Nonlinear orbital stability of traveling wave solutions to an elasto-chemical model*

Ramón G. Plaza

IIMAS - UNAM (México)

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Mathematisches Institut, Universität Leipzig
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1 The equations

2 Spectral stability

3 Exponential decay of constructed C_0-semigroup

4 Nonlinear orbital stability

5 Discussion

Modelling

- Coupled mechano-chemical system: calcium diffuses freely on a cytoplasmic material (e.g. *cytoskeleton* model of MURRAY, J. D. AND OSTER, G. F., IMA J. Math. Appl. Med. Biol. **1**, pp. 51–75; gel-like substance, elastic properties (deformable)).
- Coupling via: *(i)* Actomyosin molecules, exert stress on the material; sensitive to calcium concentration; and, *(ii)* activation of calcium due to deformation of cytoplasm.
- Slow diffusion of calcium: quasi-static balance of forces, inertial terms neglected in the elastic equation.

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Chemical model: Calcium released by autocatalytic process (self-estimulated), diffuses freely, and more calcium is seen as cytoplasm is stretched (stretch activation)

$$c_t = \underbrace{D\Delta c}_{\text{Fick's diffusion}} + \underbrace{R(c)}_{\text{autocatalytic term}} + \underbrace{\epsilon(\mathrm{div}_x\, u)}_{\text{stretch activation}}$$

$$\begin{aligned} c &= \text{free calcium concentration} \\ \mathbb{R}^3 \ni u &= \text{elastic displacement vector of citoplasm} \\ 0 \leq \epsilon &= \text{stretch factor or contraction stress coefficient} \\ 0 < D &= \text{Fick's diffusion coefficient of calcium} \end{aligned}$$

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Mechanical model: Slow diffusion of calcium, quasi-static balance of forces. Inertial terms negligible!

$$\nabla \cdot \sigma + \rho F = 0,$$

$$\sigma = \underbrace{\mu_1 e_t + \mu_2 (\mathrm{div}_x u)_t I}_{\text{viscous stress}} + \underbrace{Ee + (1-E)(\mathrm{div}_x u) I}_{\text{elastic stress}} + \underbrace{\tau(c) I}_{\text{active contraction stress due to calcium}},$$

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Body forces proportional to elastic displacement:

$$F = -su, \qquad s > 0,$$

$$\begin{aligned} \rho &= \text{cytogel density} \\ e = \tfrac{1}{2}(\nabla u + \nabla u^{\top}) &= \text{strain tensor} \\ \mu_i &= \text{bulk and shear viscosities} \\ E &= \text{elastic modulus} \end{aligned}$$

System of equations

One dimensional version of the model by Lane *et al.* :

$$\begin{aligned} \mu u_{xxt} + u_{xx} - \tau(c)_x - su &= 0, \\ c_t - Dc_{xx} - R(c) - \epsilon u_x &= 0, \end{aligned}$$

$(x,t) \in \mathbb{R} \times [0, +\infty)$, where:

$$\begin{aligned} \mathbb{R} \ni u &= \text{elastic displacement}, \\ \mathbb{R} \ni c &= \text{concentration of free calcium}, \\ \mu = \mu_1 + \mu_2 &= \text{combined shear and bulk viscosities}, \\ 0 < s &= \text{restoring force}, \\ 0 \leq \epsilon &= \text{contraction stress on the increase of } c, \\ 0 < D &= \text{Fick's diffusion constant of calcium}. \end{aligned}$$

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Spectral stability

Exponential decay of constructed C_0-semigroup

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Nonlinear terms:

$$R(c) = \text{autocatalytic term},$$
$$\tau(c) = \text{contractile forces acting on the medium due to } c.$$

Assumptions:

1. $R(0) = R(1) = R(c_0) = 0$, for some $c_0 \in (0,1)$.
2. $\int_0^1 R(c)\, dc > 0$.
3. $R'(0) < 0$, and $R'(1) < 0$ (bistable shape),
4. $\tau'(c)$ bounded for all $c \in (0,1)$,
5. τ has compact support in $(0,1)$, with $\tau \equiv 1$ for $c \in (\delta, 1-\delta)$, $1 \gg \delta > 0$.

