On the Stability of Degenerate Viscous Shock Profiles

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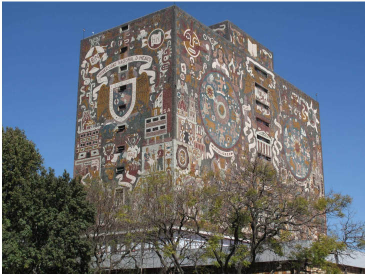

1. Introduction
2. The scalar case: L^p stability with sharp decay rates
3. The systems case: L^2 stability

Degenerate viscous shock waves

Scalar viscous conservation law:

$$u_t + f(u)_x = (b(u)u_x)_x,$$

where $(x,t) \in \mathbb{R} \times [0,+\infty), f, b \in C^2$, $b > 0$.

Viscous shock profile:

$$u(x,t) = \bar{u}(x - st),$$

$$(b(\bar{u})\bar{u}')' = f(\bar{u})' - s\bar{u}',$$

$$\bar{u}(\xi) \to u_\pm, \text{ as } \xi \to \pm\infty.$$

$$' = d/d\xi, \quad \xi = x - st.$$

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$$' = d/d\xi, \quad \xi = x - st.$$

Generalized shock front: $(u_+, u_-, s) \in \mathbb{R}^3$, $u_- \neq u_+$.

Rankine-Hugoniot condition:

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0,$$

Generalized entropy condition:

$$-s(u - u_-) + f(u) - f(u_-) < 0, \qquad \forall u \in (u_+, u_-).$$

Degenerate shocks

f changes convexity in (u_+, u_-).

Characteristic (sonic) shock speed:

$$s = f'(u_+), \quad \text{or,} \quad s = f'(u_-).$$

Toy model for sonic gas waves (e.g. Chapman-Jouguet combustion). Case boundary bet. undercompressive and Lax shock.

History (abridged)

- Mei (1995). One inflection point for f.
- Matsumura-Nishihara (1994). Asymmetric, non-bounded on the sonic side, weight function.
- Nishikawa (1998). Multi-d.
- Mei (1999). Exponentially decaying perturbations.

All results: L^2 energy estimates; not sharp decay rates in L^∞.

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Pointwise Green's function bounds methods

Under spectral stability assumption, resolvent kernel bounds; ponitwise bounds for the Green function.

- Non degenerate case: Zumbrun-Howard (1998): parabolic viscosity. Mascia-Zumbrun (2002-2008), relaxation and real viscosity. Zumbrun (2000-2008) Multi-d.
- Howard (2002a), (2002b). Degenerate viscous shocks: scalar case, first order degeneracy.
- Howard (2004). Degenerate shocks in 2×2 systems.

Howard (2002a,b): Sharper decay rates in L^p, slower decaying data. Spectral problem: $L(\bar{u}(x))v = \lambda v$, large t behavior gov. by small $|\lambda|$ behavior. Drawback: In deg. case $\lambda = 0$ is a branch point of the Evans function. Very technical estimates. Not exact estimates produce $\log t$ term in the decay rate. Degeneracy of order one only.

For zero-mass perturbations $\int \bar{u} - u_0 = 0$, $u_0 - \bar{u}$ decaying as $(1+|x|)^{-r}$, there holds (Howard (2002a))

$$\|u - \bar{u}\|_{L^p} \leq C(1+t)^{(1-r)/2} \log(2+t), \qquad 1 < r < 3, p > 1.$$

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Preliminaries

Normalization:

$x \to x - st, \;\; f \to f - su - f(u_\pm) + su_\pm.$

Assumptions:

$$f, b \;\; \text{smooth}, \quad b > 0, \;\; \text{(regularity and positive diffusion)}, \tag{A1}$$

$$f(u_-) = f(u_+) = 0, \;\; \text{(Rankine-Hugoniot condition)}, \tag{A2}$$

$$f(u) < 0 \;\forall\; u \in (u_+, u_-), \;\; \text{(generalized entropy condition)}. \tag{A3}$$

