SPECTRAL STABILITY OF TRAVELING FRONTS FOR REACTION DIFFUSION-DEGENERATE FISHER-KPP EQUATIONS

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ABSTRACT. This paper establishes the spectral stability in exponentially weighted spaces of smooth traveling monotone fronts for reaction diffusion equations of Fisher-KPP type with nonlinear degenerate diffusion coefficient. It is assumed that the former is degenerate, that is, it vanishes at zero, which is one of the equilibrium points of the reaction. A parabolic regularization technique is introduced in order to locate a subset of the compression spectrum of the linearized operator around the wave, whereas the point and approximate spectra are proved to be stable with the use of energy estimates. Detailed asymptotic decay estimates of solutions to resolvent type equations are required in order to close the energy estimates. It is shown that all fronts traveling with speed above a threshold value are spectrally stable in an appropriately chosen exponentially weighted L^2-space.

1. INTRODUCTION

In this paper we study scalar reaction-diffusion equations of the form

$$u_t = (D(u)u_x)_x + f(u), \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}$, $x \in \mathbb{R}$, $t > 0$, and the diffusion coefficient $D = D(u)$ is a nonlinear, non-negative density dependent function which is *degenerate* at $u = 0$. More precisely, it is assumed that D satisfies

$$\begin{aligned} &D(0) = 0, \quad D(u) > 0 \ \text{ for all}\, u \in (0,1], \\ &D \in C^2([0,1];\mathbb{R}) \ \text{ with } D'(u) > 0 \text{ for all } u \in [0,1]. \end{aligned} \tag{1.2}$$

As an example we have the quadratic function

$$D(u) = u^2 + bu, \tag{1.3}$$

for some constant $b > 0$, as proposed by Shigesada *et al.* [51] to model dispersive forces due to mutual interferences between individuals of an animal population.

The nonlinear reaction function is supposed to be of *Fisher-KPP type* [11, 23], that is, $f \in C^2([0,1];\mathbb{R})$ has one stable and one unstable equilibrium points in $[0,1]$; more precisely,

$$\begin{aligned} &f(0) = f(1) = 0, \\ &f'(0) > 0, \ \ f'(1) < 0, \\ &f(u) > 0 \text{ for all } u \in (0,1). \end{aligned} \tag{1.4}$$

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An example of a reaction of Fisher-KPP type is the classical logistic function,

$$f(u) = u(1-u), \tag{1.5}$$

which is used to model the dynamics of a population in an environment with limited resources.

Nonlinear reaction-diffusion equations arise as models in several natural phenomena, with applications to population dynamics, chemical reactions with diffusion, fluid mechanics, action potential propagation, and flow in porous media (see [12, 32, 33] and the references therein). The celebrated equation $u_t = u_{xx} + u(1-u)$, known as the *Fisher-KPP reaction diffusion equation*, was introduced in seminal works by Fisher [11] and by Kolmogorov, Petrosky and Piskunov [23] as a model to describe the spatial one-dimensional spreading when mutant individuals with higher adaptability appear in populations. Since then, very intensive research has been carried out to extend their model to take into account many other physical, chemical or biological factors. The first reaction-diffusion models of the form (1.1) considered constant diffusion coefficients $D \equiv D_0 > 0$ (cf. [32, 52]). It is now clear, however, that there are situations in which the diffusion coefficient must be a function of the unknown u. For example, in the context of population biology it has been reported (cf. [2, 15, 34]) that motility varies with the population density, requiring density-dependent dispersal coefficients. Such feature has been incorporated into mathematical models in spatial ecology [51], and eukaryotic cell biology [47], to mention a few. Density dependent diffusion functions seem to be particularly important in bacterial aggregation, where motility appears as an increasing function of bacterial density as well as of nutrients or other substrate substances (cf. [4, 22, 24]). A significant example from a different field is the equation $u_t = \Delta(u^m)$, with $m > 0$, which is very well-known in chemical engineering for the description of porous media [33].

An interesting phenomenon occurs when the nonlinearity in the diffusion coefficient is *degenerate*, meaning that diffusion approaches zero when the density does also. Among the new mathematical features one finds that equations with degenerate diffusion possess finite speed of propagation of initial disturbances, in contrast with the strictly parabolic case. Another property is the emergence of traveling waves of "sharp" type (cf. [43, 48]). Reaction-diffusion models with degenerate nonlinear diffusion are widely used nowadays to describe biological phenomena (see, for example, [15, 32] and the references therein).

In all these models, one of the most important mathematical solution types is the traveling front. Traveling fronts (or wave fronts) are solutions to equations (1.1), of the form

$$u(x,t) = \varphi(x - ct),$$

where $c \in \mathbb{R}$ is the speed of the wave and $\varphi : \mathbb{R} \to \mathbb{R}$ is the wave profile function. For pattern formation problems it is natural to consider infinite domains and to neglect the influence of boundary conditions. Thus, these fronts usually have asymptotic limits, $u_\pm = \lim_{\xi \to \pm\infty} \varphi(\xi)$, which are equilibrium points of the reaction function under consideration, $f(u_\pm) = 0$. They are widely used to model, for example, invasions in theoretical ecology [16], the advancing edges of cell populations like growing tumors [49], or the envelope fronts of certain bacterial colonies which extend effectively as one-dimensional fronts [22, 24]. Since the classical work of Kolmogorov *et al.* [23], which set the foundations of their existence theory, traveling waves solutions to equations of the from (1.1) have attracted a great deal of attention. The existence of fronts for reaction-diffusion equations with degenerate diffusion was first studied in particular cases (see, e.g., [3, 36, 37]). The first general existence results for degenerate diffusions satisfying hypotheses (1.2), and generic reaction functions of Fisher-KPP type satisfying (1.4), is due to Sanchez-Garduño and Maini [41, 42]. In these works, the authors prove the existence of a positive threshold speed $c_* > 0$ such that: (i) there exist no traveling fronts with speed $0 < c < c_*$; (ii) there exists a traveling wave of *sharp* type traveling with speed $c = c_*$, with $\varphi(-\infty) = 1$ and $\varphi(\xi) = 0$ for all $\xi \geq \xi_*$ with some $\xi_* \in \mathbb{R}$; and, (iii) there exists a family of smooth monotone decreasing traveling fronts, each of which travels with speed $c > c_*$, and is such that $\varphi(-\infty) = 1$ and $\varphi(+\infty) = 0$ (see Proposition 2.1 below). The present paper pertains to the stability properties of monotone Fisher-KPP degenerate fronts (case (iii) above).

The stability of traveling wave solutions is a fundamental issue. It has been addressed for strictly parabolic reaction-diffusion equations using methods that range from comparison principles for super- and sub-solutions (see, e.g., the pioneer work of Fife and McLeod [10]), linearization techniques and generation of stable semigroups [46], and dynamical systems techniques for PDEs [17], among others. The modern stability theory of nonlinear waves links the functional analysis approach with dynamical systems techniques, by setting a program leading to the spectral stability properties (the analysis of the spectrum of the linearized differential operator around the wave) and their relation to the nonlinear (orbital) stability of the waves under the dynamical viewpoint of the equations of evolution. The reader is referred to the seminal paper by Alexander, Gardner and Jones [1], the review article by Sandstede [45], and the recent book by Kapitula and Promislow [20] for further information. A recent contribution by Meyries *et al.* [30] follows the same methodology and presents rigorous results for quasi-linear systems with density-dependent diffusion tensors which are strictly parabolic (non-degenerate).

In this paper we take a further step in this general stability program by considering degenerate diffusion coefficients. We start with the study of scalar equations and specialize the analysis to the spectral stability of the fronts. The former property, formally defined as the absence of spectra with positive real part of the linearized differential operator around the wave (see Definition 3.3 below), can be seen as a first step of the general program. The degeneracy of the diffusion coefficient in one of the asymptotic limits poses some technical difficulties which are not present in the standard parabolic case. As far as we know, this is the first time that the spectral stability of a degenerate front is addressed in the literature. The contributions of this paper can be summarized as follows.

- Due to the degeneracy of the diffusion at one of the equilibrium points of the reaction, the hyperbolicity of the asymptotic coefficients at one of the end points, which arise when the spectral problem is written in first order form, is lost. This precludes a direct application of the standard methods to locate the essential spectrum of the linearized operator around the front (cf. [20, 30, 45]). To circumvent this difficulty, we propose an equivalent (but *ad hoc*) partition of the spectrum of the linearized operator in the form $\sigma = \sigma_{\mathrm{pt}} \cup \sigma_\delta \cup \sigma_\pi$, where σ_{pt} is the point spectrum, σ_π is a subset of the approximate spectrum, and σ_δ is a subset of the compression spectrum (see Definition 3.1 below for details).

