Spectral and nonlinear stability of traveling fronts for a hyperbolic Allen-Cahn model with relaxation

Ramón G. Plaza

Institute of Applied Mathematics (IIMAS),
National Autonomous University of Mexico (UNAM)

10th IMACS International Conference in Nonlinear Waves. Athens, GA. March 29-31, 2017.

Collaborators:

- Corrado Mascia (Univ. di Roma, 'La Sapienza')
- Corrado Lattanzio (Univ. dell'Aquila)
- Chiara Simeoni (Univ. of Nice, Sophia-Antipolis)

Reference: Lattanzio, Mascia, P, Simeoni, *Math. Models Methods Appl. Sci.* 26 (2016), 931-985.

Sponsors:

MIUR (Italy) CONACyT (Mexico)

Ministero dell'Istruzione, dell'Università e della Ricerca

CONACYT

Joint project no. 146529

1. Allen-Cahn model with relaxation
2. Hyperbolic reaction-diffusion fronts
3. Stability
4. Numerical experiments

Hyperbolic Allen-Cahn equation with relaxation

Particles or individuals react or interact according to a rate law $f(u)$ and diffuse. Reaction-diffusion model with "relaxation": flux $\boldsymbol{J}$ relaxes to ∇u with relaxation time $\tau > 0$, small,

$$u_t + \operatorname{div} \boldsymbol{J} = f(u),$$
$$\tau \boldsymbol{J}_t + \nabla u = -\boldsymbol{J}.$$

$x \in \Omega \subset \mathbb{R}^n$, $t > 0$, $u = u(x,t)$ scalar (population density), $\boldsymbol{J} \in \mathbb{R}^n$ (flux function). Here diffusion coefficient is $D = 1$.

When $\tau \to 0^+$ one formally recovers the standard parabolic Allen-Cahn equation: $u_t = \Delta u + f(u)$

Allen-Cahn (bi-stable) reaction

$f \in C^2([0,1])$ satisfies:

$$f(0) = f(\alpha) = f(1), \qquad f'(0), f'(1) < 0, \; f'(\alpha) > 0$$
$$f(u) > 0, u \in (0,1), \qquad f(u) < 0, u \in (-\infty, 0) \cup (1, +\infty)$$

Bi-stable reaction:

$$u = 0, u = 1, \quad \text{stable}, \qquad u = \alpha \in (0,1), \quad \text{unstable}$$

e.g. $f(u) = u(1-u)(u-\alpha).$

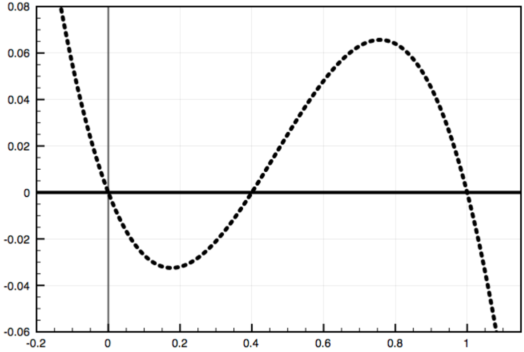


Figure : Bistable reaction $f = f(u)$.

Prototype for:

- phase separation (Cahn–Hilliard, Ginzburg–Landau)
- nerve conduction (Hogdkin-Huxley, Nagumo)
- kinetics of biomolecular reactions (Mikhailov, Murray)
- population dynamics (Allee effect)
- phase trasitions (Allen-Cahn)

Hyperbolic Allen-Cahn system in one dimension

Make $\boldsymbol{J} = -v \in \mathbb{R}^1$, $n = 1$, $x \in \mathbb{R}$, $t > 0$:

$$\begin{aligned} u_t - v_x &= f(u) \\ \tau v_t - u_x &= -v. \end{aligned} \tag{HAC}$$

Hyperbolic system of equations for u, v, scalars.

Motivation

- **Hyperbolic theory of heat conduction:**
Fourier (Fick) empirical heat transfer law:

$$\boldsymbol{J} = -\kappa \nabla u$$

replaced by a heat transfer law of Cattaneo-Maxwell type:

$$\tau \boldsymbol{J}_t + \boldsymbol{J} = -\kappa \nabla u, \qquad 1 \gg \tau > 0$$

J. C. Maxwell, Trans. Soc. London 157 (1867); **Cattaneo**, “Sulla conduzione del calore”, *Atti. Sem. Mat. Fis. Univ. Modena* 3 (1948)

- **Reaction correlated random walk (1d)**

Taylor, *Proc. London Math. Soc.* (1920); **Fürth**, *Z. Phys.* (1920); **Goldstein**, *Quart. J. Mech. Appl. Math.* (1951);

Kac, *Rocky Mountain J. Math.* (1974).

Features:

- Particles or individuals take steps of length Δx and duration Δt
- Particles density: $u = u^+ + u^-$, right and left moving particles.
- Particles continue previous direction with probability $\alpha = 1 - \mu\Delta t$, reverse direction with prob. $\beta = \mu\Delta t$; $\mu =$ frequency of “turns”.
- Particles, in addition, react with each other: correlated random walk is Markovian, it is legitimate to add reaction terms.

- In the continuum limit

$$\gamma = \lim_{\Delta t, \Delta x \to 0} \frac{\Delta x}{\Delta t} = \text{constant}$$

- Particles travel with speed γ

Reaction correlated random walk equations

(Goldstein-Kac)

$$\begin{cases} \partial_t u^+ + \gamma \partial_x u^+ = \mu(u^- - u^+) + F_+(u^+, u^-) \\ \partial_t u^- - \gamma \partial_x u^- = \mu(u^+ - u^-) + F_-(u^+, u^-) \end{cases}$$

Isotropic reaction:

$$F_+(u^+, u^-) = F_-(u^+, u^-) = \tfrac{1}{2}F(u^+ + u^-) = \tfrac{1}{2}F(u),$$

i.e. reaction does not depend on direction of motion.