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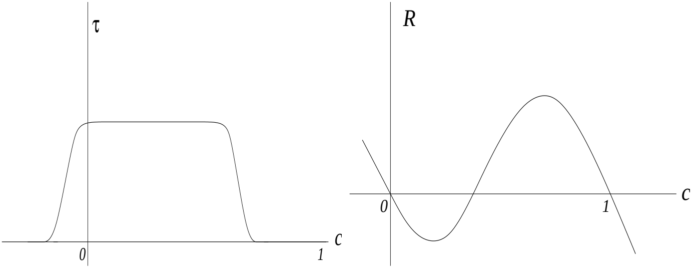

Form of the nonlinear terms $R(c)$ and $\tau(c)$.

Traveling wave solutions

$$(u, c)(x, t) = (\bar{u}, \bar{c})(x + \theta t),$$

$$\theta = \text{ wave speed},$$

with

$$\bar{u}(\pm\infty) = 0, \qquad \bar{c}(+\infty) = 1, \qquad \bar{c}(-\infty) = 0.$$

Wave equations:

$$\begin{aligned} \mu\theta\bar{u}''' + \bar{u}'' - (\tau(\bar{c}))' - s\bar{u} &= 0, \\ \theta\bar{c}' - D\bar{c}'' - R(\bar{c}) - \epsilon\bar{u}' &= 0. \end{aligned}$$

$x \to x + \theta t$, $\;' = d/dx$ (galilean variable).

Existence of traveling waves for small $\epsilon \geq 0$

- G. Flores, A. Minzoni, K. Mischiakow, and V. Moll, Nonlinear Anal. **36** (1999), no. 1, Ser. A: Theory Methods, pp. 45–62.

Proposition

For $\epsilon \geq 0$ suff. small, there exist $(\bar{u}^\epsilon, \bar{c}^\epsilon)$ such that $\bar{u}^\epsilon(\pm\infty) = 0$, $\bar{c}^\epsilon(+\infty) = 1$, $\bar{c}^\epsilon(-\infty) = 0$, and the speed is uniquely determined by

$$\theta(\epsilon) = \theta_0 + o(1),$$

$$\theta_0 := \frac{\int_0^1 R(c)\,dc}{\int_{\mathbb{R}} \bar{c}'(x)^2\,dx} > 0,$$

When $\epsilon = 0$, $\bar{c}^0$ is the bistable Nagumo front, with speed θ_0.

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Corollary (Exponential decay)

$$
\begin{aligned}
|\partial_x^j \bar{u}^\epsilon(x)| &\lesssim e^{-|x|/C_1}, \quad \text{as } |x| \to +\infty, \quad j = 0,1,2,\\
|\partial_x^i (\bar{c}^\epsilon(x) - 1)| &\lesssim e^{-x/C_1}, \quad \text{as } x \to +\infty, \quad i = 0,1,\\
|\partial_x^i \bar{c}^\epsilon(x)| &\lesssim e^{+x/C_1}, \quad \text{as } x \to -\infty, \quad i = 0,1,
\end{aligned}
$$

uniform $C_1 > 0$, *for all* $\epsilon \sim 0^+$.

Perturbed problem

Let $u + \bar{u}$, $c + \bar{c}$ be solutions, with (u, c) perturbations.

Nonlinear perturbation equations

$$\begin{aligned} \mu\theta u_{xxx} + \mu u_{xxt} + u_{xx} - su - (\tau(c+\bar{c}) - \tau(\bar{c}))_x &= 0, \\ c_t + \theta c_x - Dc_{xx} - \epsilon u_x - (R(c+\bar{c}) - R(\bar{c})) &= 0. \end{aligned} \tag{NL}$$

Linearized (around the waves) system for the perturbation

$$\begin{aligned} \mu\theta u_{xxx} + \mu u_{xxt} + u_{xx} - su - (\tau'(\bar{c})c)_x &= 0, \\ c_t + \theta c_x - Dc_{xx} - \epsilon u_x - R'(\bar{c})c &= 0. \end{aligned} \tag{L}$$

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Results

- G. Flores and R. P., Journal of Differential Eqs. **247** (2009), no. 5, pp. 1529–1590.