- With the use of energy estimates of solutions to resolvent type equations, we control the point spectrum and the subset σ_π of the approximate spectrum. For that purpose, we introduce a suitable transformation that gets rid of the advection terms, allowing us to close the energy estimate and to locate these subsets of spectra along the non-positive real line. A detailed analysis (included in the Appendix A) of the decay properties of the solutions to resolvent equations is necessary to justify such transformation.
- In order to locate the subset σ_δ of the compression spectrum, we introduce a regularization technique which circumvents the degeneracy of the diffusion at one of the equilibrium points. It is shown that the family of regularized operators converge, in the generalized sense, to the degenerate operator as the regularization parameter tends to zero. This convergence allows us, in turn, to relate the standard Fredholm properties of the regularized operators to those of the original degenerate operator.
- As it is known from the parabolic Fisher-KPP case [17, 20], spectral stability of fronts holds only in exponentially weighted spaces. We thus show that one can choose an appropriate weighted L^2-space (provided that a certain condition on the speed holds) in which degenerate-diffusion fronts are spectrally stable. We profit from the invariance of the point and approximate spectra under conjugation, and from the particular technique to locate σ_δ based on its Fredholm borders.

The main result of this paper can be expressed in the following

Theorem 1.1. *For any monotone traveling front for Fisher-KPP reaction diffusion-degenerate equations* (1.1)*, under hypotheses* (1.2) *and* (1.4)*, and traveling with speed $c \in \mathbb{R}$ satisfying the condition*

$$c > \max\left\{ c_*,\ \frac{f'(0)\sqrt{D(1)}}{\sqrt{f'(0) - f'(1)}} \right\} > 0,$$

there exists an exponentially weighted space $L^2_a(\mathbb{R};\mathbb{C})$, with $a \in \mathbb{R}$, such that the front is L^2_a-spectrally stable. Here $c_* > 0$ denotes the minimum threshold speed (the velocity of the sharp wave).

There exist previous results on the stability of diffusion-degenerate fronts in the literature. In an early paper, Hosono [18] addresses the convergence to traveling fronts for reaction diffusion equations in the "porous medium" form, $u_t = (u^m)_{xx} + f(u)$, with $m > 0$ (that is, for $D(u) = mu^{m-1}$) and reaction function f of Nagumo (or bistable type). His method is based on the construction of super- and sub-solutions to the parabolic problem and the use of the comparison principle. Hosono establishes the asymptotic convergence of solutions to the nonlinear equation to a translated front when the initial data is close to the stationary front profile. We observe that the diffusion coefficient does not satisfy assumptions (1.2); in addition, the method of proof relies heavily on the particular properties of solutions to the porous medium equation. Hosono's paper, however, warrants note as the first work containing a rigorous proof of convergence to a traveling front for reaction diffusion-degenerate equations. For the Fisher-KPP case, Sherratt and Marchant [50] numerically studied the convergence to traveling fronts of solutions with particular initial data in the case of diffusion given by $D(u) = u$. Biró [5] and Medvedev *et al.* [29], for degenerate diffusions of porous medium type and for their generalizations, respectively, employed similar techniques to those introduced by Hosono to show that solutions with compactly supported initial data evolve towards the Fisher-KPP degenerate front with minimum speed $c_* > 0$ (that is, towards the sharp-front wave). These results were extended by Kamin and Rosenau [19] to reaction functions of the form $f(u) = u(1 - u^m)$, with the same porous medium type of diffusion, and with initial data decaying sufficiently fast. So far, no rigorous work on stability of smooth fronts is known.

Our paper differs from the aforementioned works in several ways. On one hand our study focuses on the property of spectral stability, the first step in a general stability program. Consequently, our analysis makes no use of parabolic PDE techniques, pertains to the spectral theory of operators, and could be extrapolated to the systems case. On the other hand, we study the stability of the whole family of smooth fronts with speed $c > c_*$, unlike previous analyses which are restricted to the sharp wave case with $c = c_*$. In addition, we consider the generic class of degenerate diffusion coefficients satisfying (1.2) and introduced by Sánchez-Garduño and Maini. Finally, we conjecture that some of the ideas employed here at the spectral level, and designed to deal with the degeneracy of the diffusion, could be applied to more general situations (see section 7 below for a discussion on this point). This paper is, thus, more related in spirit to the work by Meyries *et al.* [30], by extending their agenda to the case of degenerate diffusions, and it is closer in methodology to the program initiated in [1, 20, 45].

Plan of the paper. In section 2 we briefly review the existence theory of traveling fronts due to Sánchez-Garduño and Maini [41, 40]. We focus on the main structural properties of the waves, such as monotonicity, the lower bound of the speed, as well as its asymptotic behavior. In section 3 we pose the stability problem and introduce the partition of the spectrum suitable for our needs. Section 4 contains the energy estimates which allow us to control the point spectrum and a subset of the appproximate spectrum. Section 5 contains the definition of the (parabolic) regularized operator and the proof of generalized convergence when the regularization parameter tends to zero. In addition, we compute the Fredholm boundaries for the regularized operators and link them to the location of the subset of the compression spectrum for the degenerate operator. Section 6 contains the proof of Theorem 1.1, by choosing appropriate exponentially weighted spaces in which spectral stability does hold. In the last section 7, we make some final remarks. Appendix A contains a detailed analysis of the decay of L^2-solutions to resolvent type equations, which is needed to justify the energy estimates of section 4.

2. Structure of Fisher-KPP diffusion-degenerate fronts

In this section we recall the traveling wave existence theory due to Sánchez-Garduño and Maini [41, 42]. The latter is based on the analysis of the local and global phase portraits of the associated ODE system; another approach, which applies the Conley index to prove the existence of the waves, can be found in [9, 44]. (For existence results for a more general class of equations with advection terms, see Gilding and Kersner [14].)

Let us suppose that $u(x,t) = \varphi(x - ct)$ is a traveling wave solution to (1.1) with speed $c \in \mathbb{R}$. Upon substitution, we find that the profile function $\varphi : \mathbb{R} \to \mathbb{R}$ must be a solution to the equation

$$(D(\varphi)\varphi_\xi)_\xi + c\varphi_\xi + f(\varphi) = 0, \tag{2.1}$$

where $\xi = x - ct$ denotes the translation (Galilean) variable. Let us denote the asymptotic limits of the traveling wave as

$$u_\pm := \varphi(\pm\infty) = \lim_{\xi \to \pm\infty} \varphi(\xi).$$

It is assumed that u_+ and u_- are equilibrium points of the reaction function under consideration. Written as a first order system, equation (2.1) is recast as

$$\begin{aligned} \frac{d\varphi}{d\xi} &= v \\ D(\varphi)\frac{dv}{d\xi} &= -cv - D'(\varphi)v^2 - f(\varphi). \end{aligned} \tag{2.2}$$

Notice that, due to the degenerate diffusion, this system is degenerate at $\varphi = 0$. Aronson [2] overcomes the singularity by introducing the parameter $\tau = \tau(\xi)$, such that

$$\frac{d\tau}{d\xi} = \frac{1}{D(\varphi(\xi))},$$

and, therefore, the system is transformed into

$$\begin{aligned} \frac{d\varphi}{d\tau} &= D(\varphi)v \\ \frac{dv}{d\tau} &= -cv - D'(\varphi)v^2 - f(\varphi). \end{aligned} \tag{2.3}$$

Heteroclinic trajectories of both systems are equivalent, so the analysis focuses on the study of the topological properties of equilibria for system (2.3), which depend upon the reaction function f. The existence of monotone traveling wave solutions is summarized in the following

Proposition 2.1 (monotone degenerate Fisher-KPP fronts [41])**.** *If the function $D = D(u)$ satisfies* (1.2) *and $f = f(u)$ is of Fisher-KPP type satisfying* (1.4)*, then there exists a unique speed value $c_* > 0$ such that equation* (1.1) *has a monotone decreasing traveling front for each $c > c_*$, with*

$$u_+ = \varphi(+\infty) = 0, \qquad u_- = \varphi(-\infty) = 1,$$

and $\varphi_\xi < 0$ for all $\xi \in \mathbb{R}$. Each front is diffusion degenerate at $u_+ = 0$, as $\xi \to +\infty$.

Notice that this is an infinite family of fronts parametrized by the speed $c > c_*$ connecting the equilibrium points $u_+ = 0$ and $u_- = 1$. The fronts are diffusion degenerate in the sense that the diffusion coefficient vanishes at one of the equilibrium points, in this case, at $u_+ = 0$.

Remark 2.2. Theorem 2 in [41] guarantees the absence of traveling wave solutions when $0 < c < c_*$, as well as the existence of traveling waves of "sharp" type when $c = c_*$. The latter are not considered in the present analysis. See [42, 43] for further information.

As a by-product of the analysis in [41], one can explicitly determine the asymptotic behavior of the traveling fronts as $\xi \to \pm\infty$. This information will be useful later on.