Substitute $u := u_+ + u_-$ (total mass), $v = \gamma(u^- - u^+)$ (particle flux). One recovers (HAC):

$$\begin{aligned} u_t - v_x &= f(u) \\ \tau v_t - Du_x &= -v. \end{aligned}$$

with correlation time of particle turning process: $\tau = 1/2\mu$; diffusion coefficient $D = \gamma^2/(2\mu)$.

Kac's trick

Cross differentiate (HAC) to eliminate v:

$$\tau u_{tt} + (1 - \tau f'(u))u_t = u_{xx} + f(u).$$

Telegrapher's equation: Nonlinear wave equation with "damping" term $(1 - \tau f'(u))$. Condition:

$$0 < \tau < \frac{1}{\sup |f'(u)|}$$

1. Allen-Cahn model with relaxation
2. Hyperbolic reaction-diffusion fronts
3. Stability
4. Numerical experiments

Parabolic fronts

Bistable (Allen-Cahn) reaction $f \in C^2$. Parabolic equation $u_t = u_{xx} + f(u)$ underlies **traveling front solutions**:

$$u(x,t) = U(\xi), \qquad \xi = x - ct$$

$$U(+\infty) = 1, \qquad U(-\infty) = 0.$$

Features:

- Unique wave speed c_{AC}
- Profile unique up to translations
- U monotone increasing

Example: cubic nonlinearity $f(u) = u(1-u)(u-\alpha)$, $\alpha \in (0,1)$

$$U(\xi) = \frac{1}{2}\left(1 + \tanh\left(\frac{\xi}{2\sqrt{2}}\right)\right), \qquad c_{\text{AC}} = \sqrt{2}(\alpha - \tfrac{1}{2}).$$

General case: phase plane analysis for (U, U').

Hyperbolic Allen-Cahn fronts

$$\begin{aligned} u_t - v_x &= f(u) \\ \tau v_t - u_x &= -v. \end{aligned} \qquad \text{(HAC)}$$

Traveling wave solutions: $(U,V)(\xi)$, $\xi = x - ct$. Profile equations:

$$cU' + V' + F(U) = 0, \qquad U' + c\tau V' - V = 0$$

$$(U,V)(-\infty) = (0,0), \quad (U,V)(+\infty) = (1,0)$$

Theorem (Existence)

f bistable, let τ satisfy

$$0 < \tau < \tau_m := 1/ \sup_{u\in[0,1]} |f'(u)|. \qquad \text{(SC)}$$

Then, there exists a unique value $c_* \in (-1/\sqrt{\tau}, 1/\sqrt{\tau})$ for which system (HAC) possesses a traveling wave $(U,V)(\xi)$ connecting $(0,0)$ with $(1,0)$. Moreover,

- *U is monotone increasing*
- *U and V are positive and converge to their asymptotic states exponentially fast*
- *c_* depends continuously on $\tau \in (0,\tau_m)$, converges to c_{AC} as $\tau \to 0^+$*

Sketch of proof

- Assuming there is a tws: $\operatorname{sgn} c = -\operatorname{sgn} \int_0^1 f(u)\,du$ and

$$c^2\tau < 1$$

  (subcharacteristic condition, same interpretation as in HCL: equilibrium wave speed cannot exceed characteristic speed of the perturbed wave eqn.)
- Under $\xi \to (1 - c^2\tau)^{-1}\xi$ study

$$U' = \phi(U,V) = c\tau f(U) + V, \quad V' = \psi(U,V) = -f(U) - cV$$

- Saddle points $(0,0)$, $(1,0)$. For $c^2 < 1/\tau$, $U_0(c)$ = unstable manifold at 0, $S_1(c)$ stable manifold at 1.

- Shooting method: there exists a unique value c_* such that the graphs of U_0 and S_1 match: $v_1(c_*) = v_0(c_*)$

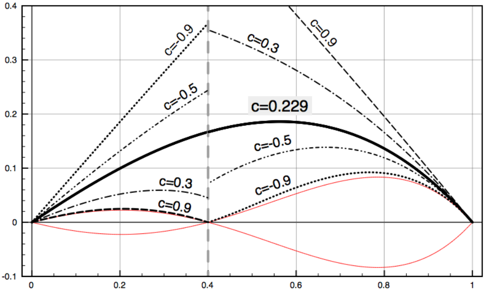


Figure : $f(u) = u(1-u)(u-\alpha)(0.5+u)$, $\alpha = 0.4$, $\tau = 1$. Manifolds U_0 and S_1 are represented for different values of $c \in (-1,1)$. (Monotonicity with respect to the parameter c.) For $c = 0.229$ the two curves intersect at $u = \alpha = 0.4$. Thin red lines = graphs of $\pm\sqrt{\tau f}$.

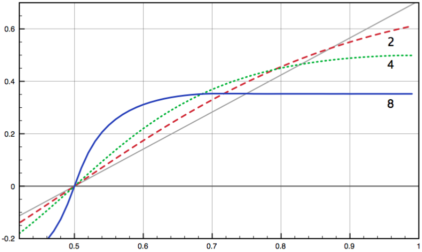
- Monotonicity by contradiction: U' never changes sign.
- By hyperbolicity of the end points, exponential decay:

$$\left| \frac{d^j}{d\xi^j}(U - U_\pm, V)(\xi) \right| \leq C \exp(-\nu|\xi|) \qquad \forall \xi \in \mathbb{R}$$

$C > 0, j = 0, 1, 2.$

- Continuity on τ: by implicit function theorem on $v_0(c,\tau) = v_1(c,\tau)$

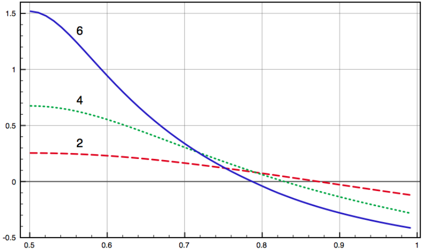
Numerics of propagating speed



Graph of the function

$$\alpha \mapsto c_*$$

for $\tau = 2$ (dashed), 4 (dots), 6 (line). Thin straight line: c_{AC}.