Existence and structure of profiles:

Under assumptions (A), let θ = deg. of degeneracy, $(d/du)^k f(u_+) = 0$, for all $0 \leq k \leq \theta$, $(d/du)^{\theta+1} f(u_+) \neq 0$. Then there exists a profile $\bar{u}$, $\bar{u}(\pm\infty) = u_\pm$, unique up to translations, monotone decreasing, $\bar{u}_x < 0$, decaying as

$$|\partial_x^j(\bar{u}(x) - u_-)| \leq Ce^{-c|x|}, \quad \text{as } x \to -\infty,$$
$$|\partial_x^j(\bar{u}(x) - u_+)| \leq C|x|^{-1/\theta}, \quad \text{as } x \to +\infty,$$

$j = 0, 1, 2$, uniform $C > 0$, all $x \in \mathbb{R}$ cf. Howard (2002a)

Perturbation equations

Zero-mass perturbations.

$$\int_{\mathbb{R}} (u_0(x) - \bar{u}(x-\delta))\, dx = 0.$$

W.l.o.g. $\delta = 0$. Integrated perturbation:

$$u(x,t) = v_x(x,t) + \bar{u}(x)$$

Perturbation equations

$$v_t = b(\bar{u})v_{xx} - a(x)v_x + F,$$

$$v(x,0) = v_0(x) = \int_{-\infty}^{x} (u_0(y) - \bar{u}(y))\, dy$$

$$a(x) = f'(\bar{u}) - b(\bar{u})_x,$$

$$\begin{aligned} F := & -(f(v_x + \bar{u}) - f(\bar{u}) - f'(\bar{u})v_x) + \\ & + (b(v_x + \bar{u}) - b(\bar{u}) - b'(\bar{u})v_x)(\bar{u}_x + v_{xx}) + \\ & + b'(\bar{u})v_x v_{xx} = b'(\bar{u})v_x v_{xx} + O(v_x^2), \end{aligned}$$

The Matsumura-Nishihara weight function

$$\eta(x) := \bar{\eta}(\bar{u}(x)),$$

$$\bar{\eta}(u) := \frac{(u-u_+)(u-u_-)}{f(u)} > 0, \qquad u \in (u_+, u_-).$$

$$\eta \sim |\bar{u} - u_+|^{-\theta}, \qquad \text{as } x \to +\infty,$$
$$\eta \sim C > 0, \qquad \text{as } x \to -\infty,$$

$u \in L^p_\eta$ iff $\eta^{1/p} u \in L^p$.

Properties:

$$\eta \sim \langle x \rangle_+ := \begin{cases} (1+x^2)^{1/2}, & x \geq 0, \\ 1, & x < 0. \end{cases}$$

$$\eta \geq \bar{C}^{-1} > 0, \qquad |\eta_x| \leq C\eta,$$

Right sign for energy estimates:

$$\begin{aligned} \Phi(x) &= ((b(\bar{u})\eta)_x + a(x)\eta)_x \\ &= \bar{u}_x (d^2/du^2)(f(u)\bar{\eta}(u))_{|u=\bar{u}} = -2|\bar{u}_x| < 0 \end{aligned}$$

Main result

- P, J. Math Anal. Appl. (2011).

Theorem. Under assumptions, with $u_+ < u_-$. Suppose zero mass condition and that

$$v_0 := \int_{-\infty}^{x} (u_0(x) - \bar{u}(x))\, dx \in Z_{\eta,p},$$

$Z_{\eta,p} = L^1_\eta \cap L^2_\eta \cap L^p_\eta \cap W^{2,p}$, for some $2 \leq p < +\infty$.

