v_\varepsilon)) \longrightarrow (u,v)$ in $(L^2(Q_T))^2$ and

$$|a_\varepsilon(u_\varepsilon, v_\varepsilon)| \leq C(1 + f_\varepsilon(u_\varepsilon) + f_\varepsilon(v)), \tag{3.14}$$

$$|a(u,v)| \leq C(1 + u + v) \tag{3.15}$$

for some uniform constant $C > 0$ (compare with (2.30)), thus it follows using the Generalized Dominated Convergence Theorem that

$$a_\varepsilon(u_\varepsilon, v_\varepsilon) \longrightarrow a(u,v) \quad \text{in } \; L^2(Q_T).$$

The same arguments shows that

$$b_\varepsilon(u_\varepsilon, v_\varepsilon) \longrightarrow b(u,v) \quad \text{in } \; L^2(Q_T),$$

since (see (2.31))

$$|b_\varepsilon(u_\varepsilon, v_\varepsilon)| \leq C(1 + f_\varepsilon(u_\varepsilon) + f_\varepsilon(v_\varepsilon)),$$
$$|b(u,v)| \leq C(1 + u + v).$$

b) Lemma 3.1 implies that for a.e. $t \in [0,T]$,

$$(f_\varepsilon(u_\varepsilon(t)), f_\varepsilon(v_\varepsilon(t))) \longrightarrow (u(t), v(t)) \qquad \text{a.e. in } \; (L^2(\Omega))^2$$

and therefore, by the continuity of F_i, we obtain that

$$F_{i,\varepsilon}(u_\varepsilon, v_\varepsilon)[t] \longrightarrow F_i(u,v)[t] \qquad \text{a.e. in } \; \Omega.$$

Applying again the Generalized Dominated Convergence Theorem in Ω and the hypothesis (1.20) we obtain

$$\|F_i(u,v)[t] - F_{i,\varepsilon}(u_\varepsilon, v_\varepsilon)[t]\|_{L^1(\Omega)} \longrightarrow 0 \qquad \text{for a.e. } \; t \in [0,T].$$

Moreover, using again the hypothesis (1.20), the definition of f_ε and (2.25)

$$\begin{aligned}\|F_i(u,v)[t]-F_{i,\varepsilon}(u_\varepsilon,v_\varepsilon)[t]\|_{L^1(\Omega)} \leq & C_F\left(2\,|\Omega| + \|f_\varepsilon(u_\varepsilon(t))\|^2_{L^2(\Omega)} + \|f_\varepsilon(v_\varepsilon(t))\|^2_{L^2(\Omega)}\right.\\ &\left. + \|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)}\right)\\ \leq & C_F\left(2\,|\Omega| + \|u_\varepsilon(t)\|^2_{L^2(\Omega)} + \|v_\varepsilon(t)\|^2_{L^2(\Omega)}\right.\\ &\left. + \|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)}\right)\\ \leq & C_F\left(2\,|\Omega| + 4C_1^2\right),\end{aligned}$$

for a.e. $t \in [0,T]$, by an application of the Bounded Convergence Theorem in $[0,T]$ we have that,

$$\|F_i(u,v) - F_{i,\varepsilon}(u_\varepsilon,v_\varepsilon)\|_{L^2(0,T;\,L^1(\Omega))} \longrightarrow 0,$$

concluding the proof of this lemma.

□

3.1 Proof of the Theorem 1.3

We are ready to show that (u,v) is a weak solution to the problem (1.4)-(1.8). As we noticed previously, in order to show (3.5) and (3.6) for any $\varphi,\psi \in L^2(0,\,T;\,W^{1,\infty}(\Omega))$, we just need to demonstrate the convergences (3.8), (3.9), (3.10) and (3.13).

Since (3.2) holds, then, to demonstrate (3.8) and (3.9), it is sufficient to prove that, for any $\varphi \in L^2(0,\,T;\,W^{1,\infty}(\Omega))$,

$$a_\varepsilon(u_\varepsilon,v_\varepsilon)\nabla\varphi \longrightarrow a(u,v)\nabla\varphi \quad \text{in} \quad L^2(Q_T), \tag{3.16}$$

$$b_\varepsilon(u_\varepsilon,v_\varepsilon)\nabla\varphi \longrightarrow b(u,v)\nabla\varphi \quad \text{in} \quad L^2(Q_T). \tag{3.17}$$