Lemma 2.3 (asymptotic decay)**.** *Let $\varphi = \varphi(\xi)$ a monotone decreasing Fisher-KPP degenerate front, traveling with speed $c > c_* > 0$, and with $u_+ = 0$, $u_- = 1$. Then φ behaves asymptotically as*

$$|\partial_\xi^j(\varphi - u_+)| = |\partial_\xi^j \varphi| = O(e^{-f'(0)\xi/c}), \quad \textit{as } \ \xi \to +\infty, \ j = 0, 1,$$

on the degenerate side; and as,

$$|\partial_\xi^j(\varphi - u_-)| = |\partial_\xi^j(\varphi - 1)| = O(e^{\eta\xi}), \quad \textit{as } \ \xi \to -\infty, \ j = 0, 1,$$

on the non-degenerate side, with $\eta = (2D(1))^{-1}(-c + \sqrt{c^2 - 4D(1)f'(1)}) > 0$.

Proof. To verify the exponential decay on the the non-degenerate side as $\xi \to -\infty$, note that the end point $u_- = 1$ is not diffusion-degenerate and, therefore, the equilibrium point $P_1 = (1, 0)$ for system (2.2) is hyperbolic. Indeed, writing (2.2) as $\partial_\xi(\varphi, v) = F(\varphi, v)$, then the linearization around P_1 is given by

$$DF_{|(1,0)} = \begin{pmatrix} 0 & 1 \\ -f'(1)/D(1) & -c/D(1) \end{pmatrix},$$

with positive eigenvalue $\eta = (2D(1))^{-1}(-c + \sqrt{c^2 - 4D(1)f'(1)}) > 0$. The exponential decay as $\xi \to -\infty$ follows by standard ODE estimates around a hyperbolic rest point.

To verify the exponential decay on the degenerate side, notice that $P_0 = (0, 0)$ is a non-hyperbolic point for system (2.3) for all admissible values of the speed $c > c_*$, and we need higher order terms to approximate the trajectory along a center manifold. Let us denote the latter as $v = h(\varphi)$; after an application of the center manifold theorem, we find that P_0 is locally a saddle-node, and the center manifold has the form

$$h(\varphi) = -\frac{f'(0)}{c}\varphi - \frac{1}{2c^3}\big(f''(0)c^2 + 4D'(0)f'(0)^2\big)\varphi^2 + O(\varphi^3),$$

for $\varphi \sim 0$ (see [41] for details). The trajectory leaves the saddle-node along the center manifold for $\varphi \sim 0$. Therefore, for $\xi \to +\infty$, the trajectory behaves as

$$\varphi_\xi = h(\varphi) \approx -\frac{f'(0)}{c}\varphi \le 0,$$

yielding

$$\varphi = O(e^{-f'(0)\xi/c}), \quad \text{as } \ \xi \to +\infty.$$

This proves the result. □

Remark 2.4. We finish this section by observing that, due to their asymptotic decay, the monotone fronts satisfy $\varphi_\xi \in L^2(\mathbb{R})$ ($\varphi_\xi \to 0$ as $\xi \to \pm\infty$ fast enough). Upon substitution into system (2.2) and by a bootstrapping argument, it can be verified that $\varphi_\xi \in H^2(\mathbb{R})$ for all monotone fronts under consideration. We omit the details.

3. The stability problem

3.1. **Perturbation equations.** Suppose that $u(x, t) = \varphi(x - ct)$ is a monotone traveling front solution to the diffusion degenerate Fisher-KPP equation (1.1), traveling with speed $c > c_* > 0$. With a slight abuse of notation we make the change of variables $x \to x - ct$, where now x denotes the Galilean variable of translation. We shall keep this notation for the rest of the paper. In the new coordinates, equation (1.1) reads

$$u_t = (D(u)u_x)_x + cu_x + f(u), \tag{3.1}$$

for which traveling fronts are stationary solutions, $u(x,t) = \varphi(x)$, satisfying,

$$(D(\varphi)\varphi_x)_x + c\varphi_x + f(\varphi) = 0. \tag{3.2}$$

The front connects asymptotic equilibrium points of the reaction: $\varphi(x) \to u_\pm$ as $x \to \pm\infty$, where $u_+ = 0$, $u_- = 1$, and the front is monotone decreasing $\varphi_x < 0$.

Let us consider solutions to (3.1) of the form $\varphi(x) + u(x,t)$, where now u denotes a perturbation. Substituting we obtain the nonlinear perturbation equation

$$u_t = (D(\varphi + u)(\varphi + u)_x)_x + cu_x + c\varphi_x + f(u + \varphi).$$

Linearizing around the front and using the profile equation (3.2) we get

$$u_t = (D(\varphi)u)_{xx} + cu_x + f'(\varphi)u. \tag{3.3}$$

The right hand side of equation (3.3), regarded as a linear operator acting on an appropriate Banach space X, naturally defines the spectral problem

$$\lambda u = \mathcal{L}u, \tag{3.4}$$

where $\lambda \in \mathbb{C}$ is the spectral parameter, and

$$\begin{aligned} &\mathcal{L} : \mathcal{D}(\mathcal{L}) \subset X \to X, \\ &\mathcal{L}u = (D(\varphi)u)_{xx} + cu_x + f'(\varphi)u, \end{aligned} \tag{3.5}$$

is the linearized operator around the wave, acting on X with domain $\mathcal{D}(\mathcal{L}) \subset X$.

Formally, a necessary condition for the front to be stable is that there are no solutions $u \in X$ to equation (3.4) for $\operatorname{Re}\lambda \geq 0$ and $\lambda \neq 0$, precluding the existence of solutions to the linear equation (3.3) of the form $e^{\lambda t}u$ that grow exponentially in time. This condition is known as *spectral stability*, which we define rigorously below.

3.2. **Resolvent and spectra.** We shall define a particular partition of spectrum suitable for our needs. Let X and Y be Banach spaces, and let $\mathscr{C}(X,Y)$ and $\mathscr{B}(X,Y)$ denote the sets of all closed and bounded linear operators from X to Y, respectively. For any $\mathcal{L} \in \mathscr{C}(X,Y)$ we denote its domain as $\mathcal{D}(\mathcal{L}) \subseteq X$ and its range as $\mathcal{R}(\mathcal{L}) := \mathcal{L}(\mathcal{D}(\mathcal{L})) \subseteq Y$. We say $\mathcal{L}$ is densely defined if $\overline{\mathcal{D}(\mathcal{L})} = X$.

Definition 3.1. Let $\mathcal{L} \in \mathscr{C}(X,Y)$ be a closed, densely defined operator. Its *resolvent* $\rho(\mathcal{L})$ is defined as the set of all complex numbers $\lambda \in \mathbb{C}$ such that $\mathcal{L} - \lambda$ is injective, $\mathcal{R}(\mathcal{L} - \lambda) = Y$, and $(\mathcal{L} - \lambda)^{-1}$ is bounded. Its *spectrum* is defined as $\sigma(\mathcal{L}) := \mathbb{C}\backslash\rho(\mathcal{L})$. Furthermore, we also define the following subsets of the complex plane:

$$\begin{aligned} \sigma_{\mathrm{pt}}(\mathcal{L}) :=& \{\lambda \in \mathbb{C} \, : \, \mathcal{L} - \lambda \text{ is not injective}\}, \\ \sigma_{\delta}(\mathcal{L}) :=& \{\lambda \in \mathbb{C} \, : \, \mathcal{L} - \lambda \, \text{ is injective, } \mathcal{R}(\mathcal{L} - \lambda) \text{ is closed, and } \mathcal{R}(\mathcal{L} - \lambda) \neq Y\}, \\ \sigma_{\pi}(\mathcal{L}) :=& \{\lambda \in \mathbb{C} \, : \, \mathcal{L} - \lambda \, \text{ is injective, and } \mathcal{R}(\mathcal{L} - \lambda) \text{ is not closed}\}. \end{aligned}$$

The set $\sigma_{\mathrm{pt}}(\mathcal{L})$ is called the *point spectrum* and its elements, *eigenvalues*. Clearly, $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L})$ if and only if there exists $u \in \mathcal{D}(\mathcal{L})$, $u \neq 0$, such that $\mathcal{L}u = \lambda u$.

Remark 3.2. First, notice that the sets $\sigma_{\mathrm{pt}}(\mathcal{L})$, $\sigma_\pi(\mathcal{L})$ and $\sigma_\delta(\mathcal{L})$ are clearly disjoint and, since $\mathcal{L}$ is closed, that

$$\sigma(\mathcal{L}) = \sigma_{\mathrm{pt}}(\mathcal{L}) \cup \sigma_\pi(\mathcal{L}) \cup \sigma_\delta(\mathcal{L}).$$

Indeed, in the general case it could happen that, for a certain $\lambda \in \mathbb{C}$, $\mathcal{L} - \lambda$ is invertible but $(\mathcal{L} - \lambda)^{-1}$ is not bounded. But this pathology never occurs if the operator is closed: for any $\mathcal{L} \in \mathscr{C}(X, Y)$ with $\mathcal{R}(\mathcal{L}) = Y$, if $\mathcal{L}$ is invertible then $\mathcal{L}^{-1} \in \mathscr{B}(Y, X)$ (cf. [21], pg. 167).