Theorem 1 (Spectral stability)
For each $\epsilon \geq 0$ sufficiently small, traveling waves $(\bar{u}^\epsilon, \bar{u}^\epsilon)$ are spectrally stable and $\lambda = 0$ is an isolated simple eigenvalue.

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Theorem 1 (Spectral stability)
For each $\epsilon \geq 0$ sufficiently small, traveling waves $(\bar{u}^\epsilon, \bar{u}^\epsilon)$ are spectrally stable and $\lambda = 0$ is an isolated simple eigenvalue.

Theorem 2 (Semigroup estimates)

For each $\epsilon \geq 0$ sufficiently small, there exists $\omega_0 > 0$ such that for each inicial cond. $(u_0, c_0) \in H^2 \times H^1$ there is a global solution (u_x, c) en $C([0, +\infty); H^1 \times H^1)$ to system (L) and some $\alpha_* \in \mathbb{R}$ such that

$$\|(u_x, c)(\cdot, t) - \alpha_*(\bar{u}^\epsilon_{xx}, \bar{c}^\epsilon_x)(\cdot)\|_{L^2 \times L^2} \lesssim e^{-\omega_0 t},$$

for each $t > 0$. Moreover, if $(u_0, c_0) \in (W^{1,1} \cap H^3) \times H^2$ then

$$\|(u, c)(\cdot, t) - \alpha_*(\bar{u}^\epsilon_x, \bar{c}^\epsilon_x)(\cdot)\|_{L^\infty \times L^\infty} \to 0, \qquad \text{if} \quad t \to +\infty.$$

Theorem 3 (Nonlinear stability)

For each $\epsilon \geq 0$ sufficiently small there exists $\eta_0 > 0$ such that, if $(\tilde{u}_0, \tilde{c}_0) \in H^2 \times H^1$ and $\alpha_0 \in \mathbb{R}$ satisfy

$$\|(\tilde{u}_{0x}, \tilde{c}_0)(\cdot) - (\bar{u}_x^\epsilon, \bar{c}^\epsilon)(\cdot + \alpha_0)\|_{H^1 \times H^1} < \eta \leq \eta_0,$$

Then there exists a unique global solution $(\tilde{u}_x, \tilde{c}) \in C([0, +\infty); H^1 \times H^1)$ to system (NL) and some $\alpha_\infty \in \mathbb{R}$ such that

$$|\alpha_0 - \alpha_\infty| < C_1 \eta_0,$$

$$\|(\tilde{u}_x, \tilde{c})(\cdot, t) - (\bar{u}_x^\epsilon, \bar{c}^\epsilon)(\cdot + \theta t + \alpha_\infty)\|_{H^1 \times H^1} \leq C\eta_0 e^{-\frac{1}{2}\omega_0 t} \to 0,$$

as $t \to +\infty$. Moreover, if $(\tilde{u}_0, \tilde{c}_0) \in (W^{1,1} \cap H^3) \times H^2$ then

$$\|(u, c)(\cdot, t) - (\bar{u}^\epsilon, \bar{c}^\epsilon)(\cdot + \theta t + \alpha_\infty)\|_{L^\infty \times L^\infty} \leq C\eta_0 e^{-\omega_0 t} \to 0,$$

as $t \to +\infty$.

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Spectral problem

Perturbations of form $(e^{\lambda t}u(x), e^{\lambda t}c(x))$ with $\lambda \in \mathbb{C}$:

$$\mu\theta u_{xxx} + (\mu\lambda + 1)u_{xx} - su - (\tau'(\bar{c})c)_x = 0,$$
$$\lambda c + \theta c_x - Dc_{xx} - \epsilon u_x - R'(\bar{c})c = 0.$$

Necessary condition for stability: no solutions $(u, c) \in L^2$ with $\mathrm{Re}\,\lambda > 0$.

