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"Stability of standing waves for the nonlinear Schrödinger equation with attractive delta potential and repulsive nonlinearity"

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Presenta:
Miguel Ángel Sánchez Alcántara

Director de la tesina
Dr. Ramón Gabriel Plaza Villegas

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Con todo mi amor, para Zacnité

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Chapter 1

Introduction

Consider the following nonlinear Schrödinger equation with a delta potential

$$iu_t = -\partial_x^2 u + \gamma\delta(x)u + |u|^{p-1}u, \quad x, t \in \mathbb{R}, \tag{1.1}$$

where $\gamma < 0$, $1 < p < \infty$, and $\delta(x)$ is the Dirac delta measure at the origin. The equations of the form (1.1) belong to a family of models featuring the competition between attractive ($\gamma < 0$) and repulsive ($\alpha > 0$) terms. The combination of this nonlinearities in (1.1) is well-known in optical media (cf. Refs. [4]-[8]). In particular in the context of nonlinear optics, we recall that for an effective linear potential term, $V(x)$, the general NLS model,

$$iu_t + \partial_x^2 u + V(x)u + F(u) = 0,$$

represents a trapping structure for light beams induced by an inhomogeneity of the local refractive index. In particular, the delta-function term in (1.1), adequately represents a narrow trap which is able to capture broad solitonic beams. In the description of Bose-Einstein codensates, the same equation with a delta potential models the dynamics of a condensate in the presence of an impurity of a small lenght scale (cf. Refs. [13] and [17]). Notably, in both physical theories, the most common form of the nonlinearity $F(u)$ is a single cubic.

The formal expression $-\partial_x^2 + \gamma\delta(x)$ is formulated as a lineal operator A_γ or H_γ associated with the quadratic form

$$H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R},$$

$$(u, v) \longmapsto \alpha_\gamma(u, v) := Re\left\{\int_{\mathbb{R}} \partial_x u(x)\overline{v(x)}\, dx + \gamma u(0)\overline{v(0)}\right\}.$$

Recall that $H^1(\mathbb{R}) \hookrightarrow C_b(\mathbb{R})$, where

$$C_b(\mathbb{R}) = \left\{u \in C(\mathbb{R}; \mathbb{C}) : \lim_{|x|\to\infty} u(x) = 0\right\}.$$

The lineal operator $A_\gamma : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ is defined by

$$\langle A_\gamma u, v\rangle = \alpha_\gamma(u, v), \quad u, v \in H^1(\mathbb{R}).$$

We define a linear operator $H_\gamma v = -\partial_x^2 v$ for $v \in D(H_\gamma)$, where

$$D(H_\gamma) = \{v \in H^2(\mathbb{R} \setminus \{0\}) \cap H^1(\mathbb{R}) : \partial_x v(0^+) - \partial_x v(0^-) = \gamma v(0)\}.$$

H_γ is a self-adjoint operator in $L^2(\mathbb{R})$, and satisfies

$$(H_\gamma u, v)_{L^2} = \alpha_\gamma(u, v), \quad u, v \in D(H_\gamma).$$

The following spectral properties of H_γ are known: $\sigma_{ess}(H_\gamma) = \sigma_{ac}(H_\gamma) = [0, \infty)$ and $\sigma_p(H_\gamma) = \{-\frac{\gamma^2}{4}\}$ with its positive eigenfunction $(|\gamma|/2)^{1/2} e^{-\frac{|\gamma||x|}{2}}$.

It is worth remarking that for any $p > 1$, $H^1(\mathbb{R}) \subset L^{p+1}(\mathbb{R})$. Indeed, the Gagliardo-Nirenberg interpolation inequality (see, Theorem 12.82 in [15]) yields

$$\|u\|_{L^{p+1}(\mathbb{R})} \le C \|u\|^{\theta}_{L^2(\mathbb{R})} \|\partial_x u\|^{1-\theta}_{L^2(\mathbb{R})}, \tag{1.2}$$

with $C > 0$ an uniform positive constant and $\theta = \frac{p+2}{2p+2} \in (0, 1)$.

We study the structure and the orbital stability of the standing waves solutions $e^{i\omega t}\varphi_\omega(x)$ for (1.1), where $\omega \in \mathbb{R}$ and $\varphi_\omega \in H^1(\mathbb{R})$ is a positive solution of the stationary problem

$$A_\gamma \varphi + \omega \varphi + |\varphi|^{p-1}\varphi = 0 \quad \text{in } H^{-1}(\mathbb{R}). \tag{1.3}$$

The well-posedness of the Cauchy problem for (1.1) in the space $H^1(\mathbb{R})$ follows from an abstract result in Cazenave [11].

Proposition I.1. For any $u_0 \in H^1(\mathbb{R})$ there exist $T = T^*(u_0) \in (0, \infty]$ and a unique solution $u \in C([0, T^*); H^1(\mathbb{R}))$ of (1.1) with $u(0) = u_0$ such that

$$\lim_{t \to T^*} \|u(t)\|_{H^1(\mathbb{R})} = \infty \quad \text{if } T^* < \infty.$$

Moreover, we have that $u(t)$ satisfies the conservation of charge and energy

$$\|u(t)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}, \quad E(u(t)) = E(u_0), \quad t \in [0, T^*),$$

where the energy functional E, is defined as follows

$$E(v) = \frac{1}{2}\|\partial_x v\|^2_{L^2(\mathbb{R})} + \frac{\gamma}{2}|v(0)|^2 + \frac{1}{p+1}\|v\|^{p+1}_{L^{p+1}(\mathbb{R})} \quad v \in H^1(\mathbb{R}).$$

The stability of standing waves is defined as follows.

Definition I.2. We say that a standing wave solution $e^{i\omega t}\varphi_\omega$ is orbitally stable in $H^1(\mathbb{R})$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R})$ and $\|u_0 - \varphi_\omega\| < \delta$, then the solution $u(t)$ of (1.1) with initial value $u(0) = u_0$ exists for all $t \ge 0$ and

$$\sup_{t \ge 0} \inf_{\theta \in \mathbb{R}} \|u(t) - e^{i\theta}\varphi_\omega\| < \varepsilon.$$

Otherwise, $e^{i\omega t}\varphi_\omega$ is orbitally *unstable* in $H^1(\mathbb{R})$.

We state the main results for the case $\gamma < 0$ and $\alpha = 1$.

Theorem I.3 Let $\gamma < 0$, $1 < p < \infty$ and $0 < \omega < \frac{\gamma^2}{4}$. Then, the stationary problem (1.3) has a unique positive solution $\varphi_\omega \in H^1(\mathbb{R})$ given by

$$\varphi_\omega(x) = \left(\frac{(p+1)\omega}{2}\right)^{\frac{1}{p-1}} \left\{\sinh\left(\frac{(p-1)\sqrt{\omega}}{2}|x| + c_\gamma(\omega)\right)\right\}^{-\frac{2}{p-1}} \quad x \in \mathbb{R}, \tag{1.4}$$

where $c_\gamma = \tanh^{-1}(2\sqrt{\omega}/|\gamma|)$. The standing wave solution $e^{i\omega t}\varphi_\omega$ is orbitally stable in $H^1(\mathbb{R})$.

