L^p-decay rates for perturbations of degenerate scalar viscous shock waves

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Abstract

This contribution shows how to obtain L^p-decay rates, with $2 \leq p \leq +\infty$, for zero-mass perturbations of degenerate scalar viscous shock waves using energy methods. The proof is based upon previous work by Matsumura and Nishihara (Comm. Math. Phys. 165, 1994, no. 1, pp. 83-96), by extending their weighted energy estimates to L^p-norms, and by obtaining sharp decay rates for the antiderivative of the perturbation with the aid of basic interpolation inequalities. The analysis applies to shocks of all orders of degeneracy and it is elementary.

Key words:
Viscous degenerate shocks, L^p-decay rates, energy estimates.
2000 MSC: 35K55, 35Q53, 76E15

1. Introduction

Consider a scalar conservation law with second order viscosity in one dimension

$$u_t + f(u)_x = (b(u)u_x)_x, \tag{1}$$

where $f, b \in C^2$, $b > 0$, and $(x,t) \in \mathbb{R} \times [0, +\infty)$. Suppose that the triple $(u_+, u_-, s) \in \mathbb{R}^3$, with $u_- \neq u_+$ (without loss of generality, we take $u_+ < u_-$), constitutes a generalized shock front [16], namely, a weak solution of form

$$u(x,t) = \begin{cases} u_-, & x - st < 0, \\ u_+, & x - st > 0, \end{cases}$$

to the underlying inviscid conservation law

$$u_t + f(u)_x = 0,$$

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which satisfies both the Rankine Hugoniot jump condition,

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0,$$

and the generalized entropy condition [17],

$$-s(u - u_-) + f(u) - f(u_-) < 0, \qquad \text{for all } \; u \in (u_+, u_-).$$

This paper studies traveling wave solutions (shock profiles) to equation (1) of form $u(x,t) = \bar{u}(x - st)$, where $\bar{u}$ satisfies

$$(b(\bar{u})\bar{u}')' = f(\bar{u})' - s\bar{u}',$$

$$\bar{u}(\xi) \to u_\pm, \;\text{ as }\; \xi \to \pm\infty.$$

Here, $' = d/d\xi$ denotes differentiation with respect to the moving (galilean) variable $\xi := x - st$, and s is the shock speed. In this paper we assume that the flux function $u \mapsto f(u)$ changes its convexity in $u \in (u_+, u_-)$, a hypothesis which allows us to consider *sonic (or degenerate) shocks*, namely, waves whose speed matches one of the characteristic speeds with $s = f'(u_+)$ or $s = f'(u_-)$.

Sonic shocks appear in more complicated systems, for example, for sonic Chapman-Jouguet waves in combustion theory [3], for which the scalar case considered here can be loosely regarded as a toy model; see [12, 28, 18] for an (unabridged) survey on the topic. From the mathematical point of view, sonic shocks can also be regarded as a boundary case between classical shocks and undercompressive waves in the sense of Freistühler [4].

Motivated by such considerations, during the mid-90's there appeared some works on the stability of scalar degenerate shock profiles [22, 23, 25, 21]. The first result, due to Mei [22], used the energy method of Goodman [6] and Matsumura-Nishihara [20], under the special assumption of only one inflection point for the flux function f. The seminal work of Matsumura and Nishihara [21] introduced a suitable, asymmetric weight function, not bounded on the sonic side, which accomodates properly on the compressive side, yielding the right sign in the energy estimates (see, for example, Remark 1.4 below). Their work provides stability under decaying data, with decay rates in L^∞ spaces, which are not sharp. Later on, their weight function was used in other contexts, such as multi-dimensional equations [25], exponentially decaying initial perturbations [23], and 2×2 systems [1]. All these works have in common the use of energy methods, yielding not so sharp decay rates in L^∞ spaces, and are restricted to zero-mass perturbations.

During the early years of the last decade, Howard introduced a different approach [11, 12], generalizing the methods introduced by Howard and Zumbrun [9, 10, 31] for classical shocks. Howard's analysis follows pointwise bounds on the Green's function for the linearized operator around the wave. Such bounds arise from a suitable construction of the resolvent kernel of the linear operator (from which the Green's function can be retrieved from Laplace inversion formulae), sharp ODE estimates, and under the assumption of spectral stability (or

Evans function condition), which is also proved to hold. In [12], the analysis applies to zero-mass perturbations and the integrated operator, whereas [11] treats the general case. These methods offered more information on the asymptotic behavior of such waves than energy methods, yielding sharper decay rates in all L^p spaces, with $p \geq 1$ (by a factor $t^{-1/2} \log t$ in the case of zero-mass perturbations; see [12] for details), accomodating slower decaying data. The application of the pointwise Green's function method is, however, much more complicated than in the classical shock case, due to the fact that degenerate profiles decay algebraically, in contrast to the exponential decay of Laxian shocks. This makes the Evans function non-analytic near $\lambda = 0$. Thus, the standard estimates must be replaced by difficult and very technical ODE estimates. Although technically impressive and flexible, such estimations are very hard. We also note that these methods apply to shocks of order of degeneracy equal to one, and have been extrapolated to the 2×2 systems case [14, 13].

The purpose of this contribution is simply to observe that sharper decay rates for zero-mass perturbations of viscous degenerate shocks can be obtained using energy methods and interpolation inequalities. This analysis retrieves the Matsumura-Nishihara weight function, and extends their energy estimates to L^p spaces, with $2 \leq p \leq +\infty$. The main observation is perhaps the combination of basic estimates with interpolation inequalities of Nash-type (see [2, 24]), to get sharp decay rates for the integral of the perturbation (which is assumed to have zero-mass). This general principle, extrapolable to multi-dimensions, is explained in Section 3 below. This sharp decay rate for the integrated variable representing the perturbation is then used to establish decay rates of the derivatives. This work follows the general L^p-method of Kawashima *et al.* [15], introduced in the context of non-compressive, rarefaction-type waves (see also the recent related L^p-stability analysis for degenerate waves on half spaces [30]). In contrast with these last two works, however, our analysis uses interpolation inequalities directly to obtain decay rates for the integrated variable. The method is elementary and applies to shocks of all orders of degeneracy.

Before making precise the statement of the main result, we have to make some preliminary observations.

1.1. Normalizations

By translation invariance and without loss of generality, we normalize the flux function f such that the profile is stationary (i.e. $s = 0$), obeying the equation

$$b(\bar{u})\bar{u}_x = f(\bar{u}). \tag{2}$$

This is accomplished by taking the change of coordinates $x \to x - st$, and by normalizing $f \to f(u) - su + c$, where $c = su_\pm - f(u_\pm)$ is a constant. Consequently, $f(u_\pm) = 0$ and $f(\bar{u}(\cdot)) \in L^2(\mathbb{R})$. The latter is a standard normalization in stability analyses (see, e.g., [7, 19]). Therefore, and without loss of generality, we summarize our assumptions as follows:

$$f, b \in C^2,\ b > 0, \quad \text{(regularity and positive diffusion)}, \tag{A1}$$

$$f(u_-) = f(u_+) = 0, \quad \text{(Rankine-Hugoniot condition)}, \tag{A2}$$

$$f(u) < 0 \text{ for all } u \in (u_+, u_-), \quad \text{(generalized entropy condition)}. \tag{A3}$$

Remark 1.1. It is well-known [17] that entropy condition (A3) reduces to Lax entropy condition $f'(u_+) < 0 < f'(u_-)$ when the mode is strictly convex, i.e., when $f'' > 0$. Otherwise, the generalized entropy condition (A3) implies the non-strict condition $f'(u_+) \leq 0 \leq f'(u_-)$, which allows sonic waves with $f'(u_+) = 0$ or $f'(u_-) = 0$. For concreteness, we shall consider for the rest of the paper that the shock is sonic on the positive side, namely, that $f'(u_+) = 0$. Whence, we rewrite assumption (A3) as

$$f(u) < 0, \quad \forall u \in (u_+, u_-), \quad \text{and}, \quad f'(u_+) = 0 < f'(u_-). \tag{A3'}$$

1.2. Existence and structure of profiles

It is well-known [29, 21, 5] that, under assumptions (A1) to (A3'), traveling wave solutions to (1) exist.

Proposition 1.2. *Under* (A1) - (A3')*, let us define*

$$\theta := \min\{k \in \mathbb{Z}^+ \,:\, \frac{d^k f}{du^k}(u_+) \neq 0\} \geq 1,$$

as the degree of degeneracy of the shock. Then there exists a traveling wave solution $\bar{u}$ of (2) *with $\bar{u}(\pm\infty) = u_\pm$, unique up to translations. Moreover, $\bar{u}$ is monotone decreasing, $\bar{u}_x < 0$, and $\bar{u}$ and its derivatives decay as*

$$\begin{aligned} |\partial_x^j(\bar{u}(x) - u_-)| &\leq C e^{-c|x|}, \\ |\partial_x^j(\bar{u}(x) - u_+)| &\leq C |x|^{-1/\theta}, \end{aligned} \tag{3}$$

for $j = 0, 1, 2$, some uniform $C > 0$, and all $x \in \mathbb{R}$.

Proof. See, e.g., Howard [12], Section 2. □

Remark 1.3. Observe that, as a by-product of the existence result, the profile decays algebraically on the sonic side.

1.3. Perturbation equations

Let $\bar{u}(x)$ be the stationary profile satisfying (2) and (3), under assumptions (A1) - (A3'). Since any translate of the traveling wave is also a solution, the most we can expect is *orbital stability*, or the property that a solution initially near $\bar{u}(x)$ will approach to a translate $\bar{u}(x - \delta)$ as $t \to +\infty$, with $\delta \in \mathbb{R}$ uniquely determined by the mass carried by the initial perturbation

$$m_0 = \int_{\mathbb{R}} (u_0(x) - \bar{u}(x))\, dx.$$

We shall restrict our analysis to the class of perturbations with zero-mass, and choose δ such that

$$\int_{\mathbb{R}} (u_0(x) - \bar{u}(x-\delta))\, dx = 0.$$

This choice allows us to write the perturbation as

$$u(x,t) - \bar{u}(x-\delta) = v_x(x,t), \tag{4}$$

for some function $v(\cdot,t)$ in L^2, i.e., we are able to integrate the equation [6, 8]. We suppose, without loss of generality, that $\delta = 0$, yielding the condition

$$\int_{\mathbb{R}} (u_0(x) - \bar{u}(x))\, dx = 0. \tag{5}$$

In view of last observations, substitute now $u(x,t) = v_x(x,t) + \bar{u}(x)$ into (1) to obtain

$$v_{xt} + f(v_x + \bar{u})_x = \big(b(v_x+\bar{u})(v_{xx} + \bar{u}_x)\big)_x. \tag{6}$$

Integrating in $(-\infty, x)$ we get the integrated equation for the perturbation

$$v_t + f(v_x + \bar{u}) = b(v_x + \bar{u})(v_{xx} + \bar{u}_x),$$

which, in view of the profile equation (2), can be recast as

$$v_t = b(\bar{u})v_{xx} - a(x)v_x + F, \tag{7}$$

with

$$a(x) := f'(\bar{u}) - b(\bar{u})_x, \tag{8}$$

and where

$$\begin{aligned} F := &-(f(v_x+\bar{u}) - f(\bar{u}) - f'(\bar{u})v_x) + \\ &+ (b(v_x+\bar{u}) - b(\bar{u}) - b'(\bar{u})v_x)(\bar{u}_x + v_{xx}) + \\ &+ b'(\bar{u})v_x v_{xx}, \end{aligned} \tag{9}$$

comprises the nonlinear terms. Observe that

$$F = b'(\bar{u})v_x v_{xx} + \mathcal{O}(v_x^2). \tag{10}$$

Upon differentiation,

$$F_x = b'(\bar{u})v_x v_{xxx} + \mathcal{O}(v_x^2 + v_{xx}^2). \tag{11}$$

Note that the first is the only term of order $\mathcal{O}(|v_x||v_{xxx}|)$.