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First order system formulation

J. ALEXANDER, R. A. GARDNER, AND C.K.R.T. JONES, J. Reine Angew. Math. **410** (1990), pp. 167–212.

Spectral problem:

$$W_x = \mathbb{A}^\epsilon(x, \lambda)W,$$

$$W := (u, u_x, u_{xx}, c, c_x)^\top.$$

$$W \in L^2(\mathbb{R}; \mathbb{C}^5)$$

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$$\mathbb{A}^\epsilon(x,\lambda) :=$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ s/\mu\theta & 0 & -(1+\mu\lambda)/\mu\theta & \tau''(\bar{c})\bar{c}_x/\mu\theta & \tau'(\bar{c})/\mu\theta \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\epsilon/D & 0 & (\lambda - R'(\bar{c}))/D & \theta/D \end{pmatrix}$$

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Asymptotic systems:

$$W_x = \mathbb{A}^{\epsilon}_{\pm}(\lambda) W,$$

$$\mathbb{A}^{\epsilon}(\lambda) := $$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ s/\mu\theta & 0 & -(1+\mu\lambda)/\mu\theta & 0 & \tau'(n)/\mu\theta \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -\epsilon/D & 0 & (\lambda - R'(n))/D & \theta/D \end{pmatrix}$$

$n = 0, 1$ for $x = -\infty, +\infty$.

Definition of spectra

Family of densely defined closed operators in $L^2(\mathbb{R};\mathbb{C}^5)$:

$$\mathcal{T}(\lambda) : \mathcal{D}(\mathcal{T}) \longrightarrow L^2(\mathbb{R};\mathbb{C}^5),$$
$$\mathcal{T}(\lambda)W := W_x - \mathbb{A}^\epsilon(x,\lambda)W,$$

domain $\mathcal{D}(\mathcal{T}^\epsilon) = H^1(\mathbb{R};\mathbb{C}^5)$, indexed by $\epsilon \geq 0$ and $\lambda \in \mathbb{C}$.

Definition (Spectra)

$$
\begin{aligned}
\rho &:= \{\lambda \in \mathbb{C} \,:\, \mathcal{T}(\lambda) \text{ is 1 - 1 and onto } \mathcal{T}(\lambda)^{-1} \text{ bounded}\},\\
\sigma_{\text{pt}} &:= \{\lambda \in \mathbb{C} \,:\, \mathcal{T}(\lambda) \text{ is Fredholm with index 0 and}\\
&\qquad\qquad \text{non-trivial kernel}\},\\
\sigma_{\text{ess}} &:= \{\lambda \in \mathbb{C} \,:\, \mathcal{T}(\lambda) \text{ either has non-zero index or is not Fredholm}\}
\end{aligned}
$$

The spectrum is $\sigma = \sigma_{ess} \cup \sigma_{pt}$. Since $\mathcal{T}$ closed, then $\rho = \mathbb{C} \backslash \sigma$. We say $\lambda \in \sigma_{pt}$ is an eignvalue.

Definition (Multiplicities)

For $\lambda \in \sigma_{pt}$: (i) Its geometric multiplicity (g.m.) is the maximal number of linearly independent elements in ker $\mathcal{T}^{(}\lambda)$. (ii) Suppose $\lambda \in \sigma_{pt}$ has g.m. $= 1$, so that ker $\mathcal{T}(\lambda) =$ span $\{W_1\}$. We say λ has algebraic multiplicity (a.m.) equal to m if we can solve

$$\mathcal{T}^{\epsilon}(\lambda)W_j = \tilde{\mathbb{A}}_1^{\epsilon}(x)W_{j-1},$$

for each $j = 2, \ldots, m$, with $W_j \in H^1$, but there is no H^1 solution W to

$$\mathcal{T}^{\epsilon}(\lambda)W = \tilde{\mathbb{A}}_1^{\epsilon}(x)W_m.$$

For an arbitrary eigenvalue $\lambda \in \sigma_{pt}$ with g.m. $= l$, the a.m. is defined as the sum of the multiplicities $\sum_k^l m_k$ of a maximal set of linearly independet elements in ker $\mathcal{T}(\lambda) =$ span $\{W_1, \ldots, W_l\}$.