♦

Theorem I.4 Let $\gamma < 0$, $\omega = 0$ and $1 < p < 5$. Then, the stationary problem (1.3) has a unique positive solution $\varphi_0 \in H^1(\mathbb{R})$ given by

$$\varphi_0(x) = \left(\frac{2(p+1)(p-1)^2\gamma^2}{\{4 + (p-1)|\gamma||x|^2\}}\right)^{\frac{1}{p-1}} \quad \text{for all } x \in \mathbb{R}. \tag{1.5}$$

The equilibrium solution φ_0 is orbitally stable in $H^1(\mathbb{R})$.

♦

We do not consider the case $\omega \notin (0, \gamma^2/4)$ or $\omega = 0$ and $p \geq 5$ in Theorems I.3 and I.4. We prove that there exists no trivial solutions of (1.1) in $H^1(\mathbb{R})$ for these cases.

The main purpose of this work is to obtain explicitly the standing profiles (1.4) and (1.5) using a quadrature procedure and to prove Theorem I.3 and I.4.

Chapter 2

Standing waves and equilibrium solutions

This chapter is devoted to the construction of explicit solutions of the standing waves to (1.1), with $\omega \neq 0$ and equilibrium solutions.

Next lemma establishes the principal properties of the solutions to (1.3) when $\varphi \in H^1(\mathbb{R})$. This result will be useful throughout the variational analysis,

Lemma II.1. Let $\gamma \in \mathbb{R} \setminus \{0\}$, $\omega \in \mathbb{R}$ and $\varphi \in H^1(\mathbb{R})$ a nontrivial solution of (1.3). Then

$$\varphi \in C(\mathbb{R}) \cap C^2(\mathbb{R} \setminus \{0\}), \tag{2.1}$$

$$-\varphi''(x) + \omega\varphi(x) + |\varphi(x)|^{p-1}\varphi(x) = 0, \quad x \in \mathbb{R} \setminus \{0\}, \tag{2.2}$$

$$\varphi'(0^+) - \varphi'(0^-) = \gamma\varphi(0), \tag{2.3}$$

$$\lim_{|x|\to\infty} \varphi(x) = 0, \quad \lim_{|x|\to\infty} \varphi'(x) = 0, \tag{2.4}$$

$$|\varphi'(x)|^2 = \omega|\varphi(x)|^2 + \frac{2}{p+1}|\varphi(x)|^{p+1}, \quad x \in \mathbb{R}\setminus\{0\}. \tag{2.5}$$

Proof. We give a sketch of the proof. Properties (2.1)-(2.5) are proved by a standard bootstrap argument, namely, for all $\xi \in C_0^\infty(\mathbb{R} \setminus \{0\})$ and for every $\chi \in H^1(\mathbb{R})$,

$$\langle -(\xi\varphi)'' + \omega(\xi\varphi), \chi\rangle = \langle -\xi''\varphi - 2\xi'\varphi' - |\varphi|^{p-1}\varphi\xi, \chi\rangle,$$

because φ is a solution to (1.3) with χ substituted by $\chi\xi$. Thus, we obtain the following equality in the distributional sense

$$-(\xi\varphi)'' + \omega(\xi\varphi) = -\xi''\varphi - 2\xi'\varphi' - \xi|\varphi|^{p-1}\varphi. \tag{2.6}$$

Since the right-hand side of the previous identity is in $L^2(\mathbb{R})$, then $\xi\varphi \in H^2(\mathbb{R})$, that is, $\varphi \in H^2(\mathbb{R} \setminus \{0\}) \cap C^1(\mathbb{R} \setminus \{0\})$. Using (2.6) again, we obtain (2.1) and (2.4). Moreover, since $\varphi \in C^2(\mathbb{R} \setminus \{0\})$ and $\xi \in C_0^\infty(\mathbb{R} \setminus \{0\})$, we obtain that equality (2.2) holds for all $x \in \mathbb{R} \setminus \{0\}$.

Next, since φ satisfies (1.3), for every $\chi \in H^1(\mathbb{R})$, we obtain, after integration by parts and using (2.2), that

$$0 = Re\left((\varphi(0^+) - \varphi(0^-) - \gamma\varphi(0))\overline{\chi(0)}\right).$$

Thus we obtain (2.3). Now, from (2.2), we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dx}\left(|\varphi'(x)|^2 - \omega|\varphi(x)|^2 - \frac{2}{p+1}|\varphi(x)|^{p+1}\right) &= Re\left(\varphi''(x) - \omega\varphi(x) - |\varphi(x)|^{p-1}\varphi(x)\right)\\ &= 0.\end{aligned}$$

Hence integrating respect x and using (2.4), we obtain (2.5).

□

Lemma II.2. Let $\gamma \in \mathbb{R}$ and $\omega \in \mathbb{R}$. Let φ as in the Lemma II.1. Then $\varphi(x) \neq 0$ for all $x \in \mathbb{R}$.

Proof. Suppose that there exists $x_0 \in \mathbb{R}$ such that $\varphi(x_0) = 0$. If $x_0 > 0$, by property (2.5) of Lemma II.1, we have $\varphi'(x_0) = 0$.

By the uniqueness of Cauchy's problem for (2.2), we have that $\varphi(x) = 0$ for $x > 0$, and for (2.3) we have $\varphi(0) = \varphi'(0) = 0$.

For the case $x_0 \leq 0$, we obtain that $\varphi(0) = \varphi'(0)$ in the same way. Thus, by the uniqueness of solutions of the Cauchy problem for (5), we obtain that $\varphi(x) = 0$ for all $x \in \mathbb{R}$. Since φ is a nontrivial solution, this is a contradiction. Then $\varphi(x) \neq 0$ for all $x \in \mathbb{R}$.

□

Lemma III.3. Let $\gamma \in \mathbb{R} \setminus \{0\}$ and $\omega \in \mathbb{R}$. Let φ a nontrivial solution of (2.1)-(2.5). Then we have either (i) or (ii):

(i) $Im\varphi(x) = 0$ for all $x \in \mathbb{R}$.

(ii) There exists $c \in \mathbb{R}$ such that

$$Re\varphi(x) = c\, Im\varphi(x), \quad x \in \mathbb{R}.$$

Proof. Put $u = Re\varphi$ and $v = Im\varphi$. Then, the pair (u, v) satisfies the following equations

$$\begin{cases} -u''(x) + \omega u(x) + |\varphi(x)|^{p-1}u(x) = 0, \\ -v''(x) + \omega v(x) + |\varphi(x)|^{p-1}v(x) = 0, \end{cases}$$

for $x \in \mathbb{R} \setminus \{0\}$. Then, we have

$$(u'(x)v(x) - u(x)v'(x))' = 0, \quad \text{for all } x \in \mathbb{R} \setminus \{0\}.$$

By property (2.4) of Lemma II.1, we obtain

$$u'(x)v(x) = u(x)v'(x) \quad \text{for all } x \in \mathbb{R} \setminus \{0\}. \tag{2.7}$$

If there exists $x_0 \in \mathbb{R}$, such that $v(x_0) = 0$, then by (2.7) and Lemma II.2, we have that $v'(x_0) = 0$. As in the proof of Lemma II.2, we have that $v(x) = 0$ for all $x \in \mathbb{R}$. This is the case (i).