There are different partitions of the spectrum for an unbounded operator besides the classical definition of continuous, residual and point spectra (cf. [8]). For instance, the set $\sigma_\pi(\mathcal{L})$ is contained in the *approximate spectrum*, defined as

$$\begin{aligned}\sigma_\pi(\mathcal{L}) \subset \sigma_{\mathrm{app}}(\mathcal{L}) :=& \{\lambda \in \mathbb{C} \,:\, \text{for each } n \in \mathbb{N} \text{ there exists } u_n \in \mathcal{D}(\mathcal{L}) \text{ with } \|u_n\| = 1 \\ & \text{such that } (\mathcal{L} - \lambda)u_n \to 0 \text{ in } Y \text{ as } n \to +\infty\}.\end{aligned}$$

The inclusion follows from the fact that, for any $\lambda \in \sigma_\pi(\mathcal{L})$, the range of $\mathcal{L} - \lambda$ is not closed and, therefore, there exists a *singular sequence*, $u_n \in \mathcal{D}(\mathcal{L})$, $\|u_n\| = 1$ such that $(\mathcal{L} - \lambda)u_n \to 0$, which contains no convergent subsequence (see Theorems 5.10 and 5.11 in Kato [21], pg. 233). We shall make use of this property of the set σ_π later on. It is clear that that $\sigma_{\mathrm{pt}}(\mathcal{L}) \subset \sigma_{\mathrm{app}}(\mathcal{L})$, as well.

The set $\sigma_\delta(\mathcal{L})$ is clearly contained in what is often called the *compression spectrum* [39] (or *surjective spectrum* [31]):

$$\sigma_\delta(\mathcal{L}) \subset \sigma_{\mathrm{com}}(\mathcal{L}) := \{\lambda \in \mathbb{C} \,:\, \mathcal{L} - \lambda \ \text{ is injective, and } \overline{\mathcal{R}(\mathcal{L} - \lambda)} \neq Y\}.$$

(Since $\overline{\mathcal{R}(\mathcal{L} - \lambda)} \neq Y$ it is said that the range has been compressed.)

Our partition of spectrum splits the classical residual and continuous spectra into two disjoint components, making a distinction between points for which the range of $\mathcal{L} - \lambda$ is closed and those for which it is not. These properties make this partition suitable for analyzing the stability of the spectrum of the diffusion-degenerate operator (3.5), as we shall see.

Definition 3.3. We say the traveling front φ is X-*spectrally stable* if

$$\sigma(\mathcal{L}) \subset \{\lambda \in \mathbb{C} \,:\, \operatorname{Re} \lambda \leq 0\}.$$

In this paper we shall consider $X = L^2(\mathbb{R}; \mathbb{C})$ and $\mathcal{D}(\mathcal{L}) = H^2(\mathbb{R}; \mathbb{C})$, so that $\mathcal{L}$ is a closed, densely defined operator acting on L^2. In this fashion, the stability analysis of the operator $\mathcal{L}$ pertains to *localized* perturbations. In what follows, $\sigma(\mathcal{L})$ will denote the L^2-spectrum of the linearized operator (3.5) with domain $\mathcal{D} = H^2$, except where it is otherwise computed with respect to a space X and explicitly denoted as $\sigma(\mathcal{L})_{|X}$.

We recall that $\lambda = 0$ always belongs to the point spectrum (translation eigenvalue), inasmuch as

$$\mathcal{L}\varphi_x = \partial_x\big((D(\varphi)\varphi_x)_x + c\varphi_x + f(\varphi)\big) = 0,$$

in view of the profile equation (3.2) and the fact that $\varphi_x \in H^2(\mathbb{R}) = \mathcal{D}(\mathcal{L})$. Thus, φ_x is the eigenfunction associated to the eigenvalue $\lambda = 0$.

Finally, we remind the reader that an operator $\mathcal{L} \in \mathscr{C}(X, Y)$ is said to be Fredholm if its range $\mathcal{R}(\mathcal{L})$ is closed, and both its nullity, $\operatorname{nul} \mathcal{L} = \dim \ker \mathcal{L}$, and its deficiency, $\operatorname{def} \mathcal{L} = \operatorname{codim} \mathcal{R}(\mathcal{L})$, are finite. $\mathcal{L}$ is said to be semi-Fredholm if $\mathcal{R}(\mathcal{L})$ is closed and at least one of $\operatorname{nul} \mathcal{L}$ and $\operatorname{def} \mathcal{L}$ is finite. In both cases the index of $\mathcal{L}$ is defined as $\operatorname{ind} \mathcal{L} = \operatorname{nul} \mathcal{L} - \operatorname{def} \mathcal{L}$ (cf. [21]).

Remark 3.4. For nonlinear wave stability purposes (cf. [20]), the spectrum is often partitioned into essential, $\sigma_{\rm ess}$, and point spectrum, $\widetilde{\sigma}_{\rm pt}$, being the former the set of complex numbers λ for which $\mathcal{L}-\lambda$ is either not Fredholm or has index different from zero, whereas $\widetilde{\sigma}_{\rm pt}$ is defined as the set of complex numbers for which $\mathcal{L}-\lambda$ is Fredholm with index zero and has a non-trivial kernel. Note that $\widetilde{\sigma}_{\rm pt} \subset \sigma_{\rm pt}$. This definition is due to Weyl [53], making $\sigma_{\rm ess}$ a large set but easy to compute, whereas $\widetilde{\sigma}_{\rm pt}$ is a discrete set of isolated eigenvalues (see Remark 2.2.4 in [20]). In the present context with degenerate diffusion, however, this partition is not particularly useful due to the loss of hyperbolicity of the asymptotic coefficients of the operator.

4. Energy estimates and stability of approximate spectra

In this section we show that monotone diffusion-degenerate Fisher-KPP fronts are point- and approximate-spectrally stable. For that purpose we employ energy estimates and an appropriate change of variables. The monotonicity of the front plays a key role.

4.1. **The basic energy estimate.** Let $\varphi = \varphi(x)$ be any of these monotone fronts and let $\mathcal{L}$ be the corresponding linearized operator around φ, acting on L^2. Take any fixed $\lambda \in \mathbb{C}$, any $g \in L^2(\mathbb{R};\mathbb{C})$, and assume that there exists a solution $u \in \mathcal{D}(\mathcal{L}) = H^2(\mathbb{R};\mathbb{C})$ satisfying the resolvent equation $(\mathcal{L}-\lambda)u = g$, which we write down as

$$(\mathcal{L}-\lambda)u = D(\varphi)u_{xx} + (2D(\varphi)_x + c)u_x + (D(\varphi)_{xx} + f'(\varphi) - \lambda)u = g. \tag{4.1}$$

Consider the change of variables

$$u(x) = w(x)e^{\theta(x)}, \tag{4.2}$$

where $\theta = \theta(x)$ satisfies

$$\theta_x = -\frac{c}{2D(\varphi)}, \qquad \text{for all } x \in \mathbb{R}.$$

Upon substitution we arrive at the equation

$$D(\varphi)w_{xx} + 2D(\varphi)_x w_x + H(x)w - \lambda w = e^{-\theta}g, \tag{4.3}$$

where

$$\begin{aligned} H(x) &= D(\varphi)\theta_x^2 + D(\varphi)\theta_{xx} + (2D(\varphi)_x + c)\theta_x + D(\varphi)_{xx} + f'(\varphi) \\ &= -\frac{c}{2}\frac{D(\varphi)_x}{D(\varphi)} - \frac{c^2}{4D(\varphi)} + D(\varphi)_{xx} + f'(\varphi). \end{aligned}$$

Clearly, the function θ has the form

$$\theta(x) = -\frac{c}{2}\int_{x_0}^{x} \frac{ds}{D(\varphi(s))}\,, \tag{4.4}$$

for any fixed $x_0 \in \mathbb{R}$, and it is well defined for all $x \in \mathbb{R}$. Note that $e^{-\theta(x)}$ may diverge as $x \to +\infty$. It is true, however, that $w \in H^2$ whenever $u \in H^2$.

Lemma 4.1. *If $u \in H^2(\mathbb{R};\mathbb{C})$ is a solution to the resolvent equation $(\mathcal{L}-\lambda)u = g$, for fixed $\lambda \in \mathbb{C}$ and $g \in L^2(\mathbb{R};\mathbb{C})$, then*

$$w(x) = \exp\left(\frac{c}{2}\int_{x_0}^{x} \frac{ds}{D(\varphi(s))}\right)u(x),$$

belongs to $H^2(\mathbb{R};\mathbb{C})$. Here $x_0 \in \mathbb{R}$ is fixed but arbitrary.