Graph of the function

$$\alpha \mapsto \frac{c_* - c_{\mathrm{AC}}}{c_{\mathrm{AC}}}$$

for $\tau = 2$ (dashed), 4 (dots), 6 (line).

1 Allen-Cahn model with relaxation

2 Hyperbolic reaction-diffusion fronts

3 Stability

4 Numerical experiments

Main result: nonlinear (orbital) stability

Theorem (Stability)

Let $f \in C^3$ bistable, $\tau \in [0, \tau_m)$. Let (U,V) be a traveling wave of (HAC) with speed c_*. Then, there exists $\varepsilon > 0$ such that for any $(u_0, v_0) - (U,V) \in H^1(\mathbb{R})$ with $\|(u_0, v_0) - (U,V)\|_{H^1} < \varepsilon$, the solution (u,v) to the Cauchy problem for (HAC) satisfies

$$\begin{aligned} \|(u,v)(\cdot,t) - (U,V)(\cdot - c_* t + \delta)\|_{H^1} \\ \leq C \|(u_0, v_0) - (U,V)\|_{H^1} e^{-\theta t} \end{aligned}$$

for some shift $\delta \in \mathbb{R}$ and constants $C, \theta > 0$.

Three step program:

- **Spectral stability**. Linearization around the wave. The associated operator is *spectrally stable* $\sigma(\mathcal{L}) \subset \{\text{Re}\,\lambda < 0\} \cup \{0\}$
- **Semigroup and exponential decay**. Resolvent estimates, generation of a C_0 semigroup and application of Gearhart-Prüss theorem
- **Nonlinear (orbital) stability**. Suff. small initial conditions, solutions to nonlinear eq. converge to δ-shifted profile

Spectral stability

Main idea: Analyze the problem as an asymptotic limit as $\tau \to 0^+$.

Linearized operator around the wave $(U,V)(x)$, $x \to x - ct$ (Galilean coordinate):

$$\mathcal{L}^\tau w = -B^{-1}\Big(A\frac{dw}{dx} + C(x)w\Big), \qquad w = (u,v)^\top \in H^1$$

$$A = \begin{pmatrix} -c & -1 \\ -1 & -c\tau \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & \tau \end{pmatrix}, \qquad C(x) = \begin{pmatrix} -a(x) & 0 \\ 0 & 1 \end{pmatrix},$$

$$a(x) = f'(U)$$

Densely defined closed operator with domain $\mathcal{D} = H^1$ to L^2. Stability under **localized** perturbations.

Observation: The operator is singular when $\tau \to 0^+$. (It is not defined at $\tau = 0$.) In the limit, it formally converges to a scalar perturbation equation, the linearized operator around the parabolic front:

$$L_0 u = u_{xx} + c u_x + f'(u) u.$$

Theorem (Henry, Fife-Mcleod)

There exists $\omega_0 > 0$ such that the spectrum $\sigma(L_0)$ of the operator L_0 can be decomposed as $\sigma(L_0) = \{0\} \cup \sigma_-^{(0)}$, where $\lambda = 0$ is an (isolated) eigenvalue with algebraic multiplicity equal to one and eigenspace generated by $dU/dx \in L^2(\mathbb{R})$, and $\sigma_-^{(0)}$ is contained in the half-space $\{\lambda \in \mathbb{C} : \text{Re}\, \lambda \leq -\omega_0 < 0\}$.

Observation: The operator is singular when $\tau \to 0^+$. (It is not defined at $\tau = 0$.) In the limit, it formally converges to a scalar perturbation equation, the linearized operator around the parabolic front:

$$L_0 u = u_{xx} + c u_x + f'(u) u.$$

Theorem (Henry, Fife-Mcleod)

There exists $\omega_0 > 0$ such that the spectrum $\sigma(L_0)$ of the operator L_0 can be decomposed as $\sigma(L_0) = \{0\} \cup \sigma_-^{(0)}$, where $\lambda = 0$ is an (isolated) eigenvalue with algebraic multiplicity equal to one and eigenspace generated by $dU/dx \in L^2(\mathbb{R})$, and $\sigma_-^{(0)}$ is contained in the half-space $\{\lambda \in \mathbb{C} : \text{Re}\, \lambda \leq -\omega_0 < 0\}$.

Reformulation

Eigenvalue problem: $\mathcal{L}w = \lambda w$

$$\begin{aligned} cu' + v' + (a - \lambda)u &= 0, \\ u' + c\tau v' - (1 + \tau\lambda)v &= 0. \end{aligned}$$

Apply **(spectral) Kac's trick**:

$$\begin{aligned} (1 - c^2\tau)u'' + c(1 + \tau(2\lambda - a(x))u' \\ + \big((1 + \tau\lambda)(a(x) - \lambda) - c\tau a'(x)\big)u = 0 \end{aligned}$$

Spectral scalar quadratic pencil in $\lambda \in \mathbb{C}$

Reformulation

Eigenvalue problem: $\mathcal{L}w = \lambda w$

$$cu' + v' + (a - \lambda)u = 0,$$
$$u' + c\tau v' - (1 + \tau\lambda)v = 0.$$

Apply **(spectral) Kac's trick**:

$$(1 - c^2\tau)u'' + c(1 + \tau(2\lambda - a(x))u' + \big((1 + \tau\lambda)(a(x) - \lambda) - c\tau a'(x)\big)u = 0$$

Spectral scalar quadratic pencil in $\lambda \in \mathbb{C}$

Alexander, Gardner, Jones (1990)

Spectral problem can be written as a first order system:

$$W_x = \mathbb{A}^\tau(x,\lambda)W, \qquad W = (u,u')^\top$$

$$\begin{aligned}\mathbb{A}^\tau(x,\lambda) &= \frac{1}{1-c^2\tau}\begin{pmatrix} 0 & 1-c^2\tau \\ c\tau a' + (1+\tau\lambda)(\lambda - a) & c(\tau a - (1+2\tau\lambda)) \end{pmatrix} \\ &= A_0^\tau(x) + \lambda \mathbb{A}_1^\tau(x) + \lambda^2 \mathbb{A}_2^\tau(x).\end{aligned}$$

Family of closed, densely defined operators:

$$\begin{aligned}\mathcal{T}^\tau(\lambda) : \mathcal{D} = H^2 \subset L^2 \to L^2 \\ \mathcal{T}^\tau(\lambda)W = W_x - \mathbb{A}^\tau(x,\lambda)W.\end{aligned}$$

Observation: Well defined for $\tau = 0$.