The *integrated linear operator* [6] is then defined as

$$Lv := b(\bar{u})v_{xx} - a(x)v_x. \tag{12}$$

This operator was introduced by Goodman [6, 8], who recognized that compressivity of the wave (in the convex case $f'' > 0$) yields "good" energy estimates for the integrated operator, but not for the original linearized operator around the wave.

Therefore, after these reformulations, the Cauchy problem for the perturbation v is written as follows,

$$v_t = Lv + F, \qquad \text{for } (x,t) \in \mathbb{R} \times (0, +\infty), \tag{13}$$

$$v(x,0) = v_0(x) = \int_{-\infty}^{x} (u_0(y) - \bar{u}(y))\, dy, \qquad \text{for } x \in \mathbb{R}. \tag{14}$$

1.4. The Matsumura-Nishihara weight function

In the case of non-convex modes, and in order to obtain energy estimates with a good sign yielding stability, Matsumura and Nishihara [21] introduced the following weight function,

$$\eta(x) := \bar{\eta}(\bar{u}(x)), \tag{15}$$

$$\bar{\eta}(u) := \frac{(u-u_+)(u-u_-)}{f(u)} > 0, \qquad u \in (u_+, u_-).$$

which clearly satisfies the conditions

$$\eta \sim |\bar{u} - u_+|^{-\theta}, \qquad \text{as } x \to +\infty, \tag{16}$$

$$\eta \sim C > 0, \qquad \text{as } x \to -\infty, \tag{17}$$

which amount to

$$\eta \sim \langle x \rangle_+ := \begin{cases} (1+x^2)^{1/2}, & x \geq 0, \\ 1, & x < 0. \end{cases}$$

(see [21] for details). In particular we have that η is bounded below,

$$\eta \geq \bar{C}^{-1} > 0, \qquad \text{for all } x \in \mathbb{R}, \tag{18}$$

for some uniform $\bar{C} > 0$. Note, however, that it is not bounded above as it blows up on the sonic side when $x \to +\infty$.

Remark 1.4. One of the remarkable properties of the Matsumura-Nishihara weight function (15) is that it leads to the right sign of the term

$$\Phi(x) := ((b(\bar{u})\eta)_x + a(x)\eta)_x < 0, \tag{19}$$

for all $x \in \mathbb{R}$, which appears in energy estimates [21]. Indeed, using (2) we get

$$\begin{aligned} \Phi(x) &= ((b(\bar{u})\eta)_x + a(x)\eta)_x \\ &= (f(\bar{u})\bar{\eta}'(\bar{u}) + f'(\bar{u})\bar{\eta}(\bar{u}))_x \\ &= ((d/du)(f(u)\bar{\eta}(u))_{|u=\bar{u}})_x \\ &= \bar{u}_x (d^2/du^2)(f(u)\bar{\eta}(u))_{|u=\bar{u}} \\ &= -2|\bar{u}_x| < 0, \end{aligned}$$

in view of monotonicity of the profile and of $(d^2/du^2)(f(u)\bar{\eta}(u)) = 2$. Observe that, when $\eta \equiv 1$ and $f'' > 0$ (that is, in the non-sonic, convex case), $\Phi = f''(\bar{u})\bar{u}_x < 0$ has, naturally, a negative sign. Property (19) and its use for stability estimates can be seen in the original L^2 estimate of [21], in the L^p estimates of Lemma 4.2 below, and, notably, in the L^1_η estimate of Lemma 4.7.

Also, observe from the definition of η and (2) that

$$\eta_x = \bar{\eta}'(\bar{u})\bar{u}_x = \frac{2\bar{u} - (u_+ + u_-) - f'(\bar{u})\eta}{b(\bar{u})},$$

yielding $|\eta_x| \leq C(1+\eta)$, for all $x \in \mathbb{R}$, as $\bar{u}$ and $f'(\bar{u})$ are uniformly bounded. Using (18) we readily find that

$$|\eta_x| \leq C\eta, \tag{20}$$

for some uniform $C > 0$ and all $x \in \mathbb{R}$.

Finally, let us specify some notation. $W^{m,p}$ will denote the standard Sobolev spaces in $\mathbb{R}$. In terms of the weight function, L^p_η will denote the space of measurable functions u such that $\eta^{1/p}u \in L^p$, or that

$$\|u\|^p_{L^p_\eta} := \int_{\mathbb{R}} \eta|u|^p \, dx \; < +\infty,$$

for each $1 \leq p < +\infty$.

1.5. Main result

After these preparations, we are ready to state the main theorem.

Theorem 1.5. *Under assumptions* (A1) - (A3'), *with $u_+ < u_-$, let $\bar{u}$ be the traveling wave solution to* (1) *of Proposition 1.2. Suppose that the integrability condition* (5) *holds, and that*

$$v_0 := \int_{-\infty}^{x} (u_0(x) - \bar{u}(x)) \, dx \; \in \; Z_{\eta,p}, \tag{21}$$

where $Z_{\eta,p} = L^1_\eta \cap L^2_\eta \cap L^p_\eta \cap W^{2,p}$, for some $2 \leq p < +\infty$, and where η denotes the Matsumura-Nishihara weight function (15). *Then there exists a positive constant $\hat{\epsilon} > 0$ such that if $\|v_0\|_{Z_{\eta,p}} < \hat{\epsilon}$, then the Cauchy problem for equation* (1) *with initial condition $u(0) = u_0$ has a unique global solution $u - \bar{u} \in C([0,+\infty]; W^{1,p})$ satisfying*

$$\|u - \bar{u}\|_{L^p} \leq CME_0 t^{-1/2}(1+t)^{-\frac{1}{2}(1-1/p)}, \qquad 2 \leq p < +\infty, \tag{22}$$

$$\|u - \bar{u}\|_{L^\infty} \leq CME_0 t^{-1/2-1/2p}(1+t)^{-\frac{1}{2}(1-1/p)}, \tag{23}$$

for all $0 < t < +\infty$, where $E_0 = \|v_0\|_{L^1_\eta} + \|v_0\|_{L^p_\eta} + \|v_0\|^2_{L^2_\eta}$, and with uniform constant $M > 0$.

Remark 1.6. Observe that the decay rates (22) seem almost sharp. In particular, we got rid of the $\log t$ term appearing in the decay rates for zero-mass perturbations in [12] (pages 24-26). Actually, Howard eliminated the $\log t$ term in his analysis of non-zero mass perturbations [11] as well. The decay rates of zero-mass perturbations, however, are inherently sharper. Such observation comes from the fact that the perturbations do not transport mass. Heuristically, in the case of the heat kernel, the perturbation

$$u(x,t) = \int_{\mathbb{R}} (4\pi t)^{-1/2} e^{-(x-y)^2/4t} u_0(y)\, dy$$

gains a t^{-1} factor under the assumption of $\int u_0 = 0$. Since it does not transport mass, the decay rate is determined by diffusion, both above and below the traveling wave (see [11] for details).

Remark 1.7. Like in [21], our results apply to zero-mass perturbations only and require very rapidly decaying data, as v_0 must belong to the weighted space $Z_{\eta,p}$. A natural direction of investigation would be to treat the general case. Nonetheless, the method presented here is simpler than that using the Evans function, and it might be possibly applied to the more complicated (and more interesting) problem of stability of degenerate viscous shock profiles for general systems, which remains open.

Plan of the paper

Section 2 collects the interpolation inequalities, as well as some useful observations which will be used throughout the analysis. We use a suitable interpolation inequality in weighted spaces for the (non-bounded) Matsumura-Nishihara function. Section 3 contains the basic principle for the establishment of the decay rates. It is based on two inequalities: one expresses some kind of diffusion, and the other expresses uniform boundedness in the L^1_η norm. The central Section 4 contains the proofs of the *a priori* energy estimates for local solutions to the integrated equations. Finally, Section 5 makes use of such estimates to provide a global existence theorem with rates of decay, leading to the proof of the main theorem.

2. Interpolation inequalities

In this section we collect some interpolation inequalities involving weighted spaces and the Matsumura-Nishihara weight function, which play a crucial role in the analysis. In the sequel we often use the following basic identities:

$$\partial_x |u| = \frac{u u_x}{|u|}, \tag{24}$$

$$\partial_x \Big(\frac{1}{p}|u|^p\Big) = |u|^{p-2} u u_x, \tag{25}$$

$$\big|(|u|^{p/2})_x\big|^2 = \frac{p^2}{4}|u|^{p-2}u_x^2, \tag{26}$$

$$(|u|^{p-2})_x u u_x = \frac{4(p-2)}{p^2}\big|(|u|^{p/2})_x\big|^2, \tag{27}$$

$$(|u|^{p-2}u)_x = (p-1)|u|^{p-2}u_x, \tag{28}$$

for all $u \in W^{2,p}$, with $2 \le p < +\infty$. We start with an elementary result.

Lemma 2.1. *Let $2 \le p < +\infty$. Then*

$$\|u\|_\infty^p \le C\|u_x\|_{L^p}\|u\|_{L^p}^{p-1}, \tag{29}$$

$$\|u_x\|_{L^p} \le C\|(|u_x|^{p/2})_x\|_{L^2}^{2/(p+2)}\|u\|_{L^p}^{2/(p+2)}, \tag{30}$$

for all $u \in W^{2,p}$, with some constants $C = C(p) > 0$.

Proof. Use (25) to obtain

$$\frac{1}{p}|u|^p = \int_{-\infty}^x |u|^{p-2} u u_x \, dx,$$

as $u \in W^{2,p}$ and $u(-\infty) = u_x(-\infty) = 0$. By Hölder's inequality for L^p norms with $1/p + (p-1)/p = 1$ we arrive at

$$|u|^p \le p\int_{\mathbb{R}} |u|^{p-1}|u_x| \, dx \le \|u_x\|_{L^p}\|u\|_{L^p}^{p-1},$$

yielding (29) with $C(p) = p$.