Remark : The definition coincides with the usual one for equations in standard form $U_t = LU$, for a given linearized operator L, when written as a first order system. This holds because *the Fredholm properties of* $\mathcal{L} - \lambda$ *and* $\mathcal{T}(\lambda)$ *are the same*. See:

- B. Sandstede, *Stability of travelling waves*, in Handbook of dynamical systems, Vol. 2, B. Fiedler, ed., North-Holland, Amsterdam, 2002, pp. 983–1055.
- B. Sandstede and A. Scheel, Proc. Roy. Soc. Edinburgh Sect. A **130** (2000), no. 2, pp. 419–448.
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Definition (Spectral stability)
We say the waves are spectrally stable if

$$\sigma \subset \{\lambda \in \mathbb{C} \,:\, Re\,\lambda < 0\} \cup \{0\},$$

i.e., there are no solutions in L^2 *with* $Re\,\lambda \geq 0$*; here* $\lambda = 0$ *is the eigenvalue associated to trasnlation invariance, with eigenfunction* $(\bar{u}_x, \bar{c}_x)$.

Idea:

- For $\epsilon = 0$, spectral stability of the Nagumo and elastic fronts follow by energy estimates.
- For $\epsilon \geq 0$ sufficiently small, stability persists due to uniform convergence of the *Evans functions* as $\epsilon \to 0^+$.

Evans function

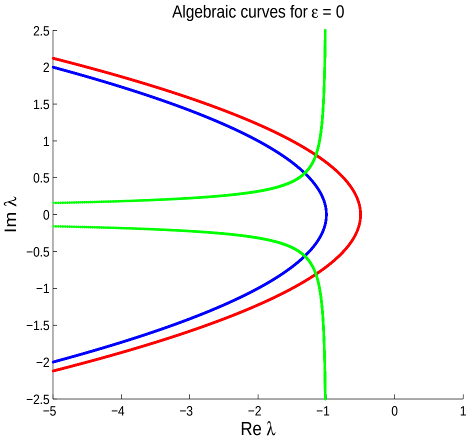
- Ω = open region in the complement of the essential spectrum containing $\{\text{Re}\,\lambda \geq 0\}$.
- The Evans function $D^\epsilon(\lambda)$ is an analytical function defined on Ω; its zeroes coincide in location and multiplicity with the eigenvalues of the spectral problem.
- D can be defined via the Wronskian of the first order system.

$$\Lambda_0 := \min\{|R'(1)|, |R'(0)|, 1/\mu\} > 0$$

$$\Omega := \{\lambda \in \mathbb{C} \; : \mathrm{Re}\,\lambda > -\tfrac{1}{2}\Lambda_0\}.$$

In Ω the dimensions of the stable and unstable manifolds of the asymptotic systems $W_x = \mathbb{A}_\pm W$ are contant:

$$\dim U_\pm^\epsilon(\lambda) = 2, \qquad \dim S_\pm^\epsilon(\lambda) = 3.$$



Algebraic curves limiting the essential spectrum for $\epsilon \sim 0^+$. Note the spectral gap.

Definition of Evans function:

$$D^\epsilon(\lambda) = \det \big(W_1^-(x,\lambda), W_2^-(x,\lambda), W_3^+(x,\lambda), W_4^+(x,\lambda), W_5^+(x,\lambda) \big)_{|x=0},$$

Properties: D^ϵ analytic in Ω, and $D^\epsilon = 0$ iff λ is an eigenvalue. The order of the zero coincides with the algebraic multiplicity.