Definition (cf. Sandstede (2002))

The *resolvent* ρ, the *point spectrum* $\sigma_{\rm pt}$ and the *essential spectrum* $\sigma_{\rm ess}$ are defined as:

$$\begin{aligned}
\rho &= \{\lambda \in \mathbb{C} : \mathcal{T}^\tau(\lambda) \text{ is one-to-one and onto, and} \\
&\quad \mathcal{T}^\tau(\lambda)^{-1} \text{ is bounded}\}, \\
\sigma_{\rm pt} &= \{\lambda \in \mathbb{C} : \mathcal{T}^\tau(\lambda) \text{ is Fredholm with zero index} \\
&\quad \text{and has a non-trivial kernel}\}, \\
\sigma_{\rm ess} &= \{\lambda \in \mathbb{C} : \mathcal{T}^\tau(\lambda) \text{ is either not Fredholm or} \\
&\quad \text{has index different from zero}\}.
\end{aligned}$$

The *spectrum* is $\sigma = \sigma_{\rm ess} \cup \sigma_{\rm pt}$. ($\mathcal{T}^\tau(\lambda)$ closed $\Rightarrow \rho = \mathbb{C} \backslash \sigma$.)

Spectral Kac's transformation:

$$K : \ker(\mathcal{L}^\tau - \lambda) \subset H^1 \to \ker \mathcal{T}^\tau(\lambda) \subset H^1,$$

$$K(u,v) = (u,u')^\top = W, \qquad \text{for each } (u,v) \in \ker(\mathcal{L}^\tau - \lambda),$$

Lemma

K is one-to-one and onto.

Lemma

K induces a one-to-one correspondence between Jordan chains with same block structure and length.

Spectral Kac's transformation:

$$K : \ker(\mathcal{L}^\tau - \lambda) \subset H^1 \to \ker \mathcal{T}^\tau(\lambda) \subset H^1,$$

$$K(u,v) = (u,u')^\top = W, \qquad \text{for each } (u,v) \in \ker(\mathcal{L}^\tau - \lambda),$$

Lemma

K is one-to-one and onto.

Lemma

K induces a one-to-one correspondence between Jordan chains with same block structure and length.

Simple eigenvalue $\lambda = 0$

Lemma

For each $\tau \in [0, \tau_m)$, $\lambda = 0$ is a simple eigenvalue associated to (U', V').

Proof sketch:

By studying the adjoint equation

$$Y_x = -\mathbb{A}^\tau(x,0)^* Y,$$

has a unique bounded solution $Y_0 = (\zeta, \eta)^\top \in H^1$, and by computing the Melnikov-type integral

$$\Gamma := \langle Y_0, \mathbb{A}_1^\tau(x) W_0 \rangle_{L^2} = \int_{-\infty}^{+\infty} \begin{pmatrix} \zeta \\ \eta \end{pmatrix}^* \mathbb{A}_1^\tau(x) \begin{pmatrix} U_x \\ U_{xx} \end{pmatrix} dx,$$

$W_0 = (U_x, U_{xx})^\top$.

$$\Gamma = a_0^{-1}(1+c^2\tau a_0^{-1}) \int_{-\infty}^{+\infty} |1-\tau a(x)| \exp\Big(\int_0^x a_1(y)/a_0\, dy \Big) |U_x|^2 dx$$
$$> 0,$$

where

$$a_0 = 1 - c^2\tau > 0, \quad a_1(x) = c(1-\tau a(x)).$$

□

No purely imaginary point spectrum

Lemma

For each $\tau \in [0, \tau_m)$: if λ is an eigenvalue and $\lambda \in i\mathbb{R}$, then $\lambda = 0$.

Proof sketch: Follows by energy estimates: make $u(x) = w(x)\,z(x)$ with $z(x) = \exp(-\int^x b)$ and $b = -a_1/2a_0$,

$$w_{xx} + \alpha\lambda w_x - \beta(x,\lambda)w = 0,$$

with

$$\alpha = \frac{2c\tau}{1 - c^2\tau}$$

$$\beta(x,\lambda) = \frac{1}{4a_0^2}\Big(a_1(x)^2 - 4a_0a_2(x) - 2a_0a_1'(x)\Big) + \frac{(1-\tau a(x))}{a_0^2}\lambda + \frac{\tau}{a_0}\lambda^2.$$

Multiply by $\bar{w}$, integrate in $\mathbb{R}$:

$$\int_{\mathbb{R}} \text{Im}\,\lambda\{2c\tau(1-c^2\tau)\text{Re }(w_x\bar{w}) + \big(1-\tau a(x)\big)|w|^2\}dx = 0.$$

(by assumption, $\lambda \in i\mathbb{R}$). For $\lambda \neq 0$, thanks to

$$\text{Re }(\bar{w}w_x) = \tfrac{1}{2}\big(|w|^2\big)_x,$$

we conclude $w = 0$ a.e. since $1-\tau a(x) > 0$.

□

Stability of σ_{ess}

Determined by asymptotic operators as $x \to \pm\infty$.
Fredholm curves λ roots of

$$\det(i\xi - \mathbb{A}^{\tau}_{\pm}(\lambda)) = 0.$$

$\mathbb{A}^{\tau}_{\pm} = \lim_{x\to\pm\infty} \mathbb{A}^{\tau}(x,\lambda)$.