Otherwise $v(x) \neq 0$ for all $x \in \mathbb{R}$. Then, by (2.7), we have

$$\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{u'(x)v(x) - u(x)v'(x)}{v(x)^2} = 0,$$

for all $x \in \mathbb{R} \setminus \{0\}$, this implies (ii).

□

2.1 Standing waves with $\omega \neq 0$.

In this subsection, we construct explicit standing wave solutions to (1.1) with $\omega \neq 0$ and $\gamma \neq 0$.

Theorem II.4. Let $p > 1$ and $\gamma < 0$ in (1.3). Then, for all

$$\omega \in \left(0, \frac{\gamma^2}{4}\right),$$

the family of standing wave solutions, $u(x,t) = e^{i\omega t}\varphi_\omega$, where φ_ω is given by (1.4) are solutions to the nonlinear Schrödinger equation (1.1).

Proof. Let us first compute positive solutions φ to (1.3) with $\gamma \equiv 0$. Under these assumptions, we obtain that $\varphi \equiv \varphi_\omega$ satisfies the nonlinear elliptic equation

$$-\varphi''(x) + \omega\varphi(x) + \varphi^p(x) = 0, \quad x \in \mathbb{R}. \tag{2.8}$$

By a quadrature procedure and using the condition for the profile $\varphi(x) \to 0$ as $|x| \to \infty$, we obtain

$$[\varphi']^2 = \omega\varphi^2 + \beta\varphi^{p+1}, \tag{2.9}$$

where we define $\beta := \frac{2}{p+1} > 0$. Upon substitution $y = \varphi^{p-1}$ into the last equality, we deduce that

$$\begin{aligned} x &= \frac{1}{p-1}\int \frac{dy}{y\sqrt{\omega+\beta y}} \\ &= -\frac{2}{p-1}\frac{\coth^{-1}\left(\frac{\sqrt{\omega+\beta y}}{\sqrt{\omega}}\right)}{\sqrt{\omega}}. \end{aligned}$$

Thus, for $x < 0$, we have

$$|x| = \frac{2}{p-1}\frac{\coth^{-1}\left(\frac{\sqrt{\omega+\beta y}}{\sqrt{\omega}}\right)}{\sqrt{\omega}}. \tag{2.10}$$

Using the substitution $y = \varphi^{p-1}$, we obtain the profile function

$$\varphi(x) = \left(\frac{(p+1)\omega}{2}\right)^{\frac{1}{p-1}} \left\{\sinh\left(\frac{(p-1)\sqrt{w}}{2}|x|\right)\right\}^{-\frac{2}{p-1}}, \quad x < 0, \tag{2.11}$$

is a positive solution defined for $x \in (-\infty, 0)$ and satisfies

$$\lim_{x\to-\infty} \varphi(x) = 0 \quad \text{and,} \quad \lim_{x\to 0^-} \varphi(x) = \infty.$$

In other words, φ decays to zero at $-\infty$ and blows up at $x = 0$. Next, we use the noncontinuous profile (2.11) to construct a continuous profile when $\gamma \neq 0$. For $d > 0$ (to be specified later), we define for $x \le 0$ the half-profile $\psi(x) = \varphi(x - d)$ and consider the even profile

$$\phi_1(x) := \begin{cases} \psi(x), & x \le 0, \\ \psi(-x), & x > 0. \end{cases}$$

In other words,

$$\phi_1(x) := \varphi(-|x| - d), \quad x \in \mathbb{R}, \tag{2.12}$$

which satisfies $\phi_1 \in H^1(\mathbb{R})$ and all the properties of Lemma II.1, except for the jump condition (2.3). Since ϕ_1 is an even function, the jump condition can be rewritten as

$$\phi_1'(0^+) = \frac{\gamma}{2}\phi_1(0), \quad \text{or equivalently} \quad \varphi'(-d) = \frac{\gamma}{2}\varphi(-d).$$

Hence

$$d = \frac{2}{(p-1)\sqrt{\omega}} \tanh^{-1}\left(\frac{2\sqrt{\omega}}{|\gamma|}\right).$$

Finally, we conclude that the even-profile $\varphi_\omega = \phi_1$ is given by

$$\begin{aligned}
\varphi_\omega(x) &= \phi_1(-|x| - d) \\
&= \left(\frac{(p+1)\omega}{2}\right)^{\frac{1}{p-1}} \left\{\sinh\left(\frac{(p-1)\sqrt{w}}{2}(|x| + d)\right)\right\}^{-\frac{2}{p-1}} \\
&= \left(\frac{(p+1)\omega}{2}\right)^{\frac{1}{p-1}} \left\{\sinh\left(\frac{(p-1)\sqrt{w}}{2}|x| + \tanh^{-1}\left(\frac{2\sqrt{\omega}}{|\gamma|}\right)\right)\right\}^{-\frac{2}{p-1}} \\
&= \left(\frac{(p+1)\omega}{2}\right)^{\frac{1}{p-1}} \left\{\sinh\left(\frac{(p-1)\sqrt{w}}{2}|x| + c_\gamma(\omega)\right)\right\}^{-\frac{2}{p-1}},
\end{aligned}$$

where $c_\gamma(\omega) = \tanh^{-1}\left(\frac{2\sqrt{\omega}}{|\gamma|}\right)$. This concludes the proof.

$\square$

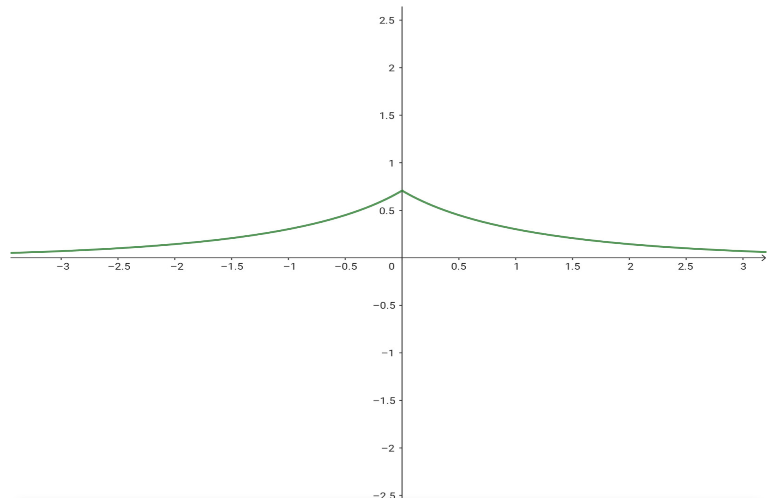

Figure 2.1: Profile function φ_ω defined in (1.4) for the parameter values $\omega = 0.5$, $\gamma = -1$ in the case of a cubic ($p = 3$) nonlinearity.

2.2 Equilibrium solutions of rational profile

In this section, we construct explicit equilibrium solutions $\omega = 0$ to the nonlinear Schrödinger equation (1.1).