To show (30), integrate by parts and apply (28); the result is

$$\|u_x\|_{L^p}^p = -\int_{\mathbb{R}} (|u_x|^{p-2}u_x)_x u \, dx = -(p-1)\int_{\mathbb{R}} |u_x|^{p-2}u_{xx}u \, dx.$$

In view of (26) and Hölder's inequality with $1/p + 1/2 + (p-2)/(2p) = 1$ we readily obtain

$$\begin{aligned} \|u_x\|_{L^p}^p &\le (p-1)\int_{\mathbb{R}} |u_x|^{(p-2)/2}|u_x||u_x|^{(p-2)/2}|u| \, dx \\ &\le (2(p-1)/p)\int_{\mathbb{R}} \big|(|u_x|^{p/2})_x\big||u_x|^{(p-2)/2}|u| \, dx \\ &\le (2(p-1)/p)\|(|u_x|^{p/2})_x\|_{L^2}\|u\|_{L^p}\|u_x\|_{L^p}^{-1+p/2}, \end{aligned}$$

which yields (30) with $C(p) = (2(p-1)/p)^{2/(p+2)}$. □

We now present a version in weighted norms of the interpolation (or Nash-type [24]) inequality which can be found in [2] (Lemma 1, pg. 129). For the proof we follow [2] closely, with the appropriate adaptations to the weighted spaces under consideration. Notably, the original inequality remains valid in weighted spaces, even though the function η is not bounded above on the sonic side.

Lemma 2.2 (Weighted interpolation inequality)**.** *For each $2 \leq p < +\infty$ there exists some constant $C = C(p) > 0$ such that*

$$\|u\|_{L^p_\eta}^{p(p+1)/(p-1)} \leq C \|u\|_{L^1_\eta}^{2p/(p-1)} \|(|u|^{p/2})_x\|_{L^2_\eta}^2, \tag{31}$$

for every $u \in W^{2,p} \cap L^p_\eta \cap L^1_\eta$, with $u_x \in W^{2,p} \cap L^p_\eta$, where η denotes the Matsumura-Nishihara weight function.

Proof. Apply the Sobolev-type inequality $\|v\|_\infty^2 \leq 2\|v\|_{L^2}\|v_x\|_{L^2}$, to the function $v = |u|^{p/2}$ and use (18) to obtain

$$\|u\|_\infty^p = \||u|^{p/2}\|_\infty^2 \leq 2\||u|^{p/2}\|_{L^2}\|(|u|^{p/2})_x\|_{L^2} \leq 2\bar{C}\|\eta^{1/2}|u|^{p/2}\|_{L^2}\|\eta^{1/2}(|u|^{p/2})_x\|_{L^2},$$

yielding

$$\|u\|_\infty^p \leq 2\bar{C}\||u|^{p/2}\|_{L^2_\eta}\|(|u|^{p/2})_x\|_{L^2_\eta}. \tag{32}$$

Notice that

$$\||u|^{p/2}\|_{L^2_\eta}^2 = \|\eta^{1/2}|u|^{p/2}\|_{L^2}^2 = \int_{\mathbb{R}} \eta |u|^p \, dx = \|\eta^{1/p}u\|_{L^p}^p = \|u\|_{L^p_\eta}^p.$$

Furthermore, we estimate

$$\|u\|_{L^p_\eta}^{2p^2/(p-1)} = \Big(\int_{\mathbb{R}} \eta |u|^p \, dx \Big)^{2p/(p-1)} \leq \Big(\|u\|_\infty^{p-1} \int_{\mathbb{R}} \eta |u| \, dx \Big)^{2p/(p-1)} = \|u\|_\infty^{2p} \|u\|_{L^1_\eta}^{2p/(p-1)}. \tag{33}$$

Combine (33) with (32) to arrive at

$$\|u\|_{L^p_\eta}^{p(p+1)/(p-1)} = \frac{\|u\|_{L^p_\eta}^{2p^2/(p-1)}}{\|u\|_{L^p_\eta}^p} \leq \frac{\|u\|_\infty^{2p} \, \|u\|_{L^1_\eta}^{2p/(p-1)}}{\||u|^{p/2}\|_{L^2_\eta}^2} \leq 4\bar{C}^2 \|(|u|^{p/2})_x\|_{L^2_\eta}^2 \|u\|_{L^1_\eta}^{2p/(p-1)},$$

yielding the result with $C := 4\bar{C}^2$, as claimed. ☐

Remark 2.3. For each $u, u_x \in W^{1,p} \cap L^p_\eta$, by Hölder's inequality and the identity (26) we have that

$$\begin{aligned} \|(|u|^{p/2})_x\|_{L^2_\eta}^2 &= \int_{\mathbb{R}} \eta |(|u|^{p/2})_x|^2 \, dx = \frac{p^2}{4} \int_{\mathbb{R}} \eta |u|^{p-2} u_x^2 \, dx \\ &\leq \frac{p^2}{4} \left(\int_{\mathbb{R}} \eta |u_x|^p \, dx \right)^{2/p} \left(\int_{\mathbb{R}} \eta |u|^p \, dx \right)^{(p-2)/p} = \frac{p^2}{4} \|u_x\|_{L^p_\eta}^2 \|u\|_{L^p_\eta}^{p-2} < +\infty. \end{aligned}$$

3. L^p-decay rates

Before obtaining the *a priori* estimates for solutions to the integrated perturbation equation, we state the basic principle for the establishment of sharp decay rates in weighted L^p spaces. We begin with an elementary lemma.

Lemma 3.1. *Suppose ρ is a nonnegative, continuously differentiable function of $t \geq 0$ that satisfies the differential inequality*

$$\frac{d\rho}{dt} \leq -C\rho^\beta,$$

for all $t \geq 0$, with $C > 0$, $\beta > 1$, and initial condition $\rho(0) = \rho_0 > 0$. Then $\rho(t) \leq \zeta(t)$ a.e. in $t \geq 0$, where ζ is the solution to

$$\frac{d\zeta}{dt} = -C\zeta^\beta, \qquad \zeta(0) = \rho_0.$$

Proof. Define $\psi(t) := (\beta - 1)^{-1}(\rho^{1-\beta} - \zeta^{1-\beta})$. Clearly $\psi(0) = 0$ and $d\psi/dt \geq 0$ in $t \geq 0$. Therefore $\psi(t) \geq 0$ a.e., that is, $\rho \leq \zeta$ a.e. ☐

Proposition 3.2. *Let $2 \leq p < +\infty$, and suppose that u is the solution to a certain evolution (linear or nonlinear) equation, which satisfies the bounds*

$$\frac{d}{dt}\|u(t)\|^p_{L^p_\eta} \leq -C_1\|(|u|^{p/2})_x(t)\|^2_{L^2_\eta}, \tag{34}$$

$$\|u(t)\|_{L^1_\eta} \leq C_0, \tag{35}$$

for all $0 < t < T \leq +\infty$, and uniform constants $C_1, C_0 > 0$. Then, there exists a positive constant $\bar{C} > 0$ such that

$$\|u(t)\|_{L^p_\eta} \leq \bar{C}(1+t)^{-\frac{1}{2}(1-1/p)}, \tag{36}$$

for all $0 < t < T$. Moreover, the constant $\bar{C}$ is of order

$$\bar{C} = \mathcal{O}(C_0 + \|u(0)\|_{L^p_\eta}). \tag{37}$$

Proof. From (34) and the interpolation inequality (31) we obtain

$$\frac{d}{dt}\|u(t)\|^p_{L^p_\eta} \leq -\frac{C_1}{C}\|u(t)\|^{p(p+1)/(p-1)}_{L^p_\eta}\|u(t)\|^{-2p/(p-1)}_{L^1_\eta}.$$

Since $2p/(p-1) > 0$, estimate (35) implies that

$$\frac{d}{dt}\|u(t)\|^p_{L^p_\eta} \leq -\frac{C_1}{C}C_0^{-2p/(p-1)}\|u(t)\|^{p(p+1)/(p-1)}_{L^p_\eta}.$$

By Lemma 3.1, $\|u(t)\|^p_{L^p_\eta}$ will then be bounded by the solution $\zeta(t)$ of the following initial value problem

$$\begin{aligned} \frac{d\zeta}{dt} &= -C_2\zeta^\beta, \\ \zeta(0) &= \|u(0)\|^p_{L^p_\eta}, \end{aligned} \tag{38}$$

with $\beta := (p+1)/(p-1) > 1$, $C_2 := (C_1/C)C_0^{-2p/(p-1)}$. The solution to (38) is

$$\zeta(t) = \left(\frac{2C_2}{p-1}\, t + \|u(0)\|_{L^p_\eta}^{-2p/(p-1)} \right)^{-(p-1)/2}. \tag{39}$$

Indeed, on one hand there holds $\zeta(0) = \|u(0)\|^p_{L^p_\eta}$. On the other hand a direct computation yields

$$\frac{d\zeta}{dt} = -C_2 \left(\zeta^{-2/(p-1)} \right)^{-(p-1)/2-1} = -C_2 \zeta^\beta$$

Therefore we have that $\|u(t)\|^p_{L^p_\eta} \leq \zeta(t)$ for all $0 < t < T$ a.e. Now, for each $t > 0$

$$\zeta(t)^{-2/(p-1)} = \frac{2C_2}{p-1}\, t + \|u(0)\|_{L^p_\eta}^{-2p/(p-1)} \geq C_3(t+1) > 0,$$

where $C_3 := \min\{2C_2/(p-1), \|u(0)\|_{L^p_\eta}^{-2p/(p-1)}\} > 0$. Henceforth,

$$\zeta(t) \leq C_3^{-\frac{1}{2}(p-1)} (1+t)^{-\frac{1}{2}(p-1)},$$

which, in turn, yields

$$\|u(t)\|^p_{L^p_\eta} \leq \tilde{C}(C_0 + \|u(0)\|_{L^p_\eta})^p (1+t)^{-\frac{1}{2}(p-1)},$$

with uniform $\tilde{C} = \tilde{C}(p) > 0$, because from definitions of C_2 and C_3 we have $C_3^{-\frac{1}{2}(p-1)} \leq (C_p C_0^p + \|u(0)\|^p_{L^p_\eta}) \leq \tilde{C}(C_0 + \|u(0)\|_{L^p_\eta})^p$, with C_p and $\tilde{C}$ depending only on p. This shows both (36) and (37). $\square$

Remark 3.3. Observe that the decay rate in (36) seems optimal for strictly parabolic equations of second order, because it is the decay rate in L^p for the heat kernel. The inequality (34) expresses diffusion in some way, whereas (35) simply expresses boundedness of the L^1_η norm.

4. A priori energy estimates

In this section we perform the energy estimates for solutions to (13) and (14). Assume $2 \leq p < +\infty$ is fixed. Let us define the suitable space for solutions as

$$Z_{\eta,p} := W^{2,p} \cap L^p_\eta \cap L^1_\eta \cap L^2_\eta,$$

$$X_{\eta,p}(0,T) := \{v \in C([0,T]; Z_{\eta,p}),\ v_x \in L^2([0,T]; Z_{\eta,p})\},$$

with $0 < T \leq +\infty$.