Lemma

For each $0 < \tau < \tau_m$, there exists a uniform

$$\chi_0 = \frac{1}{2}\min\{\delta_+, \delta_-\} > 0,$$

such that the algebraic curves $\lambda = \lambda^{\pm}_{1,2}(\xi)$, $\xi \in \mathbb{R}$, satisfy

$$\operatorname{Re}\lambda^{\pm}_{1,2}(\xi) < -\chi_0 < 0,$$

Stability of σ_{ess}

Determined by asymptotic operators as $x \to \pm\infty$.
Fredholm curves λ roots of

$$\det(i\xi - \mathbb{A}^{\tau}_{\pm}(\lambda)) = 0.$$

$\mathbb{A}^{\tau}_{\pm} = \lim_{x\to\pm\infty} \mathbb{A}^{\tau}(x,\lambda)$.

Lemma

For each $0 < \tau < \tau_m$, *there exists a uniform*

$$\chi_0 = \frac{1}{2}\min\{\delta_+, \delta_-\} > 0,$$

such that the algebraic curves $\lambda = \lambda^{\pm}_{1,2}(\xi)$, $\xi \in \mathbb{R}$, *satisfy*

$$\operatorname{Re}\lambda^{\pm}_{1,2}(\xi) < -\chi_0 < 0,$$

$$\Omega = \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda > -\chi_0\}$$

Lemma

For all $0 < \tau < \tau_m$, and all $\lambda \in \Omega$, the coefficient matrices $\mathbb{A}^\tau_\pm(\lambda)$ have no center eigenspace and, moreover,

$$\dim S^\tau_\pm(\lambda) = \dim U^\tau_\pm(\lambda) = 1.$$

Corollary (Stability of the essential spectrum)

For each $0 < \tau < \tau_m$, the essential spectrum is contained in the stable half-plane. More precisely,

$$\sigma_{ess} \subset \{\lambda \in \mathbb{C} : Re\,\lambda \leq -\chi_0 < 0\}.$$

$$\Omega = \{\lambda \in \mathbb{C} : \text{Re}\,\lambda > -\chi_0\}$$

Lemma

For all $0 < \tau < \tau_m$, and all $\lambda \in \Omega$, the coefficient matrices $\mathbb{A}^\tau_\pm(\lambda)$ have no center eigenspace and, moreover,

$$\dim S^\tau_\pm(\lambda) = \dim U^\tau_\pm(\lambda) = 1.$$

Corollary (Stability of the essential spectrum)

For each $0 < \tau < \tau_m$, the essential spectrum is contained in the stable half-plane. More precisely,

$$\sigma_{ess} \subset \{\lambda \in \mathbb{C} : Re\,\lambda \leq -\chi_0 < 0\}.$$

Stability of σ_{pt}

Continuation argument gives

$$u'' + \{a(x) + (\tau a(x) - 1)\lambda - \tau\lambda^2\}u \approx u'' + \{a(x) - \lambda\}u$$

Based on Evans function: convergence of approximate flows leads to spectral description (small τ)

Evans function

$$\mathbb{S}_+^\tau = \text{span}\{\mathbf{w}_+(\lambda)\}, \qquad \mathbb{U}_-^\tau = \text{span}\{\mathbf{w}_-(\lambda)\},$$

Definition

$$D^\tau(\lambda) := \det(\mathbf{w}_-(\lambda), \mathbf{w}_+(\lambda)), \quad \lambda \in \Omega.$$

Properties:

- D^τ is analytic in $\lambda \in \Omega$;
- $D^\tau(\lambda) = 0$ if and only if $\lambda \in \sigma_{\text{pt}} \cap \Omega$; and,
- the order of λ as a zero of D^τ is equal to its algebraic multiplicity

(cf. AGJ (1990); Sandstede (2002))

Evans function

$$\mathbb{S}_+^\tau = \text{span}\{\mathbf{w}_+(\lambda)\}, \qquad \mathbb{U}_-^\tau = \text{span}\{\mathbf{w}_-(\lambda)\},$$

Definition

$$D^\tau(\lambda) := \det(\mathbf{w}_-(\lambda), \mathbf{w}_+(\lambda)), \quad \lambda \in \Omega.$$

Properties:

- D^τ is analytic in $\lambda \in \Omega$;
- $D^\tau(\lambda) = 0$ if and only if $\lambda \in \sigma_{\text{pt}} \cap \Omega$; and,
- the order of λ as a zero of D^τ is equal to its algebraic multiplicity

(cf. AGJ (1990); Sandstede (2002))

Gap lemma: (Kapitula, Sandstede (1997); Gardner, Zumbrun (1997))
Exponential decay $\Rightarrow$ def. of Evans function near σ_{ess}.

Evans function for $\tau = 0$, i.e. stability of the Nagumo front:

Corollary

$D^0(\lambda) \neq 0$ for all Re $\lambda \geq 0$, $\lambda \neq 0$. Moreover, $\lambda = 0$ is a simple zero of $D^0(\cdot)$.

Gap lemma: (Kapitula, Sandstede (1997); Gardner, Zumbrun (1997))
Exponential decay $\Rightarrow$ def. of Evans function near σ_{ess}.

Evans function for $\tau = 0$, i.e. stability of the Nagumo front:

Corollary

$D^0(\lambda) \neq 0$ for all Re $\lambda \geq 0$, $\lambda \neq 0$. Moreover, $\lambda = 0$ is a simple zero of $D^0(\cdot)$.

Convergence of approximate flows (P, Zumbrun (2004)):

Theorem

Under suitable structural assumptions (exp. decay, limits of $\mathbb{S}_\pm^\tau$ and $\mathbb{U}_\pm^\tau$ along λ-rays, $\lambda = r\lambda_0$ as $r \to 0^+$, $\lambda_0 \in \Omega$, and

$$|(\mathbb{A}^\tau - \mathbb{A}_\pm^\tau) - (\mathbb{A}^0 - \mathbb{A}_\pm^0)| \leq C_1 \eta(\tau) e^{-\tilde{\nu}|x|},$$

then the local Evans functions D^τ converge uniformly to D^0:

$$|D^\tau - D^0| \leq C\eta(\tau) \to 0,$$

locally in λ.