Theorem II.5. Let $\gamma < 0$, $\omega = 0$ and $1 < p < 5$. The family of positive even-rational profiles defined by (1.5) are solutions to the nonlinear Schrödinger equation (1.1)

Proof. As before, we start by considering the case with $\gamma = 0$. Upon substitution of $\omega = 0$ and $\gamma = 0$ into (1.3), we obtain that $\varphi = \varphi_0$ satisfies the nonlinear elliptic equation

$$-\varphi''(x) + \varphi^p(x) = 0, \quad x \in \mathbb{R}. \tag{2.13}$$

Once again, one uses a quadrature procedure and applies the boundary condition for the profile $\varphi(x) \to 0$ as $|x| \to \infty$ to arrive at

$$[\varphi']^2 = \beta \varphi^{p+1}, \tag{2.14}$$

where we define $\beta := \frac{2}{p+1} > 0$. Let us assume that $\varphi > 0$. Upon substitution of $y = \varphi^{p-1}$ into the previous equation, we deduce that

$$x = -(2(p+1))^{1/2} y^{-1/2}. \tag{2.15}$$

It is not difficult to verify that φ has the positive profile

$$\varphi(x) = \left(\frac{2(p+1)}{|x|^2} \right)^{\frac{1}{p-1}}, \quad x < 0, \tag{2.16}$$

and satisfies

$$\lim_{x \to -\infty} \varphi(x) = 0 \quad \text{and,} \quad \lim_{x \to 0^-} \varphi(x) = \infty,$$

so that φ decays to zero at $-\infty$ and blows up at $x = 0$.

For $d > 0$ (to be determined below) we define the function

$$\phi_0(x) := \varphi(-|x| - d), \quad x \in \mathbb{R}. \tag{2.17}$$

Then $\phi_0 \in H^1(\mathbb{R})$ and satisfies all the properties of Lemma II.1, except the jump condition (2.3). Since ϕ_0 is an even function, the jump condition can be rewritten as

$$\phi_0'(0^+) = \frac{\gamma}{2}\phi_0(0), \quad \text{or equivalently} \quad \varphi'(-d) = \frac{\gamma}{2}\varphi(-d).$$

Hence $d = \frac{4}{(p-1)|\gamma|}$. We conclude that the even-rational profile $\varphi_0 = \phi_0$ is given by

$$\begin{aligned}
\varphi_0(x) &= \varphi(-|x| - d) \\
&= \left(\frac{2(p+1)}{(|x|+d)^2}\right)^{\frac{1}{p-1}} \\
&= \left(\frac{2(p+1)}{\left(|x| + \frac{4}{(p-1)|\gamma|}\right)^2}\right)^{\frac{1}{p-1}} \\
&= \left(\frac{2(p+1)(p-1)^2\gamma^2}{\{4+(p-1)|\gamma||x|\}^2}\right)^{\frac{1}{p-1}}
\end{aligned}$$

This conclude the proof.

□

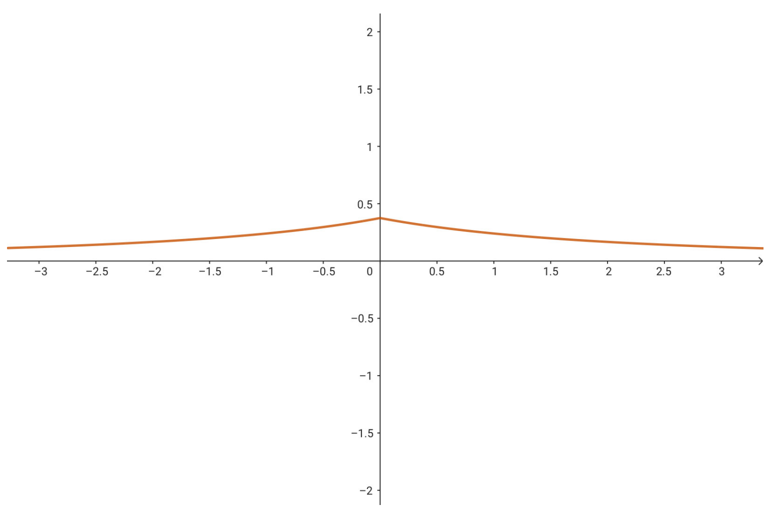

Figure 2.2: Depiction of the profile function φ_0 defined in (1.5) for the parameter values $\gamma = -1$ and $p = 2$.

Chapter 3

Stability theory

In this section, we prove Theorems I.3 and I.4, based on the minimization of the charge/energy functional and on the uniquess of the profiles in (1.4) and (1.5)

3.1 Description of the critical points

Let us consider the functional $J_\omega : H^1(\mathbb{R}) \to \mathbb{R}$, defined as

$$J_\omega(u) := \frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{R})} + \frac{\omega}{2}\|u\|^2_{L^2(\mathbb{R})} + \frac{\gamma}{2}|v(0)|^2 + \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})}, \quad u \in H^1(\mathbb{R}), \tag{3.1}$$

and the set of critical points associated with J_ω as

$$\mathcal{A}_\omega := \{u \in H^1(\mathbb{R}) : J'_\omega(u) = 0, \ \ u \neq 0\}.$$

The following lemmas show the nonexistence of nontrivial solutions for (1.3)

Lemma III.1. Let $1 < p < \infty$, $\gamma < 0$ and $\alpha > 0$. If $\omega \geq \frac{\gamma^2}{4}$ then $\mathcal{A}_\omega = \emptyset$.

Proof. Suppose that there exists $\varphi \in \mathcal{A}_\omega$. Then, we have

$$\|\partial_x\varphi\|^2_{L^2(\mathbb{R})} + \omega\|\varphi\|^2_{L^2(\mathbb{R})} - |\gamma||\varphi(0)|^2 + \|\varphi\|^{p+1}_{L^{p+1}(\mathbb{R})} = \frac{d}{d\tau}\Big|_{\tau=1} J_\omega(\tau\varphi) = 0.$$

Because the first eigenvalue of H_γ is $-\frac{\gamma^2}{4}$, then

$$\inf\left\{\|\partial_x u\|^2_{L^2(\mathbb{R})} - |\gamma||u(0)|^2 : \ u \in H^1(\mathbb{R}), \ \|u\|_{L^2(\mathbb{R})} = 1\right\} = -\frac{\gamma^2}{4}.$$

Then we have

$$\begin{aligned}
0 &= \|\partial_x\varphi\|^2_{L^2(\mathbb{R})} + \omega\|\varphi\|^2_{L^2(\mathbb{R})} - |\gamma||\varphi(0)|^2 + \|\varphi\|^{p+1}_{L^{p+1}(\mathbb{R})} \\
&\geq \left(\omega - \frac{\gamma^2}{4}\right)\|\varphi\|^2_{L^2(\mathbb{R})} + \|\varphi\|^{p+1}_{L^{p+1}(\mathbb{R})} \\
&> 0.
\end{aligned}$$

This is impossible. Then $\mathcal{A}_\omega = \emptyset$.

□

Lemma III.2. Let $1 < p < \infty$ and $\gamma < 0$. If $\omega < 0$ then $\mathcal{A}_\omega = \emptyset$.