There exists $\hat{\varepsilon} > 0$ such that if $\|v_0\|_{Z_{\eta,p}} < \hat{\varepsilon}$, then the Cauchy problem for perturbation equation with initial condition $u(0) = u_0$ has a unique global solution $u - \bar{u} \in C([0,+\infty]; W^{1,p})$ satisfying

$$\|u-\bar{u}\|_{L^p} \leq CE_0 t^{-1/2}(1+t)^{-\frac{1}{2}(1-1/p)}, \qquad 2 \leq p < +\infty,$$

$$\|u-\bar{u}\|_{L^\infty} \leq CE_0 t^{-1/2-1/p}(1+t)^{-\frac{1}{2}(1-1/p)},$$

for all $0 < t < +\infty$, $E_0 = \|v_0\|_{L^1_\eta} + \|v_0\|^2_{L^2_\eta} + \|v_0\|^p_{L^p_\eta}$, with uniform $C > 0$.

Features:

- Sharp decay rates, no $\log t$ factor.
- All orders of degeneracy
- Elementary proof.

Interpolation inequalities in weighted L^p spaces

Inequalities 1:
For $2 \leq p < +\infty$, all $u \in W^{2,p}$:

$$\|u\|_\infty^p \leq C \|u_x\|_{L^p} \|u\|_{L^p}^{p-1},$$

$$\|u_x\|_{L^p} \leq C \|(|u_x|^{p/2})_x\|_{L^2}^{2/(p+2)} \|u\|_{L^p}^{2/(p+2)},$$

some constants $C = C(p) > 0$.

Inequality 2:

For $2 \leq p < +\infty$, all $u \in W^{2,p} \cap L^p_\eta \cap L^1_\eta$, $u_x \in W^{2,p} \cap L^p_\eta$:

$$\|u\|_{L^p_\eta}^{p(p+1)/(p-1)} \leq C \|u\|_{L^1_\eta}^{2p/(p-1)} \|(|u|^{p/2})_x\|_{L^2_\eta}^2,$$

some $C = C(p) > 0$. $\eta =$ Matsumura-Nishihara function.

Reference (without η): Escobedo, Zuazua (1991).

Example: in L^2

Use $\|u\|_\infty^2 \leq 2\|u\|_{L^2}\|u_x\|_{L^2}$:

$$\|u\|_{L^2}^8 = \Big(\int_{\mathbb{R}} u^2\,dx\Big)^4 \leq \|u\|_\infty^4 \|u\|_{L^1}^4 \leq 4\|u\|_{L^2}^2\|u_x\|_{L^2}^2\|u\|_{L^1}^4$$

$$\Rightarrow \quad \|u\|_{L^2}^6 \leq C\|u_x\|_{L^2}^2\|u\|_{L^1}^4.$$

Ineq. with $p = 2$.

Decay rates

Fact: If $\rho \geq 0$, C^1 in $t > 0$,

$$\frac{d\rho}{dt} \leq -C\rho^\beta,$$

$C > 0$, $\beta > 1$, with $\rho(0) = \rho_0 > 0$. Then

$$\rho \leq \xi, \quad \text{a.e. in } t,$$

$$\frac{d\xi}{dt} = -C\xi^\beta, \qquad \xi(0) = \rho_0.$$

Example: Decay rates in L^2

Hypotheses: For $0 < t < T \leq +\infty$, some $C_0, C_1 > 0$,

$$\frac{d}{dt}\|u(t)\|_{L^2}^2 \leq -C_1\|u_x\|_{L^2}^2,$$

$$\|u(t)\|_{L^1} \leq C_0.$$

Then:

$$\|u(t)\|_{L^2} \leq C(1+t)^{-1/4},$$

$$C = O(C_0 + \|u(0)\|_{L^2})$$

Proof:

$$\frac{d}{dt}\|u\|_{L^2}^2 \leq -C_1\|u_x\|_{L^2}^2 \leq -\frac{C_1}{C}\|u\|_{L^2}^6\|u\|_{L^1}^{-4} \leq -C_2\|u\|_{L^2}^6$$

$$C_2 = \frac{C_1}{CC_0^4}$$

$$\xi = \left(2C_2t + \|u(0)\|_{L^2}^{-4}\right)^{-1/2},$$

solution to $d\xi/dt = -C_2\xi^3$, $\xi(0) = \|u(0)\|_{L^2}^2$, $\beta = 3$.