Facts:

- $(\bar{u}^0, \bar{c}^0)$ are spectrally stable (proof with energy estimates). $\lambda = 0$ is a simple eigenvalue, i.e., $D^0(\lambda) \neq 0$ for Re $\lambda \geq 0$, except for $\lambda = 0$; $(d/d\lambda)D^0(0) \neq 0$.
- For all $\epsilon \sim 0^+$, $\lambda \in \Omega$, $\mathbb{A}^\epsilon_\pm(\lambda)$ are hyperbolic with $\dim U^\epsilon_\pm(\lambda) = 2$, $\dim S^\epsilon_\pm(\lambda) = 3$. This shows $\sigma_{\text{ess}} \subset \{\text{Re}\ \lambda < 0\}$.
- For all $\epsilon \geq 0$ fixed, $\sigma_{\text{pt}} \cap \{\text{Re}\ \lambda \geq -1/2\mu\}$ is uniformly bounded.

- For $\lambda \in \Omega$, $\epsilon \geq 0$ small, $U_-^\epsilon, S_+^\epsilon \to U_-^0, S_+^0$ in angle as $\epsilon \to 0^+$ with rate $\mathcal{O}(\epsilon + \delta(\epsilon)) =: \eta(\epsilon)$, i.e.,

$$|v_{j\pm}^\epsilon - v_{j\pm}^0| \leq \eta(\epsilon).$$

for spanning bases. Moreover, by exp. decay of the waves,

$$|(\mathbb{A}^\epsilon - \mathbb{A}_\pm^\epsilon) - (\mathbb{A}^0 - \mathbb{A}_\pm^0)| \leq C_2 \eta(\epsilon) e^{-|x|/C_1}.$$

Convergence result:

- R.P. AND K. ZUMBRUN (Discr. and Cont. Dynam. Syst. **10** (2004), no. 4, pp. 885–924).

Under such structural conditions (hiperbolicity, exponential decay, regularity) the sequence of Evans functions D^ϵ converges locally and uniformly to D^0 as $\epsilon \to 0^+$ with ratio

$$|D^\epsilon - D^0| \leq \mathcal{O}(\epsilon + |\delta(\epsilon)|) = \mathcal{O}(\eta(\epsilon));$$

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$D^0(\lambda) \neq 0$ in $\mathrm{Re}\,\lambda \geq 0$, except at $\lambda = 0$. Thus, $D^\epsilon(\lambda) \neq 0$ for $\epsilon \sim 0$ small (by uniform convergence and analyticity). There are no non-zero eigenvalues with $\{\mathrm{Re}\,\lambda \geq 0\}$. The multiplicity persists by convergence of the derivatives of D^ϵ.

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Key ideas:

1. Write the linear system (L) in terms of the deformation gradient $(v, c) := (u_x, c)$,

$$\mathcal{L}(\partial_x, \partial_t)(v, c) = 0.$$

2. Construction of the semigroup associated to the above equation. Global solution operator:

$$(v_0, c_0) \in H^1 \times H^1 \mapsto \mathcal{S}(t)(v_0, c_0) \in C([0, +\infty); H^1 \times H^1).$$

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Lemma

$\mathcal{S}(t)$ is a C_0-semigroup in $H^1 \times H^1$.

3. $\mathcal{S}(t)$ has a densely defined generator $\mathcal{A}$

$$\mathcal{A} : \mathcal{D} \subset H^1 \times H^1 \longrightarrow H^1 \times H^1,$$

where

$$\mathcal{D} = (H^2 \cap L^1 \cap \mathcal{U}) \times H^3,$$

and

$$\mathcal{U} = \{u \in L^2 : \int u = 0, \int \int^x u = 0, \text{two antiderivatives in } L^2\}.$$

($\hat{u}(k)$ has a double zero in $k = 0$.)

4. Fredholm properties:

$$\mathrm{nul}\,(\mathcal{A}-\lambda) \leq \mathrm{nul}\,(\mathcal{T}(\lambda)),$$
$$\mathrm{nul}\,(\mathcal{A}^*-\lambda^*) \leq \mathrm{nul}\,(\mathcal{T}(\lambda)^*),$$

imply

$$\sigma_{\mathrm{pt}}(\mathcal{A}) \subseteq \sigma,$$
$$\sigma_{\mathrm{ess}}(\mathcal{A}) \subseteq \sigma,$$

Thus, spectral stability for small $\epsilon \geq 0$.