Proof. Suppose that there exists $\varphi \in \mathcal{A}_\omega$. Then by (2.4) in the Lemma II.1, there exists $L > 0$ such that

$$\frac{\alpha}{p+1}|\varphi(x)|^{p-1} \leq \frac{|\omega|}{4} \quad \text{if } |x| \geq L.$$

Moreover, by property (2.5) in the Lemma II.1 and Lemma II.2, we obtain

$$\begin{aligned}|\varphi'(x)|^2 &= |\varphi(x)|^2 \left(\omega + \frac{2}{p+1}|\varphi(x)|^{p-1}\right)\\ &\leq |\varphi(x)|^2 \left(\omega - \frac{\omega}{2}\right)\\ &= |\varphi(x)|^2 \frac{\omega}{2} < 0,\end{aligned}$$

if $|x| \geq L$. This is is a contradiction. Then $\mathcal{A}_\omega = \emptyset$.

□

Now, we assume $1 < p < \infty$, $\gamma < 0$ and $\omega \in \left(0, \frac{\gamma^2}{4}\right)$. The following lemma characterizes the set of nontrivial critical points in this case.

Lemma III.3. Let $1 < p < \infty$, $\gamma < 0$ and $0 < \omega < \frac{\gamma^2}{4}$. Then

$$\mathcal{A}_\omega = \{e^{i\theta}\varphi_\omega : \theta \in \mathbb{R}\},$$

where φ_ω is given by (1.4).

Proof. It is clear that for all $\theta \in \mathbb{R}$, $e^{i\theta}\varphi_\omega \in \mathcal{A}_\omega$. Conversely, if $g \in \mathcal{A}_\omega$ then g satisfies (2.1)-(2.5) and by Lemma II.2 $|g| > 0$. We will show that there exist $\theta \in \mathbb{R}$ such thar $g(x) = e^{i\theta}\varphi_\omega(x)$ for all $x \in \mathbb{R}$.

First, we show that φ_ω is the unique positive solution for (1.3). Indeed, for Lemma II.2, it is sufficient to consider $v \in H^1(\mathbb{R})$ satisfying $v(x) > 0$ for all $x \in \mathbb{R}$ and properties (2.1)-(2.5). We consider the following polynomial $P : (0, \infty) \to \mathbb{R}$ defined by

$$P(c) = \frac{1}{4}\frac{\gamma^2}{4}c^2 + F(c), \quad F(c) = -\int_0^c (\omega t + |t|^{p-1}t)\, dt,$$

and the following initial value problem on $(0, +\infty)$,

$$\begin{cases} -\psi''(x) = H(\psi(x)), \quad x > 0,\\ \quad \psi(0) = c_0,\\ \quad \psi'(0) = \frac{\gamma}{2}c_0, \end{cases} \tag{3.2}$$

where $H(\psi) = -\omega\psi - |\psi|^{p-1}\psi$ and where $c_0 > 0$ is the unique positive root of P (to be determined below). Since H is locally Lipschitz around zero, we have that the IVP problem (3.2) has a unique positive solution and it is given by φ_ω. Indeed, since v is a positive solution of $-v'' = H(v)$ we have

$$\begin{aligned} -\int_0^R v'(t)v''(t)\,dt &= \int_0^R H(v(t))v'(t)\,dt \\ &= \int_0^R F'(v(t))v'(t)\,dt \\ &= F(v(R)) - F(v(0)). \end{aligned}$$

Thus, for all $R > 0$, $\frac{1}{2}v'(0^+)^2 - \frac{1}{2}v'(R)^2 + F(v(0^+)) - F(v(R)) = 0$. Then for $R \to \infty$ we obtain

$$\frac{1}{2}v'(0^+)^2 + F(v(0^+)) = 0. \tag{3.3}$$

On the same way, we obtain

$$\frac{1}{2}v'(0^-)^2 + F(v(0^-)) = 0. \tag{3.4}$$

Next, since v is continuous in $x = 0$, we get from (3.3) and (3.4) that $|v'(0^+)| = |v'(0^-)|$. Now, suppose that $v'(0^+) = v'(0^-)$. Hence, from Lemma II.1, there holds $v(0) = 0$. Now we divide our analysis into two steps:

(i) If $v'(0^+) = v'(0^-) = 0$, then $v'(0) = 0$ and $c_0 = 0$. Then, the IVP has the unique solution $v \equiv 0$. This is a contradiction with $v(x) > 0$ for all $x \in \mathbb{R}$.

(ii) If $v'(0^+) = v'(0^-) \neq 0$, then there exists $x_0 \in \mathbb{R}$ close to zero, such that $v(x_0) < 0$, which cannot happen.

From the analysis above, we necessarily have $v'(0^+) = -v'(0^-)$ and, from the jump condition, we obtain $v'(0^+) = \frac{\gamma}{2}v(0)$. Then by (3.3), we obtain

$$\begin{aligned} P(v(0^+)) &= \frac{1}{2}\frac{\gamma^2}{4}v(0^+)^2 + F(v(0^+)) \\ &= \frac{1}{2}v'(0^+)^2 + F(v(0^+)) \\ &= 0. \end{aligned}$$

It follows that $P(v(0^+)) = 0$. Next, we determine the existence of a unique zero of P on $(0, +\infty)$. Since

$$P(c) = \frac{1}{2}\left(\frac{\gamma^2}{4} - \omega\right)c^2 - \frac{1}{p+1}c^{p+1}.$$

If we define $r = c^{p-1}$, we have that $P(c) = 0$ if and only if $\frac{1}{2}(\frac{\gamma^2}{4} - \omega) - \frac{1}{p+1}r = 0$. This last polynomial has a unique positive root in $(0, +\infty)$.

Then, since $P(v(0^+)) = 0$ with $v(0^+) > 0$, we need to have $c_0 = v(0^+)$. Then, v is the unique local solution for the IVP (3.2), for at least $x \in (0, a)$. Now, since $v \in C^j(0, \infty)$, $j = 0, 1, 2$, and $v(x) \to 0$ as $x \to +\infty$, it follows that $v \in L^\infty(0, +\infty)$. From standard ODE arguments, we can choose $a = \infty$, and consequently, the unique solution of (3.2) on $(0,+\infty)$ is positive. A similar analysis on $(-\infty,0)$ shows that v is the unique positive solution of (3.2) on $(-\infty,0)$. Therefore, since φ_ω is a continuous profile satisfying the IVP (3.2) on $(-\infty,0)$ and $(0,+\infty)$, necessarily $v=\varphi_\omega$.

Finally, since $g\in C^2(0,+\infty)$, we can write $g(x)=\rho(x)e^{i\theta(x)}$ for some $\rho,\theta\in C^2(0,+\infty)$ with $\rho>0$. Thus by substituting g in (1.3) and taking real and imaginary part, we obtain the system

$$\begin{cases}\rho\theta''+2\rho'\theta'=0, \quad x>0,\\ -\rho''+\rho(\theta')^2+\omega\rho+\rho^p=0, \quad x>0.\end{cases} \tag{3.5}$$

Therefore, there is a real constant K such that $\rho^2\theta'=K$ on $(0,+\infty)$. Now, since $|g'|$ is bounded, we get that $\rho^2(\theta')^2=\frac{K^2}{\rho^2}$ is bounded on $(0,+\infty)$. But, since $\rho(x)\to 0$ as $x\to+\infty$, we need to have $K=0$. Then, there exists $\theta_0\in\mathbb{R}$ such that $\theta(x)=\theta_0$ for all $x\in(0,+\infty)$. Thus, $g(x)=e^{i\theta_0}\rho(x)$ for all $x\in(0,+\infty)$. From the second equation in (3.5), we obtain that ρ is a positive solution of (1.3). Thus, we have $\rho(x)=\varphi_\omega(x)$ for all $x\in(0,+\infty)$. On the same way, we obtain that there exists $\theta_1\in\mathbb{R}$, such that $g(x)=e^{i\theta_1}\varphi_\omega(x)$ for all $x\in(-\infty,0)$. Finally, by continuity, we obtain $e^{i\theta_0}=e^{i\theta_1}$ and hence $g(x)=e^{i\theta_0}\varphi_\omega(x)$ for all $x\in\mathbb{R}$.