According to custom, the global existence of solutions to (13) and (14) is proved by a continuation argument based on a local existence result combined with the corresponding *a priori* energy estimates. Using the variation of constants formula and by a standard contraction mapping argument it is possible to prove the following short-time existence result. We omit the details.

Proposition 4.1 (Local existence)**.** *For any $\epsilon_0 > 0$ there exists a positive constant T_0 depending on ϵ_0 such that if $v_0 \in Z_{\eta,p}$ and $\|v_0\|_{Z_{\eta,p}} \leq \epsilon_0$, then the Cauchy problem* (13) *and* (14) *has a unique solution $v \in X_{\eta,p}(0, T_0)$ satisfying $\|v(t)\|_{Z_{\eta,p}} < 2\epsilon_0$ for each $0 \leq t \leq T_0$.*

This section is thus devoted to the establishment of *a priori* estimates for solutions to (13) and (14).

4.1. The basic energy estimate

The next lemma is the main result of this section.

Lemma 4.2 (Basic energy estimate)**.** *Let $2 \leq p < +\infty$, and let $v(t) \in X_{\eta,p}(0, T)$ be a solution to* (13)*for some $T > 0$. Then there holds the estimate*

$$\frac{1}{p}\|v(t)\|_{L^p_\eta}^p + \frac{4(p-1)}{p^2}\int_0^t \|b(\bar{u})^{1/2}(|v(\tau)|^{p/2})_x\|_{L^2_\eta}^2\,d\tau + \frac{2}{p}\int_0^t\int_{\mathbb{R}} |\bar{u}_x||v(\tau)|^p\,dx\,d\tau = \\ = \frac{1}{p}\|v(0)\|_{L^p_\eta}^p + \int_0^t\int_{\mathbb{R}} \eta F v(\tau)|v(\tau)|^{p-2}\,dx\,d\tau \tag{40}$$

for all $0 \leq t \leq T$.

Proof. Multiply equation (7) by $\eta v|v|^{p-2}$; the result is

$$\eta v|v|^{p-2}v_t = \eta v|v|^{p-2}b(\bar{u})v_{xx} - \eta v|v|^{p-2}a(x)v_x + \eta v|v|^{p-2}F.$$

Use (25), (26), (27), and rearrange the terms to find that

$$\partial_t\Big(\frac{1}{p}\eta|v|^p\Big) = \big(b(\bar{u})\eta|v|^{p-2}vv_x\big)_x - \big((b(\bar{u})\eta)_x + a(x)\eta\big)\partial_x\Big(\frac{1}{p}|v|^p\Big) + \\ - \frac{4(p-1)}{p^2}b(\bar{u})\eta\big|(|v|^{p/2})_x\big|^2 + \eta F v|v|^{p-2}.$$

Integrate last equation by parts and use (19) to obtain

$$\frac{1}{p}\frac{d}{dt}\|v(t)\|_{L^p_\eta}^p = -\frac{2}{p}\int_{\mathbb{R}} |\bar{u}_x||v(t)|^p\,dx - \frac{4(p-1)}{p^2}\|b(\bar{u})^{1/2}(|v(t)|^{p/2})_x\|_{L^2_\eta}^2 + \\ + \int_{\mathbb{R}} \eta F v(t)|v(t)|^{p-2}\,dx, \tag{41}$$

for each $t \in [0, T)$. Integration of last equation in $(0, t)$ leads to the basic energy estimate (40). □

Remark 4.3. Observe that since $b \in C^2(\mathbb{R})$, $b > 0$ and $\bar{u} \in [u_+, u_-]$ then there holds $0 < C^{-1} \leq b(\bar{u}(x)) \leq C$ for all $x \in \mathbb{R}$ and some uniform $C > 0$. This readily implies that

$$\hat{C}^{-1}\|(|v|^{p/2})_x\|_{L^2_\eta}^2 \leq \|b(\bar{u})^{1/2}(|v|^{p/2})_x\|_{L^2_\eta}^2 \leq \hat{C}\|(|v|^{p/2})_x\|_{L^2_\eta}^2, \tag{42}$$

for some uniform constant $\hat{C} > 0$ and all $v, v_x \in W^{2,p} \cap L^p_\eta$.

Let us now define

$$R(t) := \sup_{\tau\in[0,t]} \|v(\tau)\|_{Z_{\eta,p}},$$

for each $t \in [0,T]$, with $T > 0$ fixed.

Lemma 4.4. *There exists $\epsilon_1 > 0$ sufficiently small such that if $R(t) < \epsilon_1$ for $t \in [0,T]$ then we have the estimate*

$$\|v(t)\|^p_{L^p_\eta} + \hat{C}_1 \int_0^t \|(|v(\tau)|^{p/2})_x\|^2_{L^2_\eta}\, d\tau \;\leq \|v(0)\|^p_{L^p_\eta}, \tag{43}$$

for some $\hat{C}_1 > 0$ depending on p and ϵ_1, and for all $0 \leq t \leq T$.

Proof. Thanks to the Sobolev-type inequality (29), under the assumption that $R(t) < \epsilon_1$ for each $0 \leq t < T$, we have that

$$|v(\tau)| \leq \|v(\tau)\|_\infty \leq C\|v_x(\tau)\|^{1/p}_{L^p}\|v(\tau)\|^{1-1/p}_{L^p} \leq \tilde{C}\epsilon_1, \tag{44}$$

$$|v_x(\tau)| \leq \|v_x(\tau)\|_\infty \leq C\|v_{xx}(\tau)\|^{1/p}_{L^p}\|v_x(\tau)\|^{1-1/p}_{L^p} \leq \tilde{C}\epsilon_1, \tag{45}$$

for some $\tilde{C} > 0$, and all $0 < \tau \leq t \leq T$. From (10) we notice that

$$\int_{\mathbb{R}} \eta F v|v|^{p-2}\, dx = \int_R \eta\, b'(\bar{u}) v_x v_{xx} v|v|^{p-2}\, dx + \int_{\mathbb{R}} \eta \mathcal{O}(v_x^2) v|v|^{p-2}\, dx =: I_1 + I_2.$$

Using (44), the integral I_2 is easily estimated, as

$$|I_2| \leq C\tilde{C}\epsilon_1 \int_{\mathbb{R}} \eta|v|^{p-2}v_x^2\, dx = \frac{4C\tilde{C}\epsilon_1}{p^2}\int_{\mathbb{R}} \eta\big|(|v|^{p/2})_x\big|^2\, dx,$$

after applying (26). On the other hand, integrating by parts we obtain

$$\begin{aligned} I_1 &= \int_R \eta\, b'(\bar{u}) v|v|^{p-2}\partial_x\Big(\tfrac{1}{2}v_x^2\Big)\, dx \\ &= -\tfrac{1}{2}\int_{\mathbb{R}} \eta_x b'(\bar{u})v|v|^{p-2}v_x^2\, dx \,-\, \tfrac{1}{2}\int_{\mathbb{R}} \eta b'(\bar{u})_x v|v|^{p-2}v_x^2\, dx \,-\, \tfrac{1}{2}\int_{\mathbb{R}} b'(\bar{u})\big(v|v|^{p-2}\big)_x v_x^2\, dx \\ &=: I_3 + I_4 + I_5. \end{aligned}$$

The integral I_4 is estimated exactly as I_2, because $b'(\bar{u})$ is uniformly bounded. Use (28), (26) and (45) to estimate I_5 as follows:

$$|I_5| \leq \frac{p-1}{2}\int_{\mathbb{R}} \eta|b'(\bar{u})||v|^{p-2}|v_x|v_x^2\, dx \leq \frac{2(p-1)\tilde{C}\epsilon_1}{p^2}\int_{\mathbb{R}} \eta\big|(|v|^{p/2})_x\big|^2\, dx.$$

Finally, use (20), (44) and (26) to estimate I_3 in the same fashion; this yields

$$|I_3| \leq \frac{C}{2}\int_{\mathbb{R}} \eta|v|^{p-2}|v|v_x^2\, dx \leq \frac{C\tilde{C}\epsilon_1}{2}\int_{\mathbb{R}} \eta\big|(|v|^{p/2})_x\big|^2\, dx.$$

Combining last estimates together we conclude that there exists a constant $C > 0$ depending on p such that

$$\Big| \int_{\mathbb{R}} \eta F v |v|^{p-2} \, dx \Big| \leq C\epsilon_1 \int_{\mathbb{R}} \eta \big| (|v|^{p/2})_x \big|^2 \, dx. \tag{46}$$

Therefore, in view of (42) and substituting in the basic energy estimate (40), we obtain

$$\frac{1}{p} \|v(t)\|_{L^p_\eta}^p + \Big(\frac{4(p-1)}{p^2 \hat{C}} - C\epsilon_1 \Big) \int_0^t \|(|v(\tau)|^{p/2})_x\|_{L^2_\eta}^2 \, d\tau + \frac{2}{p} \int_0^t \int_{\mathbb{R}} |\bar{u}_x| |v(\tau)|^p \, dx \, d\tau \;\leq\; \frac{1}{p} \|v(0)\|_{L^p_\eta}^p,$$

which implies (43) for $\epsilon_1 > 0$ sufficiently small. □

Observe that two immediate corollaries follow.

Corollary 4.5. *Specializing (43) to the case $p = 2$ we have that if $R(t) < \epsilon_1$ for all $0 \leq t \leq T$ then*

$$\|v(t)\|_{L^2_\eta}^2 + \hat{C}_1 \int_0^t \|v_x(\tau)\|_{L^2_\eta}^2 \, d\tau \;\leq\; \|v(0)\|_{L^2_\eta}^2. \tag{47}$$

Corollary 4.6. *If $v \in X_{\eta,p}(0,T)$ is a solution with $R(t) \leq \epsilon_1$ for $0 \leq t \leq T$, then*

$$\frac{d}{dt} \|v(t)\|_{L^p_\eta}^p + C_1 \|(|v|^{p/2})_x(t)\|_{L^2_\eta}^2 \;\leq\; 0, \tag{48}$$

for some $C_1 > 0$.

4.2. L^1_η-bound

We now establish the boundedness of $\|v(t)\|_{L^1_\eta}$, which is crucial for the proof of Proposition 3.2. This a remarkable property of the solutions to (13) and the Matsumura-Nishihara weight function.

Lemma 4.7. *Assuming $R(t) < \epsilon_1$, with $\epsilon_1 > 0$ just as in Lemma 4.4, the following estimate holds*

$$\|v(t)\|_{L^1_\eta} \leq C(\|v(0)\|_{L^1_\eta} + \|v(0)\|_{L^2_\eta}^2), \tag{49}$$

for some $C > 0$, all $0 \leq t \leq T$.