Point spectral stability:

Compact subset of Ω:

$$\Omega_R := \{\lambda \in \mathbb{C} : |\lambda| \leq R, \text{Re } \lambda \geq -\tfrac{1}{2}\chi_0\}.$$

By approximation theorem, for τ small, in a Ω_R-neighborhood of λ, uniform convergence of $D^\tau(\lambda)$ to $D^0(\lambda)$ in a (possible smaller) neighborhood of λ as $\tau \to 0^+$ with rate

$$|D^\tau(\cdot) - D^0(\cdot)| = O(\eta(\tau)) = O(\tau + |\zeta(\tau)|) \to 0$$

Thus, $D^\tau(\lambda) \neq 0$ for $\lambda \in \Omega_R$, Re $\lambda \geq 0$, except only at $\lambda = 0$, and for each $0 \leq \tau \ll 1$ sufficiently small.
By continuity argument and no crossing of eigenvalues with fixed multiplicity of the imaginary axis, extends to $0 < \tau < \tau_m$.

Theorem (Spectral stability)

For each $\tau \in (0, \tau_m)$*, there exists* $\omega_0(\tau) > 0$ *such that*

$$\sigma(\mathcal{L}^\tau) = \{0\} \cup \sigma_-^{(\tau)}$$

where $\lambda = 0$ *is an (isolated) eigenvalue with algebraic multiplicity equal to one and eigenspace generated by* $(dU/dx, dV/dx) \in H^1$*, and* $\sigma_-^{(0)}$ *is contained in the half-space* $\{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq -\omega_0\}$.

Linear stability

The spectral mapping property $e^{t\sigma(\mathcal{L})} = \sigma(e^{\mathcal{L}t}) \setminus \{0\}$ is NOT always satisfied. Counterexamples can be produced using hyperbolic equation.

In the hyperbolic case, spectral stability *does not* imply linear stability!

Problems stem from the behavior of the resolvent kernel for large λ.

The Gearhart–Prüss Theorem states that, for $\Sigma_r = \{\text{Re}\,\lambda \geq 0, |\lambda| \geq r\}$,

$$\text{spectral stability} + \sup_{\lambda \in \Sigma_r} |(\lambda - \mathcal{L})^{-1}| < +\infty \quad \Rightarrow \quad \text{linear stability}$$

Linear stability

The spectral mapping property $e^{t\sigma(\mathcal{L})} = \sigma(e^{\mathcal{L}t}) \setminus \{0\}$ is NOT always satisfied. Counterexamples can be produced using hyperbolic equation.

In the hyperbolic case, spectral stability *does not* imply linear stability!

Problems stem from the behavior of the resolvent kernel for large λ.

The Gearhart–Prüss Theorem states that, for $\Sigma_r = \{\text{Re}\,\lambda \geq 0, |\lambda| \geq r\}$,

$$\text{spectral stability} + \sup_{\lambda \in \Sigma_r} |(\lambda - \mathcal{L})^{-1}| < +\infty \quad \Rightarrow \quad \text{linear stability}$$

Resolvent estimates

Technical lemma. Consider $Aw' + (\lambda B + C(x))w = \psi$; "large λ" means rapid oscillations, $C = C(x)$ is "constant" at such scale:

Lemma

There exists $M, r > 0$ such that

$$|w(\cdot;\lambda)|_{H^m} \leq M|\psi|_{H^m}$$

for $|\lambda| \geq r$.

Based on an approximate diagonalization procedure (Mascia, Zumbrun (2002)).

Resolvent estimates

Technical lemma. Consider $Aw' + (\lambda B + C(x))w = \psi$; "large λ" means rapid oscillations, $C = C(x)$ is "constant" at such scale:

Lemma

There exists $M, r > 0$ such that

$$|w(\cdot;\lambda)|_{H^m} \leq M|\psi|_{H^m}$$

for $|\lambda| \geq r$.

Based on an approximate diagonalization procedure (Mascia, Zumbrun (2002)).

By Hille-Yosida theorem:

Lemma

For each $\tau \in (0, \tau_m)$, *the operator* $\mathcal{L}^\tau : \mathcal{D} = H^1 \to L^2$ *is the infinitesimal generator of a* C_0*-semigroup of quasi-contractions* $\{T(t)\}_{t \geq 0}$.

Linear stability

By resolvent estimates we project out the eigenspace spanned by (U', V'), direct application of Gearhart-Prüss theorem:

Theorem (Linear stability)

There exists a projection operator $\pi = I - P$ with one-dimensional range $\{\kappa(U', V') \, : \, \kappa \in \mathbb{R}\}$ such that for any $t > 0$

$$T(t)\pi = \pi T(t) = \pi \qquad \textit{and} \qquad \|T(t)(I - \pi)\| \leq C e^{-\theta t}$$

for some $C, \theta > 0$.

Nonlinear stability

General orbital stability result:
$\mathcal{W}$ - Banach space, norm $|\cdot|_{\mathcal{W}}$, $B_r(\overline{W})$ open ball. F smooth function, $F : \mathcal{D} \subset \mathcal{W} \to \mathcal{W}$ such that: $F(\overline{W}) = 0$ for some $\overline{W} \in \mathcal{D}$. Assume, for some $r > 0$:

$$\{W \in \mathbb{R}^n : F(W) = 0\} \cap \{|W - \overline{W}|_{\mathcal{W}} < r\} = \phi(I)$$

for some smooth function $\phi : I \to \mathbb{R}^n$, I open interval. W.l.o.g. we may assume $0 \in I$ and $\phi(0) = \overline{W}$.

Cauchy problem:

$$\frac{dW}{dt} = F(W) \qquad W(0) = W_0 \in \mathcal{D}$$

Nonlinear stability

General orbital stability result:
$\mathcal{W}$ - Banach space, norm $|\cdot|_{\mathcal{W}}$, $B_r(\overline{W})$ open ball. F smooth function, $F : \mathcal{D} \subset \mathcal{W} \to \mathcal{W}$ such that: $F(\overline{W}) = 0$ for some $\overline{W} \in \mathcal{D}$. Assume, for some $r > 0$:

$$\{W \in \mathbb{R}^n : F(W) = 0\} \cap \{|W - \overline{W}|_{\mathcal{W}} < r\} = \phi(I)$$

for some smooth function $\phi : I \to \mathbb{R}^n$, I open interval. W.l.o.g. we may assume $0 \in I$ and $\phi(0) = \overline{W}$.