Proof. See Appendix A. □

In view that $\varphi_x \in \ker \mathcal{L}$, substitute $\lambda = 0$ and $g = 0$ into (4.3) to obtain

$$D(\varphi)\psi_{xx} + 2D(\varphi)_x\psi_x + H(x)\psi = 0, \tag{4.5}$$

where $\psi := e^{-\theta}\varphi_x \in H^2$ (by Lemma 4.1). Multiply equations (4.3) and (4.5) by $D(\varphi)$ and rearrange the terms appropriately; the result is

$$\begin{aligned} (D(\varphi)^2 w_x)_x + D(\varphi)H(x)w - \lambda D(\varphi)w = D(\varphi)e^{-\theta}g,\\ (D(\varphi)^2\psi_x)_x + D(\varphi)H(x)\psi = 0. \end{aligned} \tag{4.6}$$

Since the front is monotone, $\varphi_x < 0$, we have that $\psi \neq 0$ for all $x \in \mathbb{R}$. Therefore, we substitute

$$D(\varphi)H(x) = -\frac{(D(\varphi)^2\psi_x)_x}{\psi}$$

into the first equation in (4.6) to obtain

$$(D(\varphi)^2 w_x)_x - \frac{(D(\varphi)^2\psi_x)_x}{\psi} w - \lambda D(\varphi)w = D(\varphi)e^{-\theta}g. \tag{4.7}$$

Take the L^2-product of w with last equation, and integrate by parts. This yields,

$$\begin{aligned} \lambda \int_{\mathbb{R}} D(\varphi)|w|^2 dx + \int_{\mathbb{R}} D(\varphi)w^* e^{-\theta} g\, dx &= \int_{\mathbb{R}} w^*(D(\varphi)^2 w_x)_x dx - \int_{\mathbb{R}} \psi^{-1}(D(\varphi)^2\psi_x)_x |w|^2 dx\\ &= -\int_{\mathbb{R}} D(\varphi)^2 |w_x|^2 dx + \int_{\mathbb{R}} D(\varphi)^2\psi_x \left(\frac{|w|^2}{\psi}\right)_x dx\\ &= \int_{\mathbb{R}} D(\varphi)^2 \left(\psi_x \left(\frac{|w|^2}{\psi}\right)_x - |w_x|^2\right) dx. \end{aligned}$$

Using the identity

$$\psi^2 \left|\left(\frac{w}{\psi}\right)_x\right|^2 = -\left(\psi_x \left(\frac{|w|^2}{\psi}\right)_x - |w_x|^2\right),$$

and substituting, we obtain the estimate

$$\lambda \int_{\mathbb{R}} D(\varphi)|w|^2 dx + \int_{\mathbb{R}} D(\varphi)w^* e^{-\theta} g\, dx = -\int_{\mathbb{R}} D(\varphi)^2\psi^2 \left|\left(\frac{w}{\psi}\right)_x\right|^2 dx. \tag{4.8}$$

Let us denote, according to custom, the standard L^2-product as

$$\langle u, v\rangle_{L^2} = \int_{\mathbb{R}} u^* v\, dx, \qquad \|u\|_{L^2}^2 = \langle u, u\rangle_{L^2}.$$

Hence, we can write relation (4.8) as

$$\lambda\langle D(\varphi)w, w\rangle_{L^2} = -\|D(\varphi)\psi(w/\psi)_x\|_{L^2}^2 - \langle D(\varphi)we^{-\theta}, g\rangle_{L^2}.$$

Remark 4.2. Notice that, thanks to Lemma 4.1, $w \in H^2$, $\psi \in H^2$, and by monotonicity, $\psi < 0$, so that the L^2-products of last equation are well-defined, even the last product involving g. This last assertion holds because $D(\varphi)we^{-\theta}$ belongs to L^2, in view of the exponential decay of w at $+\infty$ as stated in Lemma A.1 (see also Remark A.3).

We summarize these calculations into the following

Proposition 4.3 (basic energy estimate)**.** *For any $\lambda \in \mathbb{C}$ and any $g \in L^2(\mathbb{R};\mathbb{C})$, suppose that there exists a solution $u \in H^2(\mathbb{R};\mathbb{C})$ to the resolvent equation $(\mathcal{L} - \lambda)u = g$. Then there holds the energy estimate*

$$\lambda \langle D(\varphi)w, w\rangle_{L^2} = -\|D(\varphi)\psi(w/\psi)_x\|_{L^2}^2 - \langle D(\varphi)we^{-\theta}, g\rangle_{L^2}, \tag{4.9}$$

where $w = e^{-\theta}u \in H^2(\mathbb{R};\mathbb{C})$, $\psi = e^{-\theta}\varphi_x \in H^2(\mathbb{R})$ is a non-vanishing real function, and $\theta = \theta(x)$ is defined in (4.4)*.*

Remark 4.4. It is to be observed that the monotonicity of the front is crucial to obtain the resolvent type equation (4.7), and consequently, the basic estimate (4.9). In addition, the transformation (4.2) is designed to eliminate the transport term cu_x in (4.1), while keeping the term $2D(\varphi)_x u_x$. A transformation that eliminates all the first order terms is not useful to close the energy estimate, as the dedicated reader may verify.

4.2. **Point spectral stability.** The first application of the energy estimate is the following

Theorem 4.5. *All monotone fronts of diffusion degenerate Fisher-KPP equations are point spectrally stable. More precisely,*

$$\sigma_{\mathrm{pt}}(\mathcal{L}) \subset (-\infty, 0] \tag{4.10}$$

that is, the L^2-point spectrum is real and non-positive.

Proof. Let $\varphi = \varphi(x)$ be a degenerate Fisher-KPP monotone front, and let $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L})$. Therefore, there exists $u \in H^2(\mathbb{R};\mathbb{C})$ such that $(\mathcal{L} - \lambda)u = 0$. Since $D(\varphi) \geq 0$ for all $x \in \mathbb{R}$, the corresponding energy estimate (4.9) with $g = 0$,

$$\lambda \langle D(\varphi)w, w\rangle_{L^2} = -\|D(\varphi)\psi(w/\psi)_x\|_{L^2}^2 \leq 0, \tag{4.11}$$

shows that λ is real and non-positive. □

Corollary 4.6. *$\lambda = 0$ has geometric multiplicity equal to one, that is,* $\ker \mathcal{L} = \mathrm{span}\{\varphi_x\}$*.*

Proof. If we suppose that $\lambda = 0$, then estimate (4.11) yields

$$\left(\frac{w}{\psi}\right)_x = 0, \quad \text{a.e. in } \mathbb{R},$$

that is, $w = \beta\psi$ for some scalar β, which implies, in turn, that $u = \beta\varphi_x$. This shows that $\lambda = 0$ has geometric multiplicity equal to one. □

4.3. **Stability of the approximate spectrum.** Next, we show that all elements in $\sigma_\pi(\mathcal{L})$ are real and non-positive, as elements of the approximate spectrum.

Lemma 4.7. *Let $\varphi = \varphi(x)$ be a degenerate Fisher-KPP monotone front and $\mathcal{L}$ the linearized operator around φ defined in* (3.5)*. Then*

$$\sigma_\pi(\mathcal{L}) \subset (-\infty, 0].$$

Proof. Take $\lambda \in \sigma_\pi(\mathcal{L})$. Then by definition $\mathcal{R}(\mathcal{L} - \lambda)$ is not closed. Since $L^2(\mathbb{R};\mathbb{C})$ is a reflexive Hilbert space, then it is known (cf. Theorems 5.10 and 5.11 in [21], pg. 233, and [8], pg. 415) that there exists a singular sequence $u_n \in \mathcal{D}(\mathcal{L}) = H^2$, $n \in \mathbb{N}$, such that $\|u_n\|_{L^2} = 1$, $u_n \rightharpoonup 0$ in L^2, and

$$g_n := (\mathcal{L} - \lambda)u_n \to 0, \quad \text{in } L^2.$$

By Lemma 4.1, $w_n := e^{-\theta} u_n$ belongs to H^2, and by the basic energy estimate (4.9) we have that

$$\lambda \langle D(\varphi) w_n, w_n \rangle_{L^2} = -\|D(\varphi)\psi (w_n/\psi)_x\|_{L^2}^2 - \langle D(\varphi) w_n e^{-\theta}, g_n \rangle_{L^2}, \tag{4.12}$$

where $\psi = e^{-\theta}\varphi_x$ as before.