Proof. Multiply equation (7) by $(\operatorname{sgn} v)\eta$ and integrate in $x \in \mathbb{R}$; noticing that

$$\int_{\mathbb{R}} \eta (\operatorname{sgn} v) v_t \, dx = \int_{\mathbb{R}} \eta v v_t / |v| \, dx = \int_{\mathbb{R}} \eta (|v|)_t \, dx = \frac{d}{dt} \|v(t)\|_{L^1_\eta},$$

we readily obtain

$$\frac{d}{dt} \|v(t)\|_{L^1_\eta} = \int_{\mathbb{R}} b(\bar{u}) \eta (\operatorname{sgn} v) v_{xx} \, dx - \int_{\mathbb{R}} a(x) \eta (\operatorname{sgn} v) v_x \, dx + \int_{\mathbb{R}} \eta (\operatorname{sgn} v) F \, dx. \tag{50}$$

The linear terms are easily controlled thanks to the choice of the weight function η. Indeed, integrate by parts twice[1] to arrive at

$$\begin{aligned}\int_{\mathbb{R}} b(\bar{u})\eta(\operatorname{sgn} v)v_{xx}\,dx - \int_{\mathbb{R}} a(x)\eta(\operatorname{sgn} v)v_x\,dx &= \int_{\mathbb{R}} ((b(\bar{u})\eta)_x + a(x)\eta)_x|v|\,dx \\ &= \int_{\mathbb{R}} \Phi(x)|v|\,dx = -2\int_{\mathbb{R}} |\bar{u}_x||v|\,dx \ < 0,\end{aligned}$$

in view of (19). Upon substitution

$$\frac{d}{dt}\|v(t)\|_{L^1_\eta} + 2\int_{\mathbb{R}} |\bar{u}_x||v|\,dx = \int_{\mathbb{R}} \eta(\operatorname{sgn} v)F\,dx. \tag{51}$$

By estimate (10), the integral on the right hand side can be written as

$$I := \int_{\mathbb{R}} b'(\bar{u})\eta(\operatorname{sgn} v)v_x v_{xx}\,dx + \int_{\mathbb{R}} \eta(\operatorname{sgn} v)\mathcal{O}(v_x^2)\,dx =: I_1 + I_2.$$

Integrate by parts and use (20) to estimate I_1 as follows:

$$\begin{aligned} I_1 &= -\tfrac{1}{2}\int_{\mathbb{R}} \eta_x(\operatorname{sgn} v)b'(\bar{u})v_x^2\,dx \ - \ \tfrac{1}{2}\int_{\mathbb{R}} \eta b'(\bar{u})_x v_x^2\,dx \\ &\leq \frac{C}{2}\int_{\mathbb{R}} \eta|b'(\bar{u})|v_x^2\,dx + \tfrac{1}{2}\int_{\mathbb{R}} \eta|b'(\bar{u})_x|v_x^2\,dx \\ &\leq C_{\bar{u}}\|v_x\|^2_{L^2_\eta}, \end{aligned}$$

where $C_{\bar{u}} = C\sup_{x\in\mathbb{R}}\{|b'(\bar{u})| + |b'(\bar{u})_x|\} > 0$. Since we readily have $I_2 \leq C\|v_x\|^2_{L^2_\eta}$, we obtain that $|I|$ is bounded by $C\|v_x\|^2_{L^2_\eta}$, for some $C > 0$. Substitute into (51) and integrate in $(0,t)$ to arrive at

$$\|v(t)\|_{L^1_\eta} + 2\int_0^t\int_{\mathbb{R}} |\bar{u}_x||v(\tau)|\,dx\,d\tau \ \leq \ \|v(0)\|_{L^1_\eta} + C\int_0^t \|v_x(\tau)\|^2_{L^2_\eta}\,d\tau.$$

If $R(t) < \epsilon_1$ for all $0 \leq t \leq T$, then we apply (47) to obtain the result. □

Remark 4.8. As a by-product of this estimation, which the reader may verify with no extra effort, the integrated operator L generates a C_0-semigroup of contractions in the Banach space L^1_η (see, e.g., [27] for the appropriate generating theorems). Under the further assumption of smallness of the nonlinear terms we obtain a bound for $\|v(t)\|_{L^1_\eta}$ in terms of $\|v(0)\|_{L^1_\eta\cap L^2_\eta}$, which is essential for the establishment of the decay rates.

The previous observations readily lead us to the following

[1] We have used the short-cut $(\operatorname{sgn} v)_x \equiv 0$, a.e. Clearly, the argument leading to (51) can be made rigurous using Friedrichs' mollifiers, but we omit it.

Corollary 4.9. *Let $v \in X_{\eta,p}(0,T)$ be a solution to* (13)*, for some $T > 0$. If $R(t) < \epsilon_1$ for all $0 \le t \le T$, then v satisfies the decay rate*

$$\|v(t)\|_{L^p_\eta}^p \le C E_0^p (1+t)^{-(p-1)/2}, \tag{52}$$

with

$$E_0 := \|v(0)\|_{L^1_\eta} + \|v(0)\|_{L^p_\eta} + \|v(0)\|_{L^2_\eta}^2. \tag{53}$$

Proof. Since $R(t) < \epsilon_1$, we may apply Lemma 4.7 and Corollary 4.6 to conclude that properties (34) and (35) hold with $C_0 = \mathcal{O}(\|v(0)\|_{L^1_\eta} + \|v(0)\|_{L^2_\eta}^2)$. Then, by Proposition 3.2 we obtain the desired decay rate (52). $\square$

4.3. Higher order estimates

The L^p estimates for the derivatives cannot be controlled as in the L^2 case (where $\|(|u|^{p/2})_x\|$ is equivalent to $\|u_x\|$ and there is a natural way to construct a decreasing norm, see [21, 26]). Thus, we follow the general method of [15] instead.

Differentiate equation (7) with respect to x, to obtain

$$v_{xt} = (b(\bar{u})v_{xx})_x - (a(x)v_x)_x + F_x. \tag{54}$$

We will use (11) and estimations of solutions to last equation in order to prove the following

Lemma 4.10. *Suppose $v \in X_{\eta,p}(0,T)$, $2 \le p < +\infty$, with $0 < T \le 1$ solves* (13) *(or, equivalently,* (54)*). Then there exists $\epsilon_2 > 0$, sufficiently small, such that if $R(t) < \epsilon_2$ for $0 \le t \le T \le 1$, then there holds the estimate*

$$\frac{1}{p} t^\alpha (1+t)^\beta \|v_x(t)\|_{L^p}^p + \hat{C}_2 \int_0^t \tau^\alpha (1+\tau)^\beta \|(|v_x|^{p/2})_x(\tau)\|_{L^2_\eta}^2 \, d\tau \le C E_0^p t^{\alpha - p/2} (1+t)^{\beta - \frac{1}{2}(p-1)} \tag{55}$$

for $0 \le t \le T$, where $\alpha, \beta > 0$ satisfy

$$\alpha > p/2, \quad \beta > \tfrac{1}{2}(p-1), \tag{56}$$

and $C, \hat{C}_2 > 0$ are constants depending on $p, \epsilon_2, \bar{u}, \alpha$ and β. Moreover, for $0 < t \le T$ there holds the decay rate

$$\|v_x(t)\|_{L^p}^p \le C E_0^p t^{-p/2} (1+t)^{-\frac{1}{2}(p-1)}. \tag{57}$$

Remark 4.11. The additional assumption $T \le 1$ means no loss of generality, as the local existence time can be chosen as $\hat{T}_0 = \min\{1, T_0(\epsilon_0)\} > 0$, with $T_0(\epsilon_0)$ as in Proposition 4.1.

Proof of Lemma 4.10. Multiply equation (54) by $|v_x|^{p-2}v_x$; use (25) and integrate by parts in $\mathbb{R}$ to obtain

$$\begin{aligned}\int_{\mathbb{R}} |v_x|^{p-2} v_x v_{xt}\,dx = \frac{1}{p}\frac{d}{dt}\|v_x(t)\|_{L^p}^p &= \int_{\mathbb{R}} |v_x|^{p-2} v_x (b(\bar{u})v_{xx})_x\,dx \\ &\quad - \int_{\mathbb{R}} |v_x|^{p-2} v_x (a(x)v_x)_x\,dx \\ &\quad + \int_{\mathbb{R}} |v_x|^{p-2} v_x F_x\,dx \\ &=: I_1 + I_2 + I_3.\end{aligned} \tag{58}$$

The first integral contributes with the negative sign, a term upon which the small terms can be absorbed. Indeed, integrate by parts and use (26) and (28) to estimate I_1 as follows:

$$\begin{aligned} I_1 &= -\int_{\mathbb{R}} (|v_x|^{p-2}v_x)_x b(\bar{u}) v_{xx}\,dx \\ &= -(p-1)\int_{\mathbb{R}} b(\bar{u})|v_x|^{p-2}v_{xx}^2\,dx \\ &= -(p-1)(4/p^2)\int_{\mathbb{R}} b(\bar{u})\big|(|v_x|^{p/2})_x\big|^2\,dx \\ &\leq -\frac{4(p-1)}{\hat{C}p^2}\|(|v_x|^{p/2})_x\|_{L^2}^2 < 0,\end{aligned} \tag{59}$$

in view of boundedness of $b(\bar{u})$ (see (42)). In order to estimate I_2, integrate by parts and make use of (25) and (28); the result is

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} (|v_x|^{p-2}v_x)_x a(x) v_x\,dx \\ &= (p-1)\int_{\mathbb{R}} a(x)|v_x|^{p-2}v_{xx}v_x\,dx \\ &= (1-1/p)\int_{\mathbb{R}} a(x)(|v_x|^p)_x\,dx \\ &= -(1-1/p)\int_{R} a'(x)|v_x|^p\,dx \leq C\|v_x\|_{L^p}^p,\end{aligned} \tag{60}$$

where $C = (1-1/p)\sup_{x\in\mathbb{R}}|a'(x)| > 0$. Now, in view of (11), I_3 can be estimated as $|I_3| \leq C(|I_4| + I_5 + I_6)$, where

$$\begin{aligned} I_4 &:= \int_{\mathbb{R}} b'(\bar{u})|v_x|^{p-2}v_x^2 v_{xxx}\,dx, \\ I_5 &:= \int_{\mathbb{R}} |v_x|^p|v_x|\,dx, \\ I_6 &:= \int_{\mathbb{R}} |v_x|^{p-1}v_{xx}^2\,dx.\end{aligned}$$

Assuming $R(t) < \epsilon_2$, use $|v_x| \leq \|v_x\|_\infty < C\epsilon_2$ and (26) to estimate

$$I_6 \leq C\epsilon_2 \int_{\mathbb{R}} |v_x|^{p-2} v_{xx}^2 \, dx \leq C_p \epsilon_2 \|(|v_x|^{p/2})_x\|_{L^2}^2.$$

Likewise, $I_5 \leq \epsilon_2 \|v_x\|_{L^p}^p$. To estimate I_4, integrate by parts and use (25) to get

$$I_4 = -\int_{\mathbb{R}} b'(\bar{u})_x |v_x|^p v_{xx} \, dx \; - \; p \int_{\mathbb{R}} b'(\bar{u}) |v_x|^{p-2} v_x v_{xx}^2 \, dx.$$