Thus,

$$\xi(t)^{-2} \geq C_3(1+t), \quad C_3 = \min\{2C_2, \|u(0)\|_{L^2}^{-4}\}.$$

$$\Rightarrow \|u(t)\|_{L^2}^2 \leq C_3^{-1/2}(1+t)^{-1/2} \leq \tilde{C}(C_0 + \|u(0)\|_{L^2})^2(1+t)^{-1/2}.$$

L^p decay rates

Proposition. *Let* $2 \leq p < +\infty$*, and* u *a solution such that*

$$\frac{d}{dt}\|u(t)\|_{L^p_\eta}^p \leq -C_1\|(|u|^{p/2})_x(t)\|_{L^2_\eta}^2,$$
$$\|u(t)\|_{L^1_\eta} \leq C_0,$$

for all $0 < t < T \leq +\infty$*, and uniform* $C_1, C_0 > 0$*. Then, there exists* $\bar{C} > 0$ *such that*

$$\|u(t)\|_{L^p_\eta} \leq \bar{C}(1+t)^{-\frac{1}{2}(1-1/p)},$$

for all $0 < t < T$*. Moreover,*

$$\bar{C} = O(C_0 + \|u(0)\|_{L^p_\eta}).$$

Local existence

$$Z_{\eta,p} := W^{2,p} \cap L^p_\eta \cap L^1_\eta \cap L^2_\eta,$$

$$X_{\eta,p}(0,T) := \{v \in C([0,T];Z_{\eta,p}),\ v_x \in L^2([0,T];Z_{\eta,p})\}.$$

$$0 < T < +\infty.$$

Proposition. *For any* $\varepsilon_0 > 0$ *there exists a positive constant* T_0 *depending on* ε_0 *such that if* $v_0 \in Z_{\eta,p}$ *and* $\|v_0\|_{Z_{\eta,p}} \leq \varepsilon_0$*, then the Cauchy problem for* v *has a unique solution* $v \in X_{\eta,p}(0,T_0)$ *satisfying* $\|v(t)\|_{Z_{\eta,p}} < 2\varepsilon_0$ *for each* $0 \leq t \leq T_0$*.*

Basic energy estimate

For $v(t) \in X_{\eta,p}(0,T)$, $T > 0$,

$$\frac{1}{p}\|v(t)\|^p_{L^p_\eta} + \frac{4(p-1)}{p^2}\int_0^t \|b(\bar{u})^{1/2}(|v(\tau)|^{p/2})_x\|^2_{L^2_\eta}\,d\tau + \frac{2}{p}\int_0^t\int_{\mathbb{R}} |\bar{u}_x||v(\tau)|^p\,dx\,d\tau = \frac{1}{p}\|v(0)\|^p_{L^p_\eta} + \int_0^t\int_{\mathbb{R}} \eta F v(\tau)|v(\tau)|^{p-2}\,dx\,d\tau$$

$$R(t) := \sup_{\tau\in[0,t]} \|v(\tau)\|_{Z_{\eta,p}},$$

Consequences:

1. If $R(t) < \varepsilon_1$ for $t \in [0,T]$, for $0 < \varepsilon_1 \ll 1$,

$$\|v(t)\|^p_{L^p_\eta} + C\int_0^t \|(|v(\tau)|^{p/2})_x\|^2_{L^2_\eta}\, d\tau \leq \|v(0)\|^p_{L^p_\eta}.$$

2. When $p = 2$, $R(t) < \varepsilon_1$,

$$\|v(t)\|^2_{L^2_\eta} + C\int_0^t \|v_x(\tau)\|^2_{L^2_\eta}\, d\tau \leq \|v(0)\|^2_{L^2_\eta}$$

3. If $R(t) < \varepsilon_1$,

$$\frac{d}{dt}\|v(t)\|^p_{L^p_\eta} + C_1\|(|v|^{p/2})_x(t)\|^2_{L^2_\eta} \leq 0.$$

Consequences:

1. If $R(t) < \varepsilon_1$ for $t \in [0, T]$, for $0 < \varepsilon_1 \ll 1$,

$$\|v(t)\|_{L^p_\eta}^p + C \int_0^t \|(|v(\tau)|^{p/2})_x\|_{L^2_\eta}^2 \, d\tau \leq \|v(0)\|_{L^p_\eta}^p.$$

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3. If $R(t) < \varepsilon_1$,

$$\frac{d}{dt} \|v(t)\|_{L^p_\eta}^p + C_1 \|(|v|^{p/2})_x(t)\|_{L^2_\eta}^2 \leq 0.$$

Consequences:

1. If $R(t) < \varepsilon_1$ for $t \in [0,T]$, for $0 < \varepsilon_1 \ll 1$,

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3. If $R(t) < \varepsilon_1$,

$$\frac{d}{dt}\|v(t)\|^p_{L^p_\eta} + C_1 \|(|v|^{p/2})_x(t)\|^2_{L^2_\eta} \leq 0.$$

L^1_η estimate

If $R(t) < \varepsilon_1$, then

$$\|v(t)\|_{L^1_\eta} \leq C(\|v(0)\|_{L^1_\eta} + \|v(0)\|^2_{L^2_\eta}),$$

Consequence: Decay rate, with
$C_0 = \|v(0)\|_{L^1_\eta} + \|v(0)\|^2_{L^2_\eta}$:
For $v \in X_{\eta,p}(0,T)$, if $R(t) < \varepsilon_1$ then

$$\|v(t)\|^p_{L^p_\eta} \leq CE^p_0(1+t)^{-(p-1)/2},$$

$$E_0 := \|v(0)\|_{L^1_\eta} + \|v(0)\|^2_{L^2_\eta} + \|v(0)\|^p_{L^p_\eta}.$$

Higher order estimates

Estimate for v_x:
Suppose $v \in X_{\eta,p}(0,T)$, $2 \leq p < +\infty$, solution. For $\varepsilon_2 > 0$ suff. small if $R(t) < \varepsilon_2$ then

$$\frac{1}{p} t^{\alpha}(1+t)^{\beta} \|v_x(t)\|_{L^p}^p + \hat{C}_2 \int_0^t \tau^{\alpha}(1+\tau)^{\beta} \|(|v_x|^{p/2})_x(\tau)\|_{L^2_\eta}^2 \, d\tau \leq CE_0^p t^{\alpha - p/2}(1+t)^{\beta - \frac{1}{2}(p-1)},$$

for $0 \leq t \leq T$, with $\alpha > p/2$, $\beta > \frac{1}{2}(p-1)$. Moreover,

$$\|v_x(t)\|_{L^p}^p \leq CE_0^p t^{-p/2}(1+t)^{-\frac{1}{2}(p-1)}.$$

Estimate for v_{xx}:
Suppose $v \in X_{\eta,p}(0,T)$, $2 \leq p < +\infty$, solution. For $0 < \varepsilon_3 \leq \varepsilon_2$ suff. small if $R(t) < \varepsilon_3$ then

$$\frac{1}{p} t^{\gamma}(1+t)^{\delta} \|v_{xx}(t)\|_{L^p}^p + \hat{C}_3 \int_0^t \tau^{\gamma}(1+\tau)^{\delta} \|(|v_{xx}|^{p/2})_x(\tau)\|_{L^2_\eta}^2 \, d\tau \leq C E_0^p t^{\gamma-p}(1+t)^{\delta-\frac{1}{2}(p-1)},$$