We claim that there exists a uniform $C_0 > 0$ such that

$$\langle D(\varphi) w_n, w_n \rangle_{L^2} \geq C_0 > 0,$$

for all $n \in \mathbb{N}$. Since $x_0 \in \mathbb{R}$ in the definition of θ is fixed but arbitrary, we set

$$\delta_0 := \inf_{x \in [-x_0, x_0]} D(\varphi(x)) = \min_{u \in I_0} D(u) > 0.$$

Since φ is monotone, here I_0 denotes the compact interval

$$I_0 = [\min\{\varphi(x_0), \varphi(-x_0)\}, \max\{\varphi(x_0), \varphi(-x_0)\}].$$

$\delta_0 > 0$ because D is strictly positive and of class C^2 in any compact interval $I_0 \subset\subset (0,1)$. It is clear that we can fix $x_0 \gg 1$, sufficiently large, such that $\|u_n\|_{L^2(-x_0,x_0)}^2 \geq 1/2$ for all $n \in \mathbb{N}$ (otherwise, if we suppose that for each $R > 0$ there exists $N = N(R) \in \mathbb{N}$ with $\|u_N\|_{L^2(-R,R)}^2 < 1/2$, then by taking the limit when $R \to +\infty$ we obtain a contradiction with $\|u_n\|_{L^2} = 1$ for all $n \in \mathbb{N}$). Therefore, we can estimate

$$\begin{aligned}
\langle D(\varphi) w_n, w_n \rangle_{L^2} &= \int_{\mathbb{R}} D(\varphi) \exp\Big(c \int_{x_0}^{x} D(\varphi(y))^{-1}\,dy\Big) |u_n(x)|^2\,dx \\
&\geq \int_{-x_0}^{x_0} D(\varphi) \exp\Big(-c \int_{x}^{x_0} D(\varphi(y))^{-1}\,dy\Big) |u_n(x)|^2\,dx \\
&\geq \int_{-x_0}^{x_0} D(\varphi) e^{-2cx_0/\delta_0} |u_n(x)|^2\,dx \\
&\geq \delta_0 e^{-2cx_0/\delta_0} \|u_n\|_{L^2(-x_0,x_0)}^2 \\
&\geq \frac{\delta_0}{2} e^{-2cx_0/\delta_0} =: C_0 > 0,
\end{aligned}$$

as claimed.

Let us denote $v_n := D(\varphi) w_n e^{-\theta}$. It can be verified that, for each $n \in \mathbb{N}$, $v_n \in L^2$ (see Remark 4.2). Since $\|u_n\|_{L^2} = 1$ for all n, we claim that there exists a uniform constant $C_1 > 0$ such that $\|v_n\|_{L^2} \leq C_1$ for all $n \in \mathbb{N}$. On the non-degenerate side, as $x \to -\infty$, there holds $|v_n| = |D(\varphi) u_n e^{-2\theta}| \leq C|u_n|$ for all $-\infty < x < -R < -x_0$, $R > |x_0|$, so that

$$\|v_n\|_{L^2(-\infty,-R)} \leq C\|u_n\|_{L^2(-\infty,-R)} \leq C\|u_n\|_{L^2} = C,$$

for some uniform $C > 0$ and all n. Also, we clearly have

$$\|v_n\|_{L^2(-R,R)} \leq C(R)\|u_n\|_{L^2(-R,R)} \leq C(R)\|u_n\|_{L^2} = C(R),$$

for all n. On the degenerate side, if we choose $R \gg |x_0|$ large enough then we know that for $x > R$, u_n decays as $|u_n(x)| \leq Ce^{\theta(x)}\zeta(x)$, where the decay of ζ is given by (A.2) (see Appendix A) as $x \to +\infty$. By Remark A.3,

$$|D(\varphi) w_n e^{-\theta}| = C_0 |D(\varphi)\zeta e^{-\theta}| \leq \bar{C} e^{-f'(0)x/2c} \to 0,$$

as $x \to +\infty$, for uniform $C_0, \bar{C} > 0$. Notice that ζ depends on n via g_n and u_n. The decay rate, however, is independent of $n \in \mathbb{N}$, because the constant $\bar{C}$ is of order $O(\|g_n\|_L^2)$ (see (A.10) in the Appendix), which is uniformly bounded in n because

$g_n \to 0$ in L^2 as $n \to +\infty$, and because $\|u_n\|_{L^2} = 1$ by hypothesis. Therefore, for $R \gg |x_0|$ sufficiently large, $\|v_n\|_{L^2(R,+\infty)} \leq C$, uniformly for all $n \in \mathbb{N}$. We conclude that there exists some uniform $C_1 > 0$ such that

$$\|v_n\|_{L^2} \leq C_1, \qquad \text{all } n \in \mathbb{N}.$$

We now take the real part of (4.12) to obtain

$$\begin{aligned} C_0 \mathrm{Re}\,\lambda &\leq (\mathrm{Re}\,\lambda)\langle D(\varphi) w_n, w_n\rangle_{L^2} \\ &= -\|D(\varphi)\psi(w_n/\psi)_x\|_{L^2}^2 - \mathrm{Re}\,\langle D(\varphi) w_n e^{-\theta}, g_n\rangle_{L^2} \\ &\leq -\mathrm{Re}\,\langle v_n, g_n\rangle_{L^2} \\ &\leq C_1 \|g_n\|_{L^2}, \end{aligned}$$

Hence,

$$\mathrm{Re}\,\lambda \leq C_1 C_0^{-1} \|g_n\|_{L^2}.$$

Taking the limit when $n \to +\infty$ we obtain $\mathrm{Re}\,\lambda \leq 0$. Likewise, take the imaginary part of (4.12) to get

$$|\mathrm{Im}\,\lambda| \langle D(\varphi) w_n, w_n\rangle_{L^2} = |\mathrm{Im}\,\langle D(\varphi) w_n e^{-\theta}, g_n\rangle_{L^2}| = |\mathrm{Im}\,\langle v_n, g_n\rangle_{L^2}| \leq C_1 \|g_n\|_{L^2}.$$

This yields,

$$0 \leq |\mathrm{Im}\,\lambda| \leq C_1 C_0^{-1} \|g_n\|_{L^2} \to 0,$$

as $n \to +\infty$. We conclude that $\lambda \in \mathbb{R}$ and $\lambda \leq 0$. ☐

5. Parabolic regularization and location of σ_δ

In this section we introduce a regularization technique that allows us to locate the subset of the compression spectrum, σ_δ, of the linearized operator around the wave. The method relies on the convergence in the generalized sense of the regularized operators.

5.1. **The regularized operator.** Let $\varphi = \varphi(x)$ be a diffusion-degenerate monotone Fisher-KPP front. Then, for any $\epsilon > 0$, we introduce the following regularization of the diffusion coefficient,

$$D^\epsilon(\varphi) := D(\varphi) + \epsilon. \tag{5.1}$$

Note that $D^\epsilon(\varphi) > 0$ for all $x \in \mathbb{R}$. Likewise, we also define the following regularized operator

$$\begin{aligned} \mathcal{L}^\epsilon \,:\, \mathcal{D} = H^2(\mathbb{R};\mathbb{C}) \subset L^2(\mathbb{R};\mathbb{C}) \,\to\, L^2(\mathbb{R};\mathbb{C}), \\ \mathcal{L}^\epsilon u := (D^\epsilon(\varphi) u)_{xx} + c u_x + f'(\varphi) u. \end{aligned} \tag{5.2}$$

Notice that, for every $\epsilon > 0$, $\mathcal{L}^\epsilon$ is a linear, closed, densely defined and *strongly elliptic* operator acting on L^2. Hence, multiplication by $D^\epsilon(\varphi)^{-1}$ is an isomorphism and the Fredholm properties of $\mathcal{L}^\epsilon - \lambda$ and those of the operator $\mathcal{J}^\epsilon(\lambda) : \mathcal{D} \to L^2(\mathbb{R};\mathbb{C})$, defined as

$$\begin{aligned} \mathcal{J}^\epsilon(\lambda) u &:= D^\epsilon(\varphi)^{-1}(\mathcal{L}^\epsilon - \lambda) u \\ &= u_{xx} + D^\epsilon(\varphi)^{-1} a_1(x) u_x + D^\epsilon(\varphi)^{-1}(a_0(x) - \lambda) u, \end{aligned} \tag{5.3}$$

for all $u \in H^2 \subset L^2$, are the same. Here the coefficients a_0 and a_1 are given by

$$a_1(x) = 2D^\epsilon(\varphi)_x + c, \qquad a_0(x) = D^\epsilon(\varphi)_{xx} + f'(\varphi).$$

Following Alexander, Gardner and Jones [1], it is now customary to recast the spectral problem (5.3) as a first order system of the form

$$W_x = \mathbb{A}^\epsilon(x,\lambda)W, \tag{5.4}$$

where

$$\mathbb{A}^\epsilon(x,\lambda) = \begin{pmatrix} 0 & 1 \\ D^\epsilon(\varphi)^{-1}(\lambda - a_0(x)) & -D^\epsilon(\varphi)^{-1}a_1(x)\end{pmatrix}, \qquad W = \begin{pmatrix} u \\ u_x \end{pmatrix} \in H^1(\mathbb{R};\mathbb{C}^2).$$