□

A similar result for $\omega=0$ is obtained.

Lemma III.4. Let $\gamma<0$. If $1<p<5$, then $\mathcal{A}_0=\{e^{i\theta}\varphi_0:\theta\in\mathbb{R}\}$, where φ_0 is defined by (1.5). If $p\geq 5$, then the set $\mathcal{A}_\omega$ is a empty set.

Proof. If $1<p<5$, in the same way of the proof of Lemma III.4, we get that $\mathcal{A}_0=\{e^{i\theta}\varphi_0\}$, where φ_0 is defined by (1.5). While if $p\geq 5$, then $\varphi_0\notin L^2(\mathbb{R})$ and thus $\mathcal{A}_0=\emptyset$.

□

3.2 Orbital stability of standing waves for $\omega\neq 0$

This section is devoted te prove Theorem I.3. Suppose that the parameter values satisfy $1<p<\infty$, $\gamma<0$ and $0<\omega<\frac{\gamma^2}{4}$. We put

$$d_\omega:=\inf_{u\in H^1(\mathbb{R})} J_\omega(u),$$

$$\mathcal{M}_\omega:=\{u\in H^1(\mathbb{R}):J_\omega(u)=d_\omega\}.$$

Lemma III.5. $-\infty<d_\omega<0$ and $\mathcal{M}_\omega\subset\mathcal{A}_\omega$.

Proof. First we will prove that there exists positive constants C_1 and C_2 such that

$$|\gamma||u(0)|^2\leq\frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{R})}+C_1\|u\|^2_{L^2(-1,1)}\leq\frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{R})}+\frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})}+C_2,$$

for each $u \in H^1(\mathbb{R})$. Indeed, if $\rho \in (0,1)$ we have the following estimate

$$\begin{aligned}
\gamma u(\rho)^2 - \gamma u(0)^2 &= \int_0^\rho \frac{d}{d\tau}(\gamma u(\tau)^2)\, d\tau \\
&= \int_0^\rho 2\gamma u(\tau)\partial_\tau u(\tau)\, d\tau \\
&\leq \frac{1}{2}\int_0^\rho |\partial_\tau u(\tau)|^2\, d\tau + 2\gamma^2 \int_0^s |u(\tau)|^2\, d\tau \\
&\leq \frac{1}{2}\int_{\mathbb{R}} |\partial_\tau u(\tau)|^2\, d\tau + 2\gamma^2 \int_0^\rho |u(\tau)|^2, d\tau.
\end{aligned}$$

Thus, we obtain

$$-\gamma|u(0)|^2 \leq \frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{R})} + 2\gamma^2\|u\|^2_{L^2(-1,1)} - \gamma u(\rho)^2.$$

As the same way, we can prove that the last inequality holds for $\rho \in (-1,0)$. Then, we have

$$-\gamma|u(0)|^2 \leq \frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{R})} + 2\gamma^2\|u\|^2_{L^2(-1,1)} - \gamma u(\rho)^2 \quad \text{if } \rho \in (-1,1).$$

We integrate respect $\rho \in (-1,1)$, to deduce

$$-2\gamma|u(0)|^2 \leq \|\partial_x u\|^2_{L^2(\mathbb{R})} + 4\gamma^2\|u\|^2_{L^2(-1,1)} + |\gamma|\|u\|^2_{L^2(-1,1)}.$$

Hence, there exists $C_1 > 0$ a positive constant, such that

$$|\gamma||u(0)|^2 \leq \frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{R})} + C_1\|u\|^2_{L^2(-1,1)}.$$

Now, we define $g : [0,\infty) \to \mathbb{R}$ given by $g(\tau) = C_1\tau^2 - \frac{1}{p+1}\tau^{p+1}$. Is easy to see that $\lim\limits_{\tau\to\infty} g(\tau) = -\infty$. Hence there exists a positive constant $C > 0$ such that $g(\tau) \leq C$ for all $\tau \geq 0$. We see that

$$C_1\|u\|^2_{L^2(-1,1)} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})} \leq 2C.$$

We conclude that there exists $C_1, C_2 > 0$ positive constants such that

$$|\gamma||u(0)|^2 \leq \frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{R})} + C_1\|u\|^2_{L^2(-1,1)} \leq \frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{R})} + \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})} + C_2.$$

Then

$$\begin{aligned}
E(u) &= \frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{R})} + \frac{\gamma}{2}|u(0)|^2 + \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})} \\
&\geq \frac{1}{4}\|\partial_x u\|^2_{L^2(\mathbb{R})} + \frac{1}{2(p+1)}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})} - C_2.
\end{aligned}$$

Thus we have the following estimate

$$E(u) \geq \frac{1}{4}\|\partial_x u\|^2_{L^2(\mathbb{R})} + \frac{1}{2(p+1)}\|u\|^{p+1}_{L^{p+1}(\mathbb{R})} - C_2, \tag{3.6}$$

this implies $d_\omega \geq -\frac{C_2}{2}$ and thus $d_\omega > -\infty$.

Next, we prove that $d_\omega < 0$. Since $\Phi(x) = e^{-\frac{|\gamma||x|}{2}}$ is a eigenfunction of H_γ corresponding to the first eigenvalue $-\frac{\gamma^2}{4}$, we have

$$\begin{aligned} d_\omega &\leq J_\omega(\lambda\Phi) \\ &= \frac{\lambda^2}{2}(\|\partial_x\Phi\|^2_{L^2(\mathbb{R})} + \gamma|\Phi(0)|^2 + \omega\|\Phi\|^2_{L^2(\mathbb{R})}) + \frac{\lambda^{p+1}}{p+1}\|\Phi\|^{p+1}_{L^{p+1}(\mathbb{R})} \\ &= \frac{\lambda^2}{2}\left(\omega - \frac{\gamma^2}{4}\right)\|\Phi\|^2_{L^2(\mathbb{R})} + \frac{\lambda^{p+1}}{p+1}\|\Phi\|^{p+1}_{L^{p+1}(\mathbb{R})} < 0, \end{aligned}$$

for sufficiently small $\lambda > 0$. Thus $d_\omega < 0$.

Let $u \in \mathcal{M}_\omega$. Then, we have $J'_\omega(u) = 0$. Moreover, since $J_\omega(u) = d_\omega < 0$, we have $u \neq 0$. Thus, $u \in \mathcal{A}_\omega$. This proves $\mathcal{M}_\omega \subset \mathcal{A}_\omega$.