Cauchy problem:

$$\frac{dW}{dt} = F(W) \qquad W(0) = W_0 \in \mathcal{D}$$

The linearized problem at $\phi(\delta)$ is

$$\frac{dZ}{dt} = dF(\phi(\delta))Z \qquad\qquad Z(0) = Z_0 \in \mathcal{D}$$

Projection:

$$Q(\delta) = I - P(\delta)$$

associated to one-dim eigenspace spanned by $r(\delta)$, e-vector of $dF(\phi(\delta))$.

General hypotheses:

H1. There exist $C, \theta > 0$ such that the solution $Z = Z(t; Z_0, \delta)$

$$|Q(\delta)Z(t; Z_0, \delta)| \leq Ce^{-\theta t}|Q(\delta)Z_0|$$

for any $Z_0 \in \mathcal{D}$.

H2. ϕ is differentiable at $\delta = 0$ and there exist $C, \delta_0, \gamma > 0$:

$$|\phi(\delta) - \phi(0) - \phi'(0)\delta|_{\mathcal{W}} \leq C\delta^{1+\gamma},$$

for $|\delta| < \delta_0$.

H3. There exist $C, M, \delta_0, \gamma > 0$ such that F is differentiable at $\phi(\delta)$ for any $\delta \in (-\delta_0, \delta_0)$ and

$$|F(\phi(\delta) + W) - F(\phi(\delta)) - dF(\phi(\delta))W|_{\mathcal{W}} \leq C|W|_{\mathcal{W}}^{1+\gamma},$$

for $|\delta| < \delta_0$ and $|W|_{\mathcal{W}} \leq M$.

Theorem

Under **H1**, **H2** *and* **H3**, *there exists* $\varepsilon > 0$ *such that for any* $W_0 \in B_\varepsilon(\bar{W})$ *there exists* $\delta \in I$ *for which the solution* $W(t;W_0)$ *to the Cauchy problem satisfies*

$$|W(t;W_0) - \phi(\delta)|_{\mathcal{W}} \leq C|W_0 - \overline{W}|_{\mathcal{W}}\, e^{-\theta t}$$

for some $C, \theta > 0$

Orbital stability of fronts for (HAC) follows by verifying hypotheses (H2) and (H3), with

$$\mathcal{W} = H^1(\mathbb{R};\mathbb{R}^2), \quad \bar{W} = 0, \quad \phi(\delta) = (U,V)(\cdot+\delta) - (U,V)(\cdot).$$

(H1) is implied by linear stability.

Theorem

Under **H1**, **H2** *and* **H3**, *there exists* $\varepsilon > 0$ *such that for any* $W_0 \in B_\varepsilon(\bar{W})$ *there exists* $\delta \in I$ *for which the solution* $W(t;W_0)$ *to the Cauchy problem satisfies*

$$|W(t;W_0) - \phi(\delta)|_{\mathcal{W}} \leq C|W_0 - \overline{W}|_{\mathcal{W}}\, e^{-\theta t}$$

for some $C, \theta > 0$

Orbital stability of fronts for (HAC) follows by verifying hypotheses (H2) and (H3), with

$$\mathcal{W} = H^1(\mathbb{R};\mathbb{R}^2), \quad \bar{W} = 0, \quad \phi(\delta) = (U,V)(\cdot + \delta) - (U,V)(\cdot).$$

(H1) is implied by linear stability.

1. Allen-Cahn model with relaxation

2. Hyperbolic reaction-diffusion fronts

3. Stability

4. **Numerical experiments**

Numerical scheme

Reactive Goldstein-Kac model correlated random walk:

$$\begin{cases} \partial_t u^+ + \gamma \partial_x u^+ = \frac{1}{2\tau}(u^- - u^+) + \frac{1}{2} f(u^+ + u^-), \\ \partial_t u^- - \gamma \partial_x u^- = \frac{1}{2\tau}(u^+ - u^-) + \frac{1}{2} f(u^+ + u^-). \end{cases}$$

with,

$$\gamma = \frac{1}{\sqrt{\tau}}, \quad u^- = \frac{1}{2}(u + \gamma^{-1} v), \quad u^+ = \frac{1}{2}(u - \gamma^{-1} v).$$

Scheme

Mesh size: $\mathrm{dx} > 0$. Discretize the space, upwind approximation of first order space derivatives, $r_j \approx u^-(j\,\mathrm{dx}, t)$, $s_j \approx u^+(j\,\mathrm{dx}, t)$.

$$\begin{cases} \dfrac{dr_j}{dt} = \dfrac{\gamma}{\mathrm{dx}}\left(r_{j+1} - r_j\right) + \dfrac{1}{2\tau}(-r_j + s_j) + \dfrac{1}{2}f(r_j + s_j), \\ \dfrac{ds_j}{dt} = -\dfrac{\gamma}{\mathrm{dx}}\left(s_j - s_{j-1}\right) + \dfrac{1}{2\tau}(r_j - s_j) + \dfrac{1}{2}f(r_j + s_j). \end{cases}$$

Time step $\mathrm{dt} > 0$, discretize time derivative by implicit-explicit approach (discretize implicitly only linear terms):

$$\frac{r_j^{n+1} - r_j^n}{\mathrm{dt}} = \frac{\gamma}{\mathrm{dx}}(r_{j+1}^{n+1} - r_j^{n+1}) + \frac{1}{2\tau}(s_j^{n+1} - r_j^{n+1}) + \frac{1}{2}f(r_j^n + s_j^n)$$
$$\frac{s_j^{n+1} - s_j^n}{\mathrm{dt}} = -\frac{\gamma}{\mathrm{dx}}(s_{j-1}^{n+1} - s_j^{n+1}) + \frac{1}{2\tau}(r_j^{n+1} - s_j^{n+1}) + \frac{1}{2}f(r_j^n + s_j^n)$$