The first integral is bounded by

$$\begin{aligned} \int_{\mathbb{R}} |b'(\bar{u})_x| |v_x|^{p-2} |v_x| |v_x| |v_{xx}| \, dx &\leq \epsilon_2 \frac{C}{2} \int_{\mathbb{R}} |v_x|^{p-2} (v_x^2 + v_{xx}^2) \, dx \\ &\leq C\epsilon_2 (\|v_x\|_{L^p}^p + \|(|v_x|^{p/2})_x\|_{L^2}^2, \end{aligned}$$

after having used (26). The second integral is estimated exactly as I_6. Combining all these estimates together we arrive at

$$|I_3| \leq C\epsilon_2 (\|v_x\|_{L^p}^p + \|(|v_x|^{p/2})_x\|_{L^2}^2), \tag{61}$$

for some $C > 0$. Upon substitution of (59), (60) and (61) into (58), we obtain

$$\frac{1}{p} \frac{d}{dt} \|v_x(t)\|_{L^p}^p + \frac{4(p-1)}{\hat{C} p^2} \|(|v_x|^{p/2})_x(t)\|_{L^2}^2 \; \leq \; C \|v_x(t)\|_{L^p}^p + C\epsilon_2 \|(|v_x|^{p/2})_x(t)\|_{L^2}^2. \tag{62}$$

The last term, of order $\mathcal{O}(\epsilon_2)$ can be absorbed into the term or order $\mathcal{O}(1)$ on the left hand side, provided ϵ_2 is sufficiently small. Note, from the estimates above, that the generic constant C does not depend on ϵ_2. We are left to estimate the $\|v_x(t)\|_{L^p}^p$ term. For that purpose choose auxiliary constants $\alpha, \beta > 0$ satisfying (56); since $\alpha > p/2 \geq 1$, then $t^{\alpha-1}(1+t)^\beta + t^\alpha (1+t)^{\beta-1} \leq 2t^{\alpha-1}(1+t)^\beta$. Whence,

$$t^\alpha (1+t)^\beta \frac{d}{dt} \|v_x(t)\|_{L^p}^p \; \geq \; \frac{d}{dt} \big(t^\alpha (1+t)^\beta \|v_x(t)\|_{L^p}^p \big) - 2C_{\alpha,\beta} t^{\alpha-1} (1+t)^\beta \|v_x(t)\|_{L^p}^p,$$

where $C_{\alpha,\beta} > 0$. Multiply (62) by $t^\alpha (1+t)^\beta$, integrate in $(0,t)$ and use last inequality to arrive at

$$\begin{aligned} \frac{1}{p} t^\alpha (1+t)^\beta \frac{d}{dt} \|v_x(t)\|_{L^p}^p + \frac{4(p-1)}{\hat{C} p^2} \int_0^t \tau^\alpha (1+\tau)^\beta \|(|v_x|^{p/2})_x(\tau)\|_{L^2}^2 \, d\tau \\ \leq \frac{\tilde{C}}{p} \int_0^t \tau^{\alpha-1} (1+\tau)^\beta \|v_x(\tau)\|_{L^p}^p \, d\tau + \\ + C \int_0^t \tau^\alpha (1+\tau)^\beta \|v_x(\tau)\|_{L^p}^p \, d\tau + \\ + C\epsilon_2 \int_0^t \tau^\alpha (1+\tau)^\beta \|(|v_x|^{p/2})_x(\tau)\|_{L^2}^2 \, d\tau \\ =: I_7 + I_8 + I_9, \end{aligned} \tag{63}$$

for some $\tilde{C} > 0$ depending on α and β. Thanks to the assumption $T \leq 1$, then clearly $I_8 \leq CI_7$, and I_7 is estimated following the method of [15]. The latter relies on the inequality

$$\begin{aligned}\tau^{\alpha-1}(1+\tau)^{\beta}\|v_x\|_{L^p}^p &\leq C\tau^{\alpha}(1+\tau)^{\beta}\big(\varepsilon\|(|v_x|^{p/2})_x\|_{L^2}^{2p/(p+2)}\big)\big(\tau^{-1}\varepsilon^{-1}\|v\|_{L^p}^{2p/(p+2)}\big)\\ &\leq C\tau^{\alpha}(1+\tau)^{\beta}\big(\varepsilon^{(p+2)/p}\|(|v_x|^{p/2})_x\|_{L^2}^2 + C_{p,\varepsilon}\tau^{-p/2-1}\|v\|_{L^p}^p\big),\end{aligned}$$

where we have used the inequality (30) and Hölder's inequality with $p/(p+2) + 2/(p+2) = 1$; here $\varepsilon > 0$ is arbitrary. Choose $\varepsilon := \epsilon_2^{p/(p+2)}$ to obtain the estimate

$$\begin{aligned}I_7 &= (\tilde{C}/p)\int_0^t \tau^{\alpha-1}(1+\tau)^{\beta}\|v_x(\tau)\|_{L^p}^p\,d\tau\\ &\leq C_p\epsilon_2\int_0^t \tau^{\alpha}(1+\tau)^{\beta}\|(|v_x|^{p/2})_x(\tau)\|_{L^2}^2\,d\tau \,+\, C_{p,\epsilon_2}\int_0^t \tau^{\alpha-p/2-1}(1+\tau)^{\beta}\|v(\tau)\|_{L^p}^p\,d\tau.\end{aligned}$$

We now use the decay rate (52) (by taking $\epsilon_2 < \epsilon_1/2$, small), to estimate

$$\|v(\tau)\|_{L^p}^p \leq C\|v(\tau)\|_{L^p_\eta}^p \leq CE_0^p(1+\tau)^{-\frac{1}{2}(p-1)},$$

and, consequently, to obtain

$$I_7 \leq C_p\epsilon_2\int_0^t \tau^{\alpha}(1+\tau)^{\beta}\|(|v_x|^{p/2})_x(\tau)\|_{L^2}^2\,d\tau \,+\, CC_{p,\epsilon_2}E_0^p t^{\alpha-p/2}(1+t)^{\beta-\frac{1}{2}(p-1)},$$

because $\beta > \frac{1}{2}(p-1)$ and $\alpha > p/2$. Since $I_8 \leq CI_7$, substitute all these estimates into (63) to obtain

$$\begin{aligned}\frac{1}{p}t^{\alpha}(1+t)^{\beta}\frac{d}{dt}\|v_x(t)\|_{L^p}^p + \Big(\frac{4(p-1)}{\hat{C}p^2} - \check{C}\epsilon_2\Big)\int_0^t \tau^{\alpha}(1+\tau)^{\beta}\|(|v_x|^{p/2})_x(\tau)\|_{L^2}^2\,d\tau\\ \leq CE_0^p t^{\alpha-p/2}(1+t)^{\beta-\frac{1}{2}(p-1)},\end{aligned}$$

for some $\check{C}$ depending on α, β and p, and some $C > 0$ depending on ϵ_2 as well; we may take $\epsilon_2 > 0$ sufficiently small such that $\hat{C}_2 := 4(p-1)/(\hat{C}p^2) - \check{C}\epsilon_2 > 0$ to obtain (55). The decay estimate (57) follows immediately from (55) for $t > 0$. □

We finally proceed with the establishment of the *a priori* estimates for the second derivatives. The proof is similar to that of the previous lemma, so we gloss over some of the details.

Lemma 4.12. *Suppose $v \in X_{\eta,p}(0,T)$, $2 \leq p < +\infty$, with $0 < T \leq 1$ solves (13). Then there exists $\epsilon_3 > 0$, sufficiently small, such that if $R(t) < \epsilon_3$ for $0 \leq t \leq T \leq 1$, then there holds the estimate*

$$\frac{1}{p}t^{\gamma}(1+t)^{\delta}\|v_{xx}(t)\|_{L^p}^p + \hat{C}_3\int_0^t \tau^{\gamma}(1+\tau)^{\delta}\|(|v_{xx}|^{p/2})_x(\tau)\|_{L^2_\eta}^2\,d\tau \leq CE_0^p t^{\gamma-p}(1+t)^{\delta-\frac{1}{2}(p-1)} \tag{64}$$

for $0 \leq t \leq T$*, where* $\gamma, \delta > 0$ *satisfy*

$$\gamma > p, \quad \delta > \tfrac{1}{2}(p-1), \tag{65}$$

and $C, \hat{C}_3 > 0$ *are constants depending on* $p, \epsilon_3, \bar{u}, \gamma$ *and* δ*. Moreover, for* $0 < t \leq T$ *there holds the decay rate*

$$\|v_{xx}(t)\|_{L^p}^p \leq C E_0^p t^{-p}(1+t)^{-\frac{1}{2}(p-1)}. \tag{66}$$

Proof. Multiply equation (54) by $|v_{xx}|^{p-2}v_{xxx}$, use (25) and (28), and integrate by parts in $\mathbb{R}$, to obtain

$$\begin{aligned} \frac{1}{p(p-1)}\frac{d}{dt}\|v_{xx}(t)\|_{L^p}^p = &-\int_{\mathbb{R}} |v_{xx}|^{p-2}v_{xxx}(b(\bar{u})v_{xx})_x\,dx \\ &+\int_{\mathbb{R}} |v_{xx}|^{p-2}v_{xxx}(a(x)v_x)_x\,dx \\ &-\int_{\mathbb{R}} |v_{xx}|^{p-2}v_{xxx}F_x\,dx \;=:\; I_1+I_2+I_3. \end{aligned} \tag{67}$$