$0 \leq t \leq T$, with $\gamma > p$, $\delta > \frac{1}{2}(p-1)$. Moreover,

$$\|v_{xx}(t)\|_{L^p}^p \leq C E_0^p t^{-p}(1+t)^{-\frac{1}{2}(p-1)}.$$

Global existence

Theorem. *Suppose* $v_0 \in Z_{\eta,p}$, $2 \le p < +\infty$. *Then there exists* $\hat{\varepsilon} > 0$ *s. t. if* $\|v_0\|_{Z_{\eta,p}} < \hat{\varepsilon}$, *then the Cauchy problem for* v *has a unique global solution* $v \in X_{\eta,p}(0,+\infty)$ *with*

$$\begin{aligned}
\|v(t)\|_{L^1} &\le CE_0, \\
\|v(t)\|_{L^p} &\le CE_0(1+t)^{-\frac{1}{2}(1-1/p)}, \\
\|v(t)\|_{L^2} &\le CE_0(1+t)^{-\frac{1}{4}}, \\
\|v_x(t)\|_{L^p} &\le CE_0 t^{-\frac{1}{2}}(1+t)^{-\frac{1}{2}(1-1/p)}, \\
\|v_{xx}(t)\|_{L^p} &\le CE_0 t^{-1}(1+t)^{-\frac{1}{2}(1-1/p)},
\end{aligned}$$

for all $0 < t < +\infty$, uniform $C > 0$, with

$$E_0 = \|v_0\|_{L^1_\eta} + \|v_0\|^2_{L^2_\eta} + \|v_0\|^p_{L^p_\eta}$$

Comparison:

- Matsumura-Nishihara (1994):
  - Non-sonic case: $v_0 \in H^2 \cap L^2_\eta$

$$\|u - \bar{u}\|_\infty \lesssim (1+t)^{-1/2}$$

  - Degenerate case: $v_0 \in H^2 \cap L^2_{\eta,\alpha}$ $(\eta^{1/2}(1+x^2)^{\alpha/4} u \in L^2)$, $0 < \alpha < 2/\theta$,

$$\|u - \bar{u}\|_\infty \lesssim (1+t)^{-\alpha/2+\varepsilon}$$

- Howard (2002a): For $\theta = 1$, set $r = 2$, init. cond. decay as $(1+|x|)^{-2}$

$$\|u - \bar{u}\|_\infty \lesssim (1+t)^{-1/2} \log(2+t),$$

- $v_0 \in H^2 \cap L^2_\eta \cap L^1_\eta$,

$$\|u - \bar{u}\|_\infty \lesssim (1+t)^{-1/2}$$

Same decay rate as the Burgers shock, decaying data, via Duhamel iterations (Hopf-Cole solution), zero-mass perturb.

Systems case

$$u_t + f(u)_x = (B(u)u_x)_x,$$

$u \in \Omega \subset \mathbb{R}^n, f \in C^2(\Omega;\mathbb{R}^n)$, $B \in C^2(\Omega;\mathbb{R}^{n\times n}$.

Non-convex principal characteristic field: $a_p(u) \in \sigma A(u)$, $1 \leq p \leq n$, $A = Df$.

Assumptions:

$$f, B \in C^2 \quad \text{(regularity)}, \tag{A1}$$

$$\begin{array}{r} a_p(u_+) = 0 < a_{p+1}(u_+), \\ a_{p-1}(u_-) < 0 < a_p(u_-), \end{array} \quad \text{(Lax + degeneracy)}, \tag{A2}$$

Exists A_0 symm., pos. def. s.t. $A_0 A$ symm., $A_0 B$ symm. pos. semi-def. (symmetric dissipativity $\Rightarrow$ non-strict hyperbolicity). (A3)

a_p is simple (simple principal field) (A4)

No eigenvector of A lies in $\ker B$ (genuine coupling), (A5)

Existing results for systems

- Mei and Nishihara (1997): viscoelastic system.
- Chern and Mei (1998): p-system with partial viscosity.
- Fries (1998, 1999): General systems. Energy estimate in L^2 for non-convex modes, $B = I$. Zero and non-zero mass perturbations.
- Howard and Zumbrun (2004), Howard (2006,2007): Pointwise bounds for 2×2 systems.

Strategy

- Goodman diagonalization method.
  Matsumura-Nishihara weight function on the principal field (Fries).
- Kawashima-type estimates to handle real viscosity.
- Apply decay estimate on each component of diagonalized system

Thank you!