□

Recall the following refinement to Fatou's lemma due to Brézis and Lieb [10].

Lemma III.6 (Brézis-Lieb). Let $1 < q < \infty$ and $\{u_n\}_{n\in\mathbb{N}}$ a bounded sequence in $L^q(\mathbb{R})$ such that $u_n(x) \to u(x)$ for a.e. $x \in \mathbb{R}$ as $n \to \infty$. Then

$$\|u_n\|^q_{L^q(\mathbb{R})} - \|u_n - u\|^q_{L^q(\mathbb{R})} - \|u\|^q_{L^q(\mathbb{R})} \to 0, \quad \text{as } n \to \infty.$$

□

The following lemma establishes the improvement from weak to strong convergence due to convergence of the charge/energy functional.

Lemma III.7. Let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$ such that $J_\omega(u_n) \to d_\omega$. Then passing to a subsequence, exists a function $u \in \mathcal{M}_\omega$ such that $u_n \to u$ in $H^1(\mathbb{R})$.

Proof. For $v \in H^1(\mathbb{R})$ we put

$$\|v\|_\omega := \frac{1}{2}\|\partial_x v\|^2_{L^2(\mathbb{R})} + \frac{\omega}{2}\|v\|^2_{L^2(\mathbb{R})} = J_\omega(v) - \frac{\gamma}{2}|v(0)|^2 - \frac{1}{p+1}\|v\|^{p+1}_{L^{p+1}(\mathbb{R})}.$$

It is easy to see that $\|\cdot\|_\omega$ is a equivalent norm to the standard norm of $H^1(\mathbb{R})$.

Let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$ such that $J_\omega(u_n) \to d_\omega$. We have the following estimate

$$\begin{aligned} J_\omega(u_n) &= E(u_n) + \frac{\omega}{2}\|u_n\|^2_{L^2(\mathbb{R})} \\ &\geq \frac{1}{4}\|\partial_x u_n\|^2_{L^2(\mathbb{R})} + \frac{\omega}{2}\|u_n\|^2_{L^2(\mathbb{R})} + \frac{1}{2(p+1)}\|u_n\|^{p+1}_{L^{p+1}(\mathbb{R})} - C_2 \\ &\geq C_3\|u_n\|^2_{H^1(\mathbb{R})} - C_2, \end{aligned}$$

when in the first inequality, we used (3.6), this implies that $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in $H^1(\mathbb{R})$. Passing to a subsequence, we may assume that there exists $u \in H^1(\mathbb{R})$ such that $u_n \rightharpoonup u$ weakly in $H^1(\mathbb{R})$ and $u_n(x) \to u(x)$ for almost everywhere $x \in \mathbb{R}$.

Moreover, since the embedding $H^1(-1,1) \hookrightarrow C[-1,1]$ is compact, we have that $u_n(0) \to u(0)$. Then, we obtain

$$d_\omega \leq J_\omega(u) \leq \liminf_{n\to\infty} J_\omega(u_n) = d_\omega,$$

this implies that $u \in \mathcal{M}_\omega$. To prove that $u_n \to u$ in $H^1(\mathbb{R})$, we use the Brezis-Lieb Lemma to obtain

$$\|u_n - u\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})} = \|u_n\|^2_{L^2(\mathbb{R})} + o(1), \tag{3.7}$$

$$\|\partial_x u_n - \partial_x u\|^2_{L^2(\mathbb{R})} + \|\partial_x u\|^2_{L^2(\mathbb{R})} = \|\partial_x u_n\|^2_{L^2(\mathbb{R})} + o(1). \tag{3.8}$$

On the other hand, by (3.6) we see that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^{p+1}(\mathbb{R})$. Therefore, by the Brezis-Lieb lemma we have

$$\|u_n - u\|^{p+1}_{L^{p+1}(\mathbb{R})} + \|u\|^{p+1}_{L^{p+1}(\mathbb{R})} = \|u_n\|^{p+1}_{L^{p+1}(\mathbb{R})} + o(1). \tag{3.9}$$

Thus, we see that

$$\begin{aligned} J_\omega(u_n - u) + J_\omega(u) &= \frac{1}{2}(\|\partial_x u_n - \partial_x u\|^2_{L^2(\mathbb{R})} + \|\partial_x u\|^2_{L^2(\mathbb{R})}) + \frac{\omega}{2}(\|u_n - u\|^2_{L^2(\mathbb{R})} + \|u\|^2_{L^2(\mathbb{R})}) \\ &+ \frac{1}{p+1}(\|u_n - u\|^{p+1}_{L^{p+1}(\mathbb{R})} + \|u\|^{p+1}_{L^{p+1}+(\mathbb{R})}) + \frac{\gamma}{2}(|u_n(0) - u(0)|^2 + |u(0)|^2) \\ &= J_\omega(u_n) + o(1). \end{aligned}$$

Therefore, we have

$$J_\omega(u_n - u) + J_\omega(u) = J_\omega(u_n) + o(1). \tag{3.10}$$

Finally, we have

$$\begin{aligned} \|u_n - u\|_\omega &= J_\omega(u_n - u) - \frac{\gamma}{2}|u_n(0) - u(0)|^2 - \frac{1}{p+1}\|u_n - u\|^{p+1}_{L^{p+1}(\mathbb{R})} \\ &\leq J_\omega(u_n - u) - \frac{\gamma}{2}|u_n(0) - u(0)|^2 \\ &= J_\omega(u_n) - J_\omega(u) - \frac{\gamma}{2}|u_n(0) - u(0)|^2 + o(1), \end{aligned}$$

where in the first inequality we used that $\alpha > 0$. Then $u_n \to u$ in $H^1(\mathbb{R})$.

□

Lemma III.8. $\mathcal{M}_\omega = \mathcal{A}_\omega = \{e^{i\theta}\varphi_\omega : \theta \in \mathbb{R}\}$.

Proof. By Lemmas III.5 and III.7, we have $\emptyset \neq \mathcal{M}_\omega \subset \mathcal{A}_\omega$. By Lemma III.3, we have $\mathcal{A}_\omega = \{e^{i\theta}\varphi_\omega : \theta \in \mathbb{R}\}$. Then $\mathcal{M}_\omega = \mathcal{A}_\omega$.

□

Now we give the proof of Theorem I.3.

Proof of Theorem I.3. Suppose that $e^{i\omega t}\varphi_\omega$ is orbitally unstable in $H^1(\mathbb{R})$. Then there exists $\varepsilon_0 > 0$, a sequence $\{u_n(t)\}_{n\in\mathbb{N}}$ of solutions of (1.1) and a sequence $\{t_n\} \subset (0,\infty)$ such that

$$\|u_n(0) - \varphi_\omega\|_{H^1(\mathbb{R})} \to 0, \tag{3.11}$$

$$\inf_{\theta\in\mathbb{R}} \|u_n(t_n) - e^{i\theta}\varphi_\omega\|_{H^1(\mathbb{R})} \geq \varepsilon_0. \tag{3.12}$$

By (3.11) and the law of conservation of charge and energy, we see that

$$J_\omega(u_n(t_n)) = J_\omega(u_n(0)) \to J_\omega(\varphi_\omega) = d_\omega.$$

By Lemmas III.7 and III.8, passing to a subsequence, we can suppose that there exists $\eta \in \mathbb{R}$ such that

$$u_n(t_n) \to e^{i\eta}\varphi_\omega \quad \text{in } \; H^1(\mathbb{R}),$$

this contradicts (3.12). Then $e^{i\omega t}\varphi_\omega$ is orbitally stable in $H^1(\mathbb{R})$.