Integrate by parts, and use (25), (26) and (42), to estimate I_1 as follows:

$$\begin{aligned} I_1 &= -\int_{\mathbb{R}} b(\bar{u})|v_{xx}|^{p-2}v_{xxx}^2\,dx \;-\; \int_{\mathbb{R}} b(\bar{u})_x|v_{xx}|^{p-2}v_{xxx}v_{xx}\,dx \\ &= -(4/p^2)\int_{\mathbb{R}} b(\bar{u})|(|v_{xx}|^{p/2})_x|^2\,dx \;-\; (1/p)\int_{\mathbb{R}} b(\bar{u})_x(|v_{xx}|^p)_x\,dx \\ &\leq -(4/(\hat{C}p^2))\|(|v_{xx}|^{p/2})_x\|_{L^2}^2 + C_{p,\bar{u}}\|v_{xx}\|_{L^p}^p, \end{aligned}$$

with $C_{p,\bar{u}} = (1/p)\sup_{x\in\mathbb{R}}|b(\bar{u})_{xx}| > 0$. This integral provides the negative term which will absorb the small terms, plus a term of order $\|v_{xx}\|_{L^p}^p$. In the same fashion, integrate I_2 by parts, and use (25), (28) to obtain

$$\begin{aligned} I_2 &= (1/p)\int_{\mathbb{R}} a(x)(|v_{xx}|^p)_x\,dx \;+\; \int_{\mathbb{R}} a'(x)|v_{xx}|^{p-2}v_{xxx}v_x\,dx \\ &= -(1/p)\int_{\mathbb{R}} a'(x)|v_{xx}|^p\,dx \;+\; (p-1)^{-1}\int_{\mathbb{R}} a'(x)(|v_{xx}|^{p-2}v_{xx})_x v_x\,dx \\ &\leq C_{p,\bar{u}}\|v_{xx}\|_{L^p}^p \;-\; (p-1)^{-1}\int_{\mathbb{R}} a'(x)|v_{xx}|^p\,dx \;-(p-1)^{-1}\int_{\mathbb{R}} a''(x)|v_{xx}|^{p-2}v_{xx}v_x\,dx \\ &\leq \tilde{C}_{p,\bar{u}}\|v_{xx}\|_{L^p}^p + C_p\int_{\mathbb{R}} |v_{xx}|^{p-1}|v_x|\,dx \\ &\leq \tilde{C}_{p,\bar{u}}\|v_{xx}\|_{L^p}^p + C_p\varepsilon^p\|v_x\|_{L^p}^p + C_{\varepsilon,p}\|v_{xx}\|_{L^p}^p, \end{aligned}$$

where we have used Hölder's inequality with $(p-1)/p + 1/p = 1$, and for any arbitrary $\varepsilon > 0$. Choose $\varepsilon = \epsilon_3^{1/p}$ to get the estimate

$$I_2 \leq C\|v_{xx}\|_{L^p}^p + C\epsilon_3\|v_x\|_{L^p}^p.$$

In order to control the nonlinear terms, we proceed similarly. From (11), we may write $I_3 = I_6 + I_7 + I_8$, with

$$
\begin{aligned}
I_6 &:= -\int_{\mathbb{R}} b'(\bar{u})|v_{xx}|^{p-2} v_{xxx}^2 v_x \, dx, \\
I_7 &\leq C \int_{\mathbb{R}} |v_{xx}|^{p-2} |v_{xxx}| v_x^2 \, dx, \\
I_8 &\leq C \int_{\mathbb{R}} |v_{xx}|^{p-2} |v_{xxx}| v_{xx}^2 \, dx.
\end{aligned}
$$

Use (26) and $|v_x| \leq \|v_x\|_\infty < C\epsilon_3$ to get

$$
I_6 \leq -(4/p^2) \int_{\mathbb{R}} b'(\bar{u})|v_x| |(|v_{xx}|^{p/2})_x|^2| \, dx \leq C\epsilon_3 \|(|v_{xx}|^{p/2})_x\|_{L^2}^2,
$$

an absorbable term. In the same fashion, use (26) and Hölder inequality once again, now with $(p-2)/p + 2/p = 1$, to obtain

$$
\begin{aligned}
I_7 &\leq \frac{C}{2}\Big(\varepsilon^2 \int_{\mathbb{R}} |v_{xx}|^{p-2}|v_{xxx}|^2 + \varepsilon^{-2} \int_{\mathbb{R}} |v_{xx}|^{p-2} |v_x|^2 \, dx \Big) \\
&\leq C_p \varepsilon^2 \|(|v_{xx}|^{p/2})_x\|_{L^2}^2 + C_\varepsilon \big(((p-2)/p) \|v_{xx}\|_{L^p}^p + (2/p)\|v_x\|_{L^p}^p \big) \\
&\leq C_p \epsilon_3 \|(|v_{xx}|^{p/2})_x\|_{L^2}^2 + C_{p,\epsilon_3} (\|v_{xx}\|_{L^p}^p + \|v_x\|_{L^p}^p),
\end{aligned}
$$

after putting $\varepsilon^2 = \epsilon_3$. Finally, we estimate I_8 as follows

$$
I_8 \leq C\varepsilon^2 \int_{\mathbb{R}} |v_{xx}|^{p-2} v_{xxx}^2 \, dx + C\varepsilon^{-2} \|v_{xx}\|_{L^p}^p \leq C\epsilon_3 \|(|v_{xx}|^{p/2})_x\|_{L^2}^2 + C_{p,\epsilon_3} \|v_{xx}\|_{L^p}^p.
$$

Substitute back all estimates into (67) and multiply by $p-1$ to arrive at

$$
\frac{1}{p}\frac{d}{dt}\|v_{xx}(t)\|_{L^p}^p + \Big(\frac{4(p-1)}{\hat{C}p^2} - \tilde{C}\epsilon_3 \Big) \|(|v_{xx}|^{p/2})_x(t)\|_{L^2}^2 \leq \check{C}_{\epsilon_3} (\|v_{xx}(t)\|_{L^p}^p + \|v_x(t)\|_{L^p}^p), \tag{68}
$$

where $\tilde{C}$ is independent of ϵ_3, but, unlike in the proof of the previous lemma, now the constant $\check{C}_{\epsilon_3}$ depends upon ϵ_3 at order $\mathcal{O}(1/\epsilon_3)$ (this comes from the estimates of I_7, I_8 and I_2). Thus, we write $\check{C}_{\epsilon_3} =: \bar{C}/\epsilon_3$. This poses no further complications as we shall see.

To control the term $\|v_{xx}(t)\|_{L^p}^p$ we proceed exactly as in the proof of the previous Lemma 4.10. First, let us choose $\epsilon_3 < \frac{1}{2}\epsilon_2$ so that estimate (57) holds. Now, choose the auxiliary constants $\gamma, \delta > 0$ satisfying (65) and multiply (68) by $t^\gamma (1+t)^\delta$; integrate in $(0,t)$ to obtain

$$
\begin{aligned}
\frac{1}{p} t^\gamma (1+t)^\delta \frac{d}{dt} \|v_{xx}(t)\|_{L^p}^p + \Big(\frac{4(p-1)}{\hat{C}p^2} - \tilde{C}\epsilon_3 \Big) \int_0^t \tau^\gamma (1+\tau)^\delta \|(|v_{xx}|^{p/2})_x(\tau)\|_{L^2}^2 \, d\tau \\
\leq \frac{\bar{C}}{\epsilon_3} \int_0^t \tau^{\gamma-1}(1+\tau)^\delta \|v_{xx}(\tau)\|_{L^p}^p \, d\tau + \\
+ \frac{\bar{C}}{\epsilon_3} \int_0^t \tau^\gamma (1+\tau)^\delta \|v_x(\tau)\|_{L^p}^p \, d\tau,
\end{aligned} \tag{69}
$$

after having absorbed the $\int \tau^\gamma (1+\tau)^\delta \|v_{xx}\|^p_{L^p}$ term under the assumption $T \leq 1$ as before.

The first term on the right hand side of (69) is controlled (almost) exactly as in the Lemma 4.10, yielding

$$\int_0^t \tau^{\gamma-1}(1+\tau)^\delta \|v_{xx}(\tau)\|^p_{L^p}\, d\tau \leq \varepsilon^{(p+2)/p} C_p \int_0^t \tau^\gamma (1+\tau)^\delta \|(|v_{xx}|^{p/2})_x(\tau)\|^2_{L^2}\, d\tau + \\ + C_{p,\varepsilon} \int_0^t \tau^{\gamma-1-p/2}(1+\tau)^\delta \|v_x(\tau)\|^p_{L^p}\, d\tau \\ \leq \epsilon_3^2 C_p \int_0^t \tau^\gamma (1+\tau)^\delta \|(|v_{xx}|^{p/2})_x(\tau)\|^2_{L^2}\, d\tau + \\ + C C_{p,\epsilon_3} E_0^p t^{\gamma-p}(1+t)^{\delta - \frac{1}{2}(p-1)},$$

with the only difference that, here, we have chosen $\varepsilon := \epsilon_3^{2p/(p+2)}$ to keep the first term absorbable, and we have used the decay estimate (57) for v_x, with $\gamma > p$ and $\delta > (p-1)/2$.

In order to estimate the second term of the right hand side of (69) we apply the decay estimate (57) directly. This yields,

$$\int_0^t \tau^\gamma (1+\tau)^\delta \|v_x(\tau)\|^p_{L^p} \leq C E_0^p (1+t)^{\delta - \frac{1}{2}(p-1)} t^{\gamma - p/2+1}.$$

Substituting these two estimates back into (69), we obtain

$$\frac{1}{p} t^\gamma (1+t)^\delta \frac{d}{dt}\|v_{xx}(t)\|^p_{L^p} + \Big(\frac{4(p-1)}{\hat{C}p^2} - \tilde{C}\epsilon_3 \Big) \int_0^t \tau^\gamma (1+\tau)^\delta \|(|v_{xx}|^{p/2})_x(\tau)\|^2_{L^2}\, d\tau \\ \leq \bar{C} C_p \epsilon_3 \int_0^t \tau^\gamma (1+\tau)^\delta \|(|v_{xx}|^{p/2})_x(\tau)\|^2_{L^2}\, d\tau + \\ + C C_{p,\epsilon_3} E_0^p (1+t)^{\delta - \frac{1}{2}(p-1)} \big(t^{\gamma-p} + t^{\gamma-p/2+1} \big).$$

Noticing that $t^{\gamma-p} \geq t^{\gamma-p/2+1}$ for each $0 < t \leq T \leq 1$, we merge the last two terms into the slower decay rate. The result is estimate (64), where $0 < \epsilon_3 < \epsilon_2/2 < \epsilon_1/4$ is chosen sufficiently small such that $\hat{C}_3 := 4(p-1)/(\hat{C}p^2) - (\tilde{C} + \bar{C})\epsilon_3 > 0$. The decay estimate (66) follows directly from (64), for $t > 0$ and under assumption (65). This proves the Lemma. □

5. Stability and proof of Theorem 1.5

Finally, in this section we apply the previous *a priori* estimates to obtain a global solution, leading to stability and the proof of the main Theorem.

5.1. Global existence

Here we perform the (almost) standard continuation argument to obtain global existence of solutions to the Cauchy problem (13) and (14). We have to pay special attention to the fact that estimates for the derivatives (57) and (66) apply only in time intervals of measure one and, thus, the proof deviates slightly from the standard argument. Notably, the energy E_0 does not involve norms of the derivatives, allowing us to extend the decay rates globally in time.