It is a well-known fact [20, 45] that the associated first order operators

$$\mathcal{T}^\epsilon(\lambda) = \partial_x - \mathbb{A}^\epsilon(\cdot,\lambda), \qquad \mathcal{T}^\epsilon(\lambda) : H^1(\mathbb{R};\mathbb{C}^2) \subset L^2(\mathbb{R};\mathbb{C}^2) \to L^2(\mathbb{R};\mathbb{C}),$$

are endowed with the same Fredholm properties as $\mathcal{J}^\epsilon(\lambda)$ and, consequently, as $\mathcal{L}^\epsilon - \lambda$; see, e.g., Theorem 3.2 in [30], as well as the references [20] and [45]. Moreover, these Fredholm properties depend upon the hyperbolicity of the asymptotic matrices (cf. [45]),

$$\mathbb{A}^\epsilon_\pm(\lambda) = \lim_{x\to\pm\infty} \mathbb{A}^\epsilon(x,\lambda) = \begin{pmatrix} 0 & 1 \\ D^\epsilon(u_\pm)^{-1}(\lambda - f'(u_\pm)) & -D^\epsilon(u_\pm)^{-1}c\end{pmatrix}.$$

For each fixed $\lambda \in \mathbb{C}$, let us denote the characteristic polynomial of $\mathbb{A}^\epsilon_\pm(\lambda)$ as $\pi^\epsilon_\pm(\lambda, z) := \det(\mathbb{A}^\epsilon_\pm(\lambda) - zI)$. Then for each $k \in \mathbb{R}$, the λ-roots of

$$\pi^\epsilon_\pm(\lambda, ik) = -k^2 + ikcD^\epsilon(u_\pm)^{-1} + D^\epsilon(u_\pm)^{-1}(f'(u_\pm) - \lambda) = 0,$$

define algebraic curves in the complex plane parametrized by $k \in \mathbb{R}$, more precisely,

$$\lambda^\epsilon_\pm(k) := -D^\epsilon(u_\pm)k^2 + ick + f'(u_\pm), \quad k \in \mathbb{R}.$$

Consider the following open connected subset in the complex plane,

$$\Omega := \{\lambda \in \mathbb{C} \,:\, \operatorname{Re}\lambda > \max\{f'(u_+), f'(u_-)\}\}. \tag{5.5}$$

This is called the region of consistent splitting. Finally, for each fixed $\lambda \in \mathbb{C}$ and $\epsilon > 0$, let us denote $S^\epsilon_\pm(\lambda)$ and $U^\epsilon_\pm(\lambda)$ as the stable and unstable eigenspaces of $\mathbb{A}^\epsilon_\pm(\lambda)$ in $\mathbb{C}^2$, respectively.

Lemma 5.1. *For each $\lambda \in \Omega$ and all $\epsilon > 0$, the coefficient matrices $\mathbb{A}^\epsilon_\pm(\lambda)$ have no center eigenspace and* $\dim S^\epsilon_\pm(\lambda) = \dim U^\epsilon_\pm(\lambda) = 1$.

Proof. Notice that

$$\max_{k\in\mathbb{R}} \operatorname{Re}\lambda^\epsilon_\pm(k) = f'(u_\pm),$$

independently of $\epsilon > 0$. Therefore, for each $\lambda \in \Omega$ it is clear that $\mathbb{A}^\epsilon_\pm(\lambda)$ has no center eigenspace, for all $\epsilon > 0$. By continuity on λ and by connectedness of Ω, the dimensions of $S^\epsilon_\pm(\lambda)$ and $U^\epsilon_\pm(\lambda)$ remain constant in Ω. To compute them, set $\lambda = \eta \in \mathbb{R}$, with η sufficiently large. The characteristic equation $\pi^\epsilon_\pm(\eta, z) = 0$ has one positive and one negative root:

$$z_1 = \tfrac{1}{2}D^\epsilon(u_\pm)^{-1}\big(-c - \sqrt{c^2 + 4D^\epsilon(u_\pm)(\eta - f'(u_\pm))}\big) < 0,$$
$$z_2 = \tfrac{1}{2}D^\epsilon(u_\pm)^{-1}\big(-c + \sqrt{c^2 + 4D^\epsilon(u_\pm)(\eta - f'(u_\pm))}\big) > 0,$$

as long as $\eta > 0$ is large, say, $\eta > \max\{|f'(u_\pm)|\}$. The conclusion follows. □

The following lemma characterizes the Fredholm properties of $\mathcal{L}^\epsilon - \lambda$ for λ in the region of consistent splitting.

Lemma 5.2. *For all $\epsilon > 0$ and for each $\lambda \in \Omega$, the operator $\mathcal{L}^\epsilon - \lambda$ is Fredholm with index zero.*

Proof. Let $\lambda \in \Omega$. Since the matrices $\mathbb{A}_\pm^\epsilon(\lambda)$ are hyperbolic, by standard exponential dichotomies theory [7] (see also Theorem 3.3 in [45]), system (5.4) is endowed with exponential dichotomies on both rays $[0, +\infty)$ and $(-\infty, 0]$, with Morse indices $i_+(\lambda) = \dim U_+^\epsilon(\lambda) = 1$ and $i_-(\lambda) = \dim S_-^\epsilon(\lambda) = 1$, respectively. Therefore, by Theorem 3.2 in [45], we conclude that the operators $\mathcal{T}^\epsilon(\lambda)$ are Fredholm with index

$$\operatorname{ind} \mathcal{T}^\epsilon(\lambda) = i_+(\lambda) - i_-(\lambda) = 0,$$

showing that $\mathcal{J}^\epsilon(\lambda)$ and $\mathcal{L}^\epsilon - \lambda$ are Fredholm with index zero, as claimed. $\square$

5.2. **Generalized convergence.** We are going to profit from the independence of the Fredholm properties of $\mathcal{L}^\epsilon - \lambda$ with respect to $\epsilon > 0$ in order to conclude some useful information about the Fredholm properties of $\mathcal{L} - \lambda$. First, we recall the succeeding standard definitions (cf. Kato [21]): Let Z be a Banach space, and let M and N be any two nontrivial closed subspaces of Z. Let S_M be the unitary sphere in M. Then we define

$$\delta(M, N) = \sup_{u \in S_M} \operatorname{dist}(u, N),$$

$$\hat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}.$$

$\hat{\delta}$ is called the *gap* between M and N. (Here the funcion $\operatorname{dist}(u, M)$ is the usual distance function from u to any closed manifold M.) Since in general $\hat{\delta}(\cdot, \cdot)$ does not satisfy the triangle inequality, one defines

$$\mathrm{d}(M, N) = \sup_{u \in S_M} \operatorname{dist}(u, S_N),$$

$$\hat{\mathrm{d}}(M, N) = \max\{\mathrm{d}(M, N), \mathrm{d}(N, M)\}.$$

$\hat{\mathrm{d}}(M, N)$ is called the *distance* between M and N, and satisfies the triangle inequality. Furthermore, there hold the inequalities [21],

$$\begin{aligned} \delta(M, N) &\leq \mathrm{d}(M, N) \leq 2\delta(M, N), \\ \hat{\delta}(M, N) &\leq \hat{\mathrm{d}}(M, N) \leq 2\hat{\delta}(M, N), \end{aligned} \tag{5.6}$$

for any closed manifolds M and N.

Definition 5.3. Let X, Y be Banach spaces. If $\mathcal{T}, \mathcal{S} \in \mathscr{C}(X, Y)$, then the graphs $G(\mathcal{T}), G(\mathcal{S})$ are closed subspaces of $X \times Y$, and we set

$$\mathrm{d}(\mathcal{T}, \mathcal{S}) = \mathrm{d}(G(\mathcal{T}), G(\mathcal{S})),$$

$$\hat{\mathrm{d}}(\mathcal{T}, \mathcal{S}) = \max\{\mathrm{d}(\mathcal{T}, \mathcal{S}), \mathrm{d}(\mathcal{S}, \mathcal{T})\}.$$

It is said that a sequence $\mathcal{T}_n \in \mathscr{C}(X, Y)$ *converges in generalized sense* to $\mathcal{T} \in \mathscr{C}(X, Y)$ provided that $\hat{\mathrm{d}}(\mathcal{T}_n, \mathcal{T}) \to 0$ as $n \to +\infty$.