Set

$$\alpha = \gamma\frac{\mathrm{dt}}{\mathrm{dx}}, \quad \beta = \frac{\mathrm{dt}}{2\tau}, \quad f_j^n = f(r_j^n + s_j^n)$$

Time step $\mathrm{dt} > 0$, discretize time derivative by implicit-explicit approach (discretize implicitly only linear terms):

$$\frac{r_j^{n+1} - r_j^n}{\mathrm{dt}} = \frac{\gamma}{\mathrm{dx}}(r_{j+1}^{n+1} - r_j^{n+1}) + \frac{1}{2\tau}(s_j^{n+1} - r_j^{n+1}) + \frac{1}{2}f(r_j^n + s_j^n)$$

$$\frac{s_j^{n+1} - s_j^n}{\mathrm{dt}} = -\frac{\gamma}{\mathrm{dx}}(s_{j-1}^{n+1} - s_j^{n+1}) + \frac{1}{2\tau}(r_j^{n+1} - s_j^{n+1}) + \frac{1}{2}f(r_j^n + s_j^n)$$

Set

$$\alpha = \gamma\frac{\mathrm{dt}}{\mathrm{dx}}, \quad \beta = \frac{\mathrm{dt}}{2\tau}, \quad f_j^n = f(r_j^n + s_j^n)$$

Algebraic manipulations furnish the explicit iterative algorithm

$$
\begin{aligned}
r^{n+1} &= (\mathbb{S} - \alpha^2 \mathbb{D}_- \mathbb{D}_+)^{-1} \Big\{ [(1+\beta)\mathbb{I} + \alpha \mathbb{D}_-] r^n + \beta s^n + \\
&\quad + \tfrac{1}{2} [(1+2\beta)\mathbb{I} + \alpha \mathbb{D}_-] f^n \mathrm{dt} \Big\} \\
s^{n+1} &= (\mathbb{S} - \alpha^2 \mathbb{D}_+ \mathbb{D}_-)^{-1} \Big\{ \beta r^n + [(1+\beta)\mathbb{I} - \alpha \mathbb{D}_+] s^n + \\
&\quad + \tfrac{1}{2} [(1+2\beta)\mathbb{I} - \alpha \mathbb{D}_+] f^n \mathrm{dt} \Big\}
\end{aligned}
$$

where $\mathbb{D}_- = (\delta_{i,j} - \delta_{i,j+1})$, $\mathbb{D}_+ = (\delta_{i+1,j} - \delta_{i,j})$ (discrete derivatives), and

$$
\mathbb{S} := (1+2\beta)\mathbb{I} + \alpha(1+\beta)(\mathbb{D}_- - \mathbb{D}_+)
$$

$\mathbb{S}$ symmetric.

The Riemann problem

Conjecture: any bounded initial data such that

$$\limsup_{x\to-\infty} u_0(x) < \alpha < \liminf_{x\to+\infty} u_0(x),$$

gives raise to a solution, asymptotically convergent to a member of the traveling fronts connecting equilibria $u = 0$ with $u = 1$.

Parameter values: $\tau = 4$, $\ell = 25$, $\mathrm{dx} = 0.125$, $\mathrm{dt} = 0.01$.

Riemann initial data: $u_0(x) = \chi_{(0,\ell)}(x)$, in $(-\ell, \ell)$, non-flux b.c. at $x = \pm\ell$.

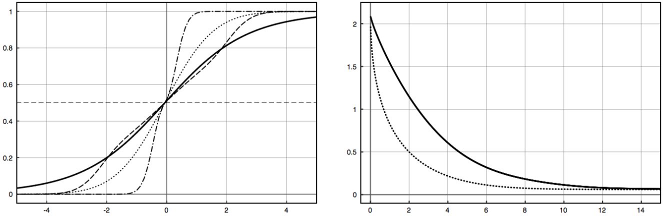


Figure : Riemann problem with initial datum $\chi_{(0,\ell)}$ in $(-\ell,\ell)$, $\ell = 25$. Left: solution profiles zoomed in the interval $(-5,5)$ at time $t = 1$ (dash-dot), $t = 5$ (dash), $t = 15$ (continuous), for comparison, solution to the parabolic Allen–Cahn equation at time $t = 1$ (dot). Right: Decay of the L^2 distance to the exact equilibrium solution for the hyperbolic (continuous) and parabolic (dot) Allen–Cahn equations.

Randomly perturbed initial data

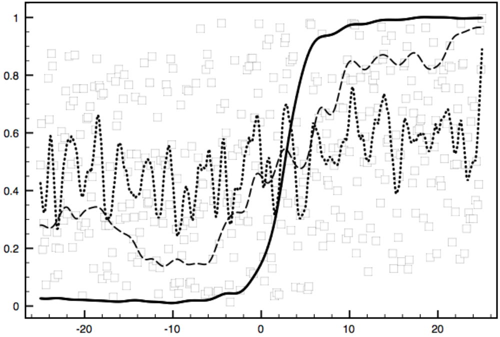


Figure : Random initial data in $(-\ell, \ell)$, $\ell = 25$ (squares). Solution profiles for the **hyperbolic** Allen–Cahn equation with relaxation at time $t = 0.5$ (dot), $t = 7.5$ (dash), $t = 15$ (continuous).

Comparison to parabolic Allen–Cahn

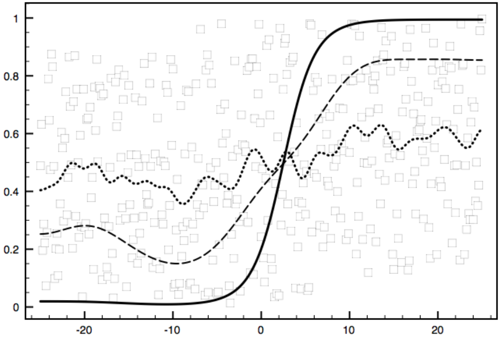


Figure : Random initial data in $(-\ell,\ell)$, $\ell = 25$ (squares). Solution profiles for the **parabolic** Allen–Cahn equation at time $t = 0.5$ (dot), $t = 7.5$ (dash), $t = 15$ (continuous).

Thank you!