Theorem 5.1. *Suppose $v_0 \in Z_{\eta,p}$, with $2 \leq p < +\infty$. Then there exists a positive constant $\hat{\epsilon} > 0$ such that if $\|v_0\|_{Z_{\eta,p}} < \hat{\epsilon}$, then the Cauchy problem (13) and (14) has a unique global solution $v \in X_{\eta,p}(0,+\infty)$ which satisfies the following estimates*

$$\|v(t)\|_{L^1_\eta} \leq M E_0, \tag{70}$$

$$\|v(t)\|_{L^p_\eta} \leq M E_0 (1+t)^{-\frac{1}{2}(1-1/p)}, \tag{71}$$

$$\|v(t)\|_{L^2_\eta} \leq M E_0 (1+t)^{-\frac{1}{4}}, \tag{72}$$

$$\|v_x(t)\|_{L^p} \leq M E_0 t^{-\frac{1}{2}} (1+t)^{-\frac{1}{2}(1-1/p)}, \tag{73}$$

$$\|v_{xx}(t)\|_{L^p} \leq M E_0 t^{-1} (1+t)^{-\frac{1}{2}(1-1/p)}, \tag{74}$$

for all $0 < t < +\infty$, with some uniform constant $M > 0$, and where

$$E_0 = \|v_0\|_{L^1_\eta} + \|v_0\|_{L^p_\eta} + \|v_0\|^2_{L^2_\eta}.$$

Proof. Let $\hat{\epsilon} > 0$ (to be chosen later) be such that $\hat{\epsilon} < \frac{1}{2}\epsilon_3$, where ϵ_3 is the fixed constant of Lemma 4.12. The local solution can be continued globally in time provided $\hat{\epsilon}$ is sufficiently small. By Proposition 4.1, for each $\epsilon > 0$ there exists $T_0 = T_0(\epsilon) > 0$ such that the solution exists in $[0, T_0(\epsilon)]$. Let us define $\hat{T}_0(\epsilon) := \min\{1, T_0(\epsilon)\} \leq 1$, so that we can apply estimates (57) and (66). Since $\hat{\epsilon} < \epsilon_3$, the short existence times clearly satisfy $\bar{T}_0 := \hat{T}_0(\epsilon_3) \leq \hat{T}_0(\hat{\epsilon})$. Therefore, we consider the solution $v(t) \in X_{\eta,p}(0, \bar{T}_0)$, which, by Proposition 4.1, satisfies $\|v(t)\|_{Z_{\eta,p}} < 2\hat{\epsilon} < \epsilon_3 < \frac{1}{2}\epsilon_2 < \frac{1}{4}\epsilon_1$, for all $t \in [0, \bar{T}_0]$. Thus, we can apply estimates (43), (49), (57) and (66). Hence, we have

$$\begin{aligned}
&\|v(\bar{T}_0)\|_{L^1_\eta} \leq C(\|v_0\|_{L^1_\eta} + \|v_0\|^2_{L^2_\eta}),\\
&\|v(\bar{T}_0)\|^2_{L^2_\eta} \leq \|v_0\|^2_{L^2_\eta},\\
&\|v(\bar{T}_0)\|^p_{L^p_\eta} \leq \|v_0\|^p_{L^p_\eta},
\end{aligned}$$

with uniform $C > 0$. Since $\bar{T}_0 \leq 1$, we apply estimates (57) and (66). For instance, from (57) we get

$$\begin{aligned}
\|v_x(\bar{T}_0)\|^p_{L^p} &\leq C E_0 \bar{T}_0^{-1/2} (1+\bar{T}_0)^{-\frac{1}{2}(1-1/p)}\\
&\leq C \bar{T}_0^{-1/2} (\|v_0\|_{L^1_\eta} + \epsilon_3 \|v_0\|_{{}^2_\eta} + \epsilon_3^{p-1} \|v_0\|_{L^p_\eta})\\
&\leq C \tilde{C}(\epsilon_3) \|v_0\|_{Z_{\eta,p}},
\end{aligned}$$

where $\tilde{C}(\epsilon_3) > 0$ is a uniform constant depending on ϵ_3, as $\bar{T}_0$ depends on ϵ_3. Similarly, from (66) we obtain $\|v_{xx}(\bar{T}_0)\|_{L^p} \leq C\tilde{C}(\epsilon_3)\|v_0\|_{Z_{\eta,p}}$.

This shows that there exists uniform constants $\hat{C}(\epsilon_3)$, and $C_* > 0$ independent of ϵ_3, such that

$$\|v(\bar{T}_0)\|_{Z_{\eta,p}} \leq C_* \hat{C}(\epsilon_3)\|v_0\|_{Z_{\eta,p}} < C_* \hat{C}(\epsilon_3)\hat{\epsilon}, \tag{75}$$

as $\|v_0\|_{Z_{\eta,p}} < \hat{\epsilon}$, by hypothesis. We now choose $\hat{\epsilon} > 0$ such that

$$0 < \hat{\epsilon} < \min\Big\{1, \frac{\epsilon_3}{2}, \frac{\epsilon_3}{2C_*\hat{C}(\epsilon_3)}, \frac{\epsilon_3}{2C_*^2\hat{C}(\epsilon_3)}\Big\}. \tag{76}$$

The reason why we took the third bound will be clear later. In this fashion, we obtain $\|v(\bar{T}_0)\|_{Z_{\eta,p}} < \frac{1}{2}\epsilon_3$. Henceforth, we can apply Proposition 4.1 once again by taking $t = \bar{T}_0$ as the initial time, to obtain a solution on the interval $[\bar{T}_0, 2\bar{T}_0]$, such that $\|v(t)\|_{Z_{\eta,p}} < \epsilon_3$ for all $t \in [\bar{T}_0, 2\bar{T}_0]$. This shows that $\|v(t)\|_{Z_{\eta,p}} < \epsilon_3$ for all $t \in [0, 2\bar{T}_0]$. Let us define

$$E(T) := \|v(T)\|_{L^1_\eta} + \|v(T)\|_{L^p_\eta} + \|v(T)\|^2_{L^2_\eta},$$

as the energy at each time step $T > 0$. Notice that estimates (43), (47) and (49) hold in $[0, 2\bar{T}_0]$. This implies, in particular, that $E(2\hat{T}_0) \leq CE_0$. In fact, we also have that $E(n\bar{T}_0) \leq CE_0$, with the same uniform constant $C > 0$, for each $n = 1, 2, \ldots$, as long as $\|v(t)\|_{Z_{\eta,p}} < \epsilon_3$ on the interval $t \in [0, n\bar{T}_0]$.

This will be achieved using estimates (57) and (66). They hold on the time interval $[\bar{T}_0, 2\bar{T}_0]$ only, which has measure less than one. For instance, from (57) we get

$$\|v_x(2\bar{T}_0)\|_{L^p} \leq CE(2\bar{T}_0)(2\bar{T}_0)^{-1/2}(1+2\bar{T}_0)^{-\frac{1}{2}(1-1/p)} \leq C^2E_0\bar{T}_0^{-1/2} \leq C^2\tilde{C}(\epsilon_3)\|v_0\|_{Z_{\eta,p}},$$

with the same uniform constants $C > 0$ and $\tilde{C}(\epsilon_3) > 0$. Similarly we have $\|v_{xx}(2\bar{T}_0)\|_{L^p} \leq C^2\tilde{C}(\epsilon_3)\|v_0\|_{Z_{\eta,p}}$. Thus, we obtain the bound

$$\|v(2\bar{T}_0)\|_{Z_{\eta,p}} \leq C_*^2\hat{C}(\epsilon_3)\|v_0\|_{Z_{\eta,p}} < C_*^2\hat{C}(\epsilon_3)\hat{\epsilon} < \tfrac{1}{2}\epsilon_3,$$

with the same uniform constants $C_*, \hat{C}(\epsilon_3)$ as in (75), and by the choice of $\hat{\epsilon}$ in (76).

Thus we can repeat the procedure to the interval $[2\bar{T}_0, 3\bar{T}_0]$, yielding $\|v(t)\|_{Z_{\eta,p}} < \epsilon_3$ up to time $t = 3\bar{T}_0$. Notice that, since $E(3\bar{T}_0) < CE_0$ with same constant $C > 0$, this yields again $\|v(3\bar{T}_0)\|_{Z_{\eta,p}} \leq C_*^2\hat{C}(\epsilon_3)\|v_0\|_{Z_{\eta,p}} < \frac{1}{2}\epsilon_3$, with same uniform constants C_* and $\hat{C}(\epsilon_3)$. Henceforth, the procedure can be repeated succesively on intervals $[(n-1)\bar{T}_0, n\bar{T}_0]$, for $n = 1, 2, \ldots$ in order to obtain a global solution.

Notice that estimates (43), (47) and (49) hold in every interval $[0, n\bar{T}_0]$, for each $n \in \mathbb{N}$. This provides estimates (71), (72) and (70) globally in time. In contrast, estimates (57) and (66) hold on each interval $[(n-1)\bar{T}_0, n\bar{T}_0]$. Thanks to the fact that $E(T)$ does not depend on the norms of the derivatives of v at previous time step, however, we can obtain the desired decay rates. Indeed, since $\|v(t)\|_{Z_{\eta,p}} < \epsilon_3$ for each $t \in [(n-1)\bar{T}_0, n\bar{T}_0]$, there holds

$$\|v_x(t)\|_{L^p} \leq CE((n-1)\bar{T}_0)t^{-1/2}(1+t)^{-\frac{1}{2}(1-1/p)} \leq C^2E_0t^{-1/2}(1+t)^{-\frac{1}{2}(1-1/p)},$$

$$\|v_{xx}(t)\|_{L^p} \leq CE((n-1)\bar{T}_0)t^{-1}(1+t)^{-\frac{1}{2}(1-1/p)} \leq C^2E_0 t^{-1}(1+t)^{-\frac{1}{2}(1-1/p)},$$

for all $t \in [(n-1)\bar{T}_0, n\bar{T}_0]$, and for each $n = 1, 2, \ldots$, with the same uniform bound C^2E_0 (depending only on p, ϵ_3 and $\|v_0\|_{Z_{\eta,p}}$). This shows that the estimates hold globally in time, providing the decay rates (73) and (74), for some uniform constant $M > 0$. This completes the proof of the Theorem. □

5.2. Proof of Theorem 1.5

The conclusions of Theorem 1.5 follow directly from the global existence Theorem 5.1. Indeed, under assumption (5) we may define the antiderivative of the perturbation v_0 as (21). We then look at the equivalent Cauchy problem (13) and (14) for v. Furthermore, if we suppose that u_0 satisfies $v_0 \in Z_{\eta,p}$, take $\hat{\epsilon} > 0$ just as in Theorem 5.1 to conclude that if $\|v_0\|_{Z_{\eta,p}} < \hat{\epsilon}$, then there exists a global solution $v \in X_{\eta,p}(0, +\infty)$ to (13) and (14), satisfying estimates (70) - (74). As $v \in X_{\eta,p}(0, +\infty)$ and $v_x = u - \bar{u}$, then $u - \bar{u} \in C([0, +\infty); W^{1,p})$, and it is a solution to (1), as v solves (6).

Decay rates (22) follows directly from (73), whereas to obtain the L^∞ estimate (23), apply the inequality of Sobolev-type (29) to v_x, together with (73) and (74) to obtain

$$\|v_x\|_\infty^p \leq C\|v_{xx}\|_{L^p}\|v_x\|_{L^p}^{p-1} \leq CM^pE_0^p t^{-1/2-p/2}(1+t)^{-\frac{p}{2}(1-1/p)},$$

yielding (23). This completes the proof.

Acknowledgements

I am grateful to Jonathan Goodman, Kevin Zumbrun and Peter Howard for many fruitful conversations along the years. In particular, I thank Jonathan, for pointing out the use of interpolation inequalities to get decay rates, and Peter, for calling my attention to the sharpening of the decay rate mechanism for zero-mass perturbations. I would also like to thank the referee for sharp observations that improved the quality of the paper.

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