5. $(\bar{u}_{xx}, \bar{c}_x)-$ eigenfunction associated to $\lambda = 0$. $(\bar{\psi}, \bar{\phi})-$ adjoint eigenfunction.

Projection:

$$\mathcal{P}_1(v,c) := (v,c) - \frac{\langle (v,c), (\bar{\psi}, \bar{\phi}) \rangle_{H^1 \times H^1}}{\Theta} (\bar{u}_{xx}, \bar{c}_x),$$

$$\Theta := \langle (\bar{u}_{xx}, \bar{c}_x), (\bar{\psi}, \bar{\phi}) \rangle_{H^1 \times H^1} \neq 0.$$

$$X_1 := \text{rank of } \ \mathcal{P}_1 \ \subset H^1 \times H^1,$$

$$\mathcal{A}_1 := \mathcal{A}_{|X_1}$$

$$\mathcal{S}_1(t) := \mathcal{S}(t)\mathcal{P}_1$$

is a C_0-semigroup in X_1.

5. Resolvent estimates:

(a) For each $\epsilon \geq 0$, Re $\lambda \geq 0$ and $|\text{Im}\,\lambda|$ suff. big,

$$\|(\lambda - \mathcal{A})^{-1}\|_{H^1 \to H^1} \leq C,$$

(b) For $\epsilon \geq 0$ suff. small and Re $\lambda \geq 0$ big enough

$$\|(\lambda - \mathcal{A})^{-1}\|_{H^1 \to H^1} \leq \frac{C}{\text{Re}\,\lambda}.$$

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6. Gearhart-Prüss criterion (C_0-semigroups in Hilbert spaces):

Since

$$\sup\{\mathrm{Re}\,\lambda : \lambda \in \sigma(\mathcal{A}_1)\} < 0,$$

$$\sup_{\mathrm{Re}\,\lambda > 0} \|(\lambda - \mathcal{A}_1)^{-1}\|_{X_1 \to X_1} < +\infty,$$

then the semigroup $\mathcal{S}_1(t)$ on the Hilbert space X_1 is *exponentially stable*:

$$\|\mathcal{S}_1(t)(v,c)\|_{H^1} \leq Ce^{-\omega_0 t}\|(v,c)\|_{H^1}, \qquad (v,c) \in X_1.$$

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Ideas:

1. Ansatz of form:

$$(\tilde{v}, \tilde{c})(x,t) = (v,c)(x + \theta t + \alpha(t), t) + (\bar{u}_x, \bar{c})(x + \theta t + \alpha(t)),$$

$\alpha(t)$ = modulated phase depending on t.

- Pego and Weinstein, Comm. Math. Phys. **164** (1994), pp. 305–349.

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Implicit function theorem applied to the functional

$$\mathcal{G}[(\tilde{v}, \tilde{c}), \alpha](t) := \langle (v, c)\, ,\ (\bar{\psi}, \bar{\phi}) \rangle_{H^1} = 0$$

3. Phase modulation equation:

$$\dot{\alpha}(t) = \frac{\Theta^{-1} \langle (\mu^{-1} N_1, N_2), (\bar{\psi}, \bar{\phi}) \rangle_{H^1 \times H^1}}{1 - \Theta^{-1} \langle (v, c), (\bar{\psi}_\xi, \bar{\phi}_\xi) \rangle_{H^1 \times H^1}}$$

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- Definition of spectra compatible with the usual one when the system is written in standard form.
- Persistence of spectral stability for $\epsilon \sim 0^+$ (uniform convergence of Evans functions).
- Persistence of multiplicity of $\lambda = 0$, eigenfunction $(\bar{u}_x, \bar{c}_x)$.
- Direct construction of the C_0-semigroup associated to the linear equations, *without* generating theorems of Hille-Yosida type.

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Thanks!