We check the first convergence (the second one is analogous). From Lemma 3.2 we can assure that, for a.e. $t \in [0,T]$,

$$\|a(u,v)[t] - a_\varepsilon(u_\varepsilon,v_\varepsilon)[t]\|^2_{L^2(\Omega)} \longrightarrow 0.$$

Moreover, from (3.14), (3.15) and (2.25), it follows that

$$\|a(u,v)[t] - a_\varepsilon(u_\varepsilon,v_\varepsilon)[t]\|^2_{L^2(\Omega)} \leq K$$

uniformly for a.e. $t \in T$, therefore

$$\|a(u,v)[t] - a_\varepsilon(u_\varepsilon,v_\varepsilon)[t]\|^2_{L^2(\Omega)}\,\|\varphi(t)\|^2_{W^{1,\infty}(\Omega)} \leq K\,\|\varphi(t)\|^2_{W^{1,\infty}(\Omega)}\,.$$

Then, in view that $\varphi \in L^2(0, T; W^{1,\infty}(\Omega))$, we can apply the Dominated Convergence Theorem to obtain that

$$\iint_{Q_T} |a(u,v) - a_\varepsilon(u_\varepsilon, v_\varepsilon)|^2 |\nabla\varphi|^2 \, dx \, dt \le \int_0^T \|a(u,v)[t] - a_\varepsilon(u_\varepsilon, v_\varepsilon)[t]\|^2_{L^2(\Omega)} \|\varphi(t)\|^2_{W^{1,\infty}(\Omega)} \, dt \longrightarrow 0.$$

This proves (3.16) and thus (3.8) holds. In the same way, one can show (3.9).

The proof of (3.10) and (3.13) are due by straightforward calculations, indeed

$$\begin{aligned}\left|\iint_{Q_T} (F_i(u,v) - F_{i,\varepsilon}(u_\varepsilon, v_\varepsilon))\, \xi \, dx \, dt\right| &\le \int_0^T \|F_i(u,v)[t] - F_{i,\varepsilon}(u_\varepsilon, v_\varepsilon)[t]\|_{L^1(\Omega)} \|\xi(t)\|_{W^{1,\infty}(\Omega)} \, dx \, dt \\ &\le \|F_i(u,v) - F_{i,\varepsilon}(u_\varepsilon, v_\varepsilon)\|_{L^2(0,T;L^1(\Omega))} \|\xi\|_{L^2(0,T;W^{1,\infty}(\Omega))} \\ &\longrightarrow 0,\end{aligned}$$

for any $\xi \in L^2(0, T; W^{1,\infty}(\Omega))$,where Lemma 3.2 was used again .

This shows that (u, v) satisfies (3.5) and (3.6).

The non negativity of (u, v) is clear due to Proposition 2.5 and (3.1a).

Finally, to see the initial conditions are satisfied in the sense of Definition 1.2, we just need to point out the clear fact that $(u(x, 0), v(x, 0)) = (u_0(x), v_0(x))$ a.e. in Ω and the fact that $u \in H^1(0, T; (W^{1,\infty}(\Omega))^*) \hookrightarrow C([0, T]; (W^{1,\infty}(\Omega))^*)$, then the initial conditions are accomplished, concluding the proof of Theorem 1.3.

3.2 Conclusions

In this work, we have proved the existence of weak solutions in the sense of definition Definition 1.2 to a chemotaxis system with a cross-diffusion term. During the proof, we used arguments of fixed point theorem and compact embeddings.

The main reasons why we needed to apply those techniques are because, on one side, the system (1.4)-(1.5) is a strongly coupled non linear system, then standard tools as maximum principle and regularity theory for parabolic equations do not apply.

On the other hand, there are no few examples of existence for quasi linear parabolic systems which draw on classic results of their type, for instance,the Amann's theorems in [1], [3], [2] and Ladyzenskaja's results in [12], however our system (1.4) is not the case because of the following aspects (cf. [6]):

(1) we assume $u_0, v_0 \in L^2(\Omega)$, meanwhile both of Amman and Ladyzenskaja's theorems hold provided initial conditions are in $L^\infty(\Omega)$;

(2) the diffusion matrix $A_\delta(u, v)$ is not uniformly upper-bounded, indeed, this fact forced us to introduce the regularized systems with diffusion matrices (2.2),

(3) the additional cross-diffusion term appearing in (1.4).

Notice that we have compensated the lack of structure in our system by demonstrating the existence of weak solutions; moreover, the definition of weak solutions in Definition 1.2 is not the usual weak solution definition in distributional sense or in $L^2(\Omega)$. Although, the biggest contribution in this work is the study of a system presenting a cross diffusion term.

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