□

3.3 Orbital stability of equilibrium solutions

This section is devoted to prove Theorem I.4. Let assume $1 < p < 5$, $\gamma < 0$ and $\alpha = 1$. We consider the space

$$X = \{u \in L^{p+1}(\mathbb{R}) : \partial_x u \in L^2(\mathbb{R})\}.$$

Thus X is a Banach reflexive space with the norm

$$\|u\|_X = \|\partial_x u\|_{L^2(\mathbb{R})} + \|u\|_{L^{p+1}(\mathbb{R})}.$$

We put

$$d = \inf_{u\in X} E(u),$$

$$\mathcal{M} = \{\varphi \in X : E(\varphi) = d\},$$

$$\mathcal{A} = \{\varphi \in X : E'(\varphi) = 0, \;\; \varphi \neq 0\}.$$

Lemma III.9. $-\infty < d < 0$ and $\mathcal{M} \subset \mathcal{A} = \{e^{i\theta}\varphi_0 : \theta \in \mathbb{R}\}$.

Proof. As the same way as in the Lemma III.5, we have $-\infty < d < 0$ and $\mathcal{M} \subset \mathcal{A}$. Remark that inequality (3.6) holds in X. If $\varphi \in \mathcal{A}$, then φ satisfies properties (2.1)-(2.5) of Lemma II.1 with $\omega = 0$. As in Lemma III.4, we obtain $\mathcal{A} = \{e^{i\theta}\varphi_0 : \theta \in \mathbb{R}\}$.

□

Lemma III.10. Let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$ such that $E(u_n) \to d$. Then, passing to a subsequence $u_n \to u$ in X, for some $u \in \mathcal{M}$.

Proof. By (3.6) $\{u_n\}_{n\in\mathbb{N}}$ is bounded in X. Since X is reflexive, passing to a subsequence, we may assume that there exists $u \in X$ such that

$$u_n \rightharpoonup u \quad \text{weakly in } \; X.$$

Then $u_n(0) \to u(0)$, $u_n \rightharpoonup u$ weakly in $L^{p+1}(\mathbb{R})$ and $\partial_x u_n \rightharpoonup \partial_x u$ weakly in $L^2(\mathbb{R})$. Thus, we have

$$d \leq E(u) \leq \liminf_{n\to\infty} E(u_n) = d.$$

This implies that $u \in \mathcal{M}$, $\|u_n\|_{L^{p+1}(\mathbb{R})} \to \|u\|_{L^{p+1}(\mathbb{R})}$ and $\|\partial_x u_n\|_{L^2(\mathbb{R})} \to \|\partial_x u\|_{L^2(\mathbb{R})}$. Then $u_n \to u$ in X.

□

Lemma III.11. Let $\{u_n\}_{n\in\mathbb{N}} \subset H^1(\mathbb{R})$ such that $E(u_n) \to E(\varphi_0)$ and $\|u_n\|_{L^2(\mathbb{R})} \to \|\varphi_0\|_{L^2(\mathbb{R})}$. Then, passing to a subsequence we may assume that there is $\theta_0 \in \mathbb{R}$ such that $u_n \to e^{i\theta_0}\varphi_0$ in $H^1(\mathbb{R})$.

Proof. By Lemmas III.9 and III.10, we have $\mathcal{M} = \mathcal{A} = \{e^{i\theta}\varphi_0 : \theta \in \mathbb{R}\}$ and $d = E(\varphi_0)$. By Lemma III.9, passing to a subsequence we assume that there exists $\theta_0 \in \mathbb{R}$ such that $u_n \to e^{i\theta_0}\varphi_0$ in X, $u_n \rightharpoonup e^{i\theta_0}\varphi_0$ weakly in $L^2(\mathbb{R})$. Thus, we have

$$\|e^{i\theta_0}\varphi_0\|_{L^2(\mathbb{R})} \leq \liminf_{n\to\infty} \|u_n\|_{L^2(\mathbb{R})} = \|\varphi_0\|_{L^2(\mathbb{R})} = \|e^{i\theta_0}\varphi_0\|_{L^2(\mathbb{R})},$$

which implies that $u_n \to e^{i\theta_0}\varphi_0$ in $L^2(\mathbb{R})$. Since $H^1(\mathbb{R})$ is a Hilbert space, we obtain that $u_n \to e^{i\theta_0}\varphi_0$ in $H^1(\mathbb{R})$.

□

Proof of Theorem I.4. Suppose that $e^{i\omega t}\varphi_0$ is orbitally unstable in $H^1(\mathbb{R})$. Then there exists $\varepsilon_0 > 0$, a sequence $\{u_n(t)\}_{n\in\mathbb{N}}$ of solutions of (1.1) and a sequence $\{t_n\} \subset (0, \infty)$ such that

$$\|u_n(0) - \varphi_0\|_{H^1(\mathbb{R})} \to 0, \tag{3.13}$$

$$\inf_{\theta\in\mathbb{R}} \|u_n(t_n) - e^{i\theta}\varphi_0\|_{H^1(\mathbb{R})} \geq \varepsilon_0. \tag{3.14}$$

By (3.13) and the law of conservation of charge and energy, we see that

$$E(u_n(t_n)) = E(u_n(0)) \to E(\varphi_0) = d.$$

By Lemma III.11, passing to a subsequence, we can suppose that there exists $\eta \in \mathbb{R}$ such that

$$u_n(t_n) \to e^{i\eta}\varphi_0 \quad \text{in } H^1(\mathbb{R}),$$

this contradicts (3.14). Then $e^{i\omega t}\varphi_0$ is orbitally stable in $H^1(\mathbb{R})$.

□

Conclusions

In this work, we have constructed explicitly the standing waves of the nonlinear Schrödinger equation (1.1) by a quadrature procedure. Also we have proved that standing waves are orbitally stable in $H^1(\mathbb{R})$ where we used that the embedding $H^1(-1,1) \hookrightarrow C[-1,1]$ is compact and properties of reflexive Banach Spaces.

On one hand, tools like the Rellich-Kondrachov theorem do not apply in this unbounded setting. We overcame this obstacle exploiting the interplay between the attractive delta potential ($\gamma < 0$) and the repulsive nonlinearity ($\alpha = 1$). This competition ensures that the energy functional (3.1) achieves its minimum in $H^1(\mathbb{R})$, thereby compensating for the lack of compactness.

On the other hand, employing techniques from the theory of Hilbert spaces and properties of reflexive Banach spaces, we established that the constructed standing waves are orbitally stable in $H^1(\mathbb{R})$. This orbital stability result confirms the robustness of these profiles under small perturbations. The biggest contribution in this work is the construction of the profiles to the nonlinear Schrödinger equation (1.3).

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