Remark 5.4. It follows from inequalities (5.6) that $\hat{\mathrm{d}}(\mathcal{T}_n, \mathcal{T}) \to 0$ is equivalent to $\hat{\delta}(\mathcal{T}_n, \mathcal{T}) \to 0$

Lemma 5.5. *For each fixed $\lambda \in \mathbb{C}$, the operators $\mathcal{L}^\epsilon - \lambda$ converge in generalized sense to $\mathcal{L} - \lambda$ as $\epsilon \to 0^+$.*

Proof. From the definition of $\mathrm{d}(\cdot,\cdot)$ we have

$$\mathrm{d}(\mathcal{L}^\epsilon-\lambda,\mathcal{L}-\lambda)=\mathrm{d}(G(\mathcal{L}^\epsilon-\lambda),G(\mathcal{L}-\lambda))=\sup_{v\in S_{G(\mathcal{L}^\epsilon-\lambda)}}\left(\inf_{w\in S_{G(\mathcal{L}-\lambda)}}\|v-w\|\right).$$

Let $v\in S_{G(\mathcal{L}^\epsilon-\lambda)}$ be such that $v=\{p,(\mathcal{L}^\epsilon-\lambda)p\}$ for $p\in\mathcal{D}(\mathcal{L}^\epsilon-\lambda)=H^2$, and

$$\|v\|^2_{L^2\times L^2}=\|p\|^2_{L^2}+\|(\mathcal{L}^\epsilon-\lambda)p\|^2_{L^2}=1.$$

Likewise, let $w\in S_{G(\mathcal{L}-\lambda)}$ be such that $w=\{u,(\mathcal{L}-\lambda)u\}$, for $u\in\mathcal{D}(\mathcal{L}-\lambda)=H^2$ and

$$\|w\|^2_{L^2\times L^2}=\|u\|^2_{L^2}+\|(\mathcal{L}-\lambda)u\|^2_{L^2}=1.$$

Now, we find a upper bound for $\|v-w\|_{L^2\times L^2}$. Consider,

$$\|v-w\|^2_{L^2\times L^2}=\|p-u\|^2_{L^2}+\|(\mathcal{L}^\epsilon-\lambda)p-(\mathcal{L}-\lambda)u\|^2_{L^2}. \tag{5.7}$$

If we keep $v\in S_{G(\mathcal{L}^\epsilon-\lambda)}$ fixed, then it suffices to take $w=\{p,(\mathcal{L}-\lambda)p\}$, inasmuch as $p\in H^2=\mathcal{D}(\mathcal{L})=\mathcal{D}(\mathcal{L}^\epsilon)$. Note that $(\mathcal{L}-\lambda)p=(\mathcal{L}^\epsilon-\lambda)p-\epsilon p_{xx}$. Therefore expression (5.7) gets simplified:

$$\|v-w\|^2_{L^2\times L^2}=\|(\mathcal{L}^\epsilon-\lambda)p-(\mathcal{L}-\lambda)p\|^2_{L^2}=\|\epsilon p_{xx}\|^2_{L^2}.$$

If we regard ∂_x^2 as a closed, densely defined operator on $L^2(\mathbb{R};\mathbb{C})$, with domain $\mathcal{D}=H^2(\mathbb{R};\mathbb{C})$, then it follows from Remark 1.5 in [21, p. 191], that ∂_x^2 is $(\mathcal{L}^\epsilon-\lambda)-$bounded, i.e., there exist a constant $C>0$ such that

$$\|p_{xx}\|_{L^2}\leq C(\|p\|_{L^2}+\|(\mathcal{L}^\epsilon-\lambda)p\|_{L^2}),$$

for all $p\in H^2$. Consequently

$$\|p_{xx}\|^2_{L^2}\leq\bar{C}(\|p\|^2_{L^2}+\|(\mathcal{L}^\epsilon-\lambda)p\|^2_{L^2})=\bar{C},$$

for some other $\bar{C}>0$ and for $v=(p,(\mathcal{L}^\epsilon-\lambda)p)\in S_{G(\mathcal{L}^\epsilon-\lambda)}$. This estimate implies, in turn, that

$$\|v-w\|^2_{L^2\times L^2}=\epsilon^2\|p_{xx}\|^2_{L^2}\leq\bar{C}\epsilon^2.$$

This yields,

$$\mathrm{d}(\mathcal{L}^\epsilon-\lambda,\mathcal{L}-\lambda)\leq\bar{C}\epsilon^2.$$

In a similar fashion it can be proved that $\mathrm{d}(\mathcal{L}-\lambda,\mathcal{L}^\epsilon-\lambda)\leq C\epsilon^2$. This shows that $\hat{\mathrm{d}}(\mathcal{L}^\epsilon-\lambda,\mathcal{L}-\lambda)\to 0$ as $\epsilon\to 0^+$, and the conclusion follows. □

For the reader's convenience we state a result from functional analysis (cf. [21], pg. 235), which will be helpful to relate the Fredholm properties of $\mathcal{L}^\epsilon-\lambda$ to those of $\mathcal{L}-\lambda$. First, we remind the reader the definition of the reduced minimum modulus of a closed operator.

Definition 5.6. For any $\mathcal{S}\in\mathscr{C}(X,Y)$ we define $\gamma=\gamma(\mathcal{S})$, as the greatest number $\gamma\in\mathbb{R}$ such that

$$\|\mathcal{S}u\|\geq\gamma\,\mathrm{dist}(u,\ker\mathcal{S}),$$

for all $u\in\mathcal{D}(\mathcal{S})$.

Theorem 5.7 (Kato [21])**.** *Let $\mathcal{T},\mathcal{S}\in\mathscr{C}(X,Y)$ and let $\mathcal{T}$ be Fredholm (semi-Fredholm). If*

$$\hat{\delta}(\mathcal{S},\mathcal{T})<\gamma(1+\gamma^2)^{-1/2},$$

where $\gamma=\gamma(\mathcal{T})$, then $\mathcal{S}$ is Fredholm (semi-Fredholm) and $\operatorname{nul}\mathcal{S}\leq\operatorname{nul}\mathcal{T}$, $\operatorname{def}\mathcal{S}\leq\operatorname{def}\mathcal{T}$*. Furthermore, there exists $\delta>0$ such that $\hat{\delta}(\mathcal{S},\mathcal{T})<\delta$ implies*

$$\operatorname{ind}\mathcal{S}=\operatorname{ind}\mathcal{T}.$$

If X, Y are Hilbert spaces then we can take $\delta = \gamma(1+\gamma^2)^{-\frac{1}{2}}$.

5.3. **Location of σ_δ.** After these preparations, we are ready to state and prove the main result of this section.

Theorem 5.8. *Let $\varphi = \varphi(x)$ be a diffusion-degenerate Fisher-KPP front, and let $\mathcal{L} \in \mathscr{C}(L^2)$ be the linearized operator around φ defined in* (3.5)*. Then,*

$$\sigma_\delta(\mathcal{L}) \subset \mathbb{C}\backslash\Omega.$$

Proof. First observe that

$$\sigma_\delta(\mathcal{L}) \subset \{\lambda \in \mathbb{C} \,:\, \mathcal{L} - \lambda \,\text{is semi-Fredholm with}\; \mathrm{ind}\,(\mathcal{L}-\lambda) \neq 0\}.$$

Indeed, by definition if $\lambda \in \sigma_\delta(\mathcal{L})$ then $\mathcal{L} - \lambda$ is injective, $\mathcal{R}(\mathcal{L}-\lambda)$ is closed and $\mathcal{R}(\mathcal{L}-\lambda) \subsetneq Y = L^2$. Hence, $\mathrm{nul}\,(\mathcal{L}-\lambda) = 0$ and $\mathcal{L}-\lambda$ is semi-Fredholm. Moreover, since $\mathrm{def}\,(\mathcal{L}-\lambda) = \mathrm{codim}\,\mathcal{R}(\mathcal{L}-\lambda) > 0$, we have that $\mathrm{ind}\,(\mathcal{L}-\lambda) \neq 0$.

Now, let us suppose that $\lambda \in \sigma_\delta(\mathcal{L}) \cap \Omega$. Since $\mathcal{L}-\lambda$ is semi-Fredholm, $\mathcal{R}(\mathcal{L}-\lambda)$ is closed, and this implies that

$$\gamma := \gamma(\mathcal{L}-\lambda) > 0,$$

(see Theorem 5.2 in [21], pg. 231). By Lemma 5.5, $\hat{\delta}(\mathcal{L}^\epsilon - \lambda, \mathcal{L}-\lambda) \to 0$ as $\epsilon \to 0^+$, so we can find $\epsilon > 0$ sufficiently small such that

$$\hat{\delta}(\mathcal{L}^\epsilon - \lambda, \mathcal{L}-\lambda) < \gamma(1+\gamma^2)^{1/2}.$$

Since $X = L^2(\mathbb{R};\mathbb{C})$ is a Hilbert space, Theorem 5.7 implies that

$$\mathrm{ind}\,(\mathcal{L}-\lambda) = \mathrm{ind}\,(\mathcal{L}^\epsilon - \lambda) = 0,$$

in view of $\lambda \in \Omega$ and Lemma 5.2. This is a contradiction with $\mathrm{ind}\,(\mathcal{L}-\lambda) \neq 0$. We conclude that $\sigma_\delta(\mathcal{L}) \subset \mathbb{C}\backslash\Omega$, as claimed. □

Remark 5.9. We observe that the location of $\sigma_\delta(\mathcal{L})$ depends upon the sign of $f'(u_\pm)$, as it lies to the left of the region of consistent splitting Ω. Since $f'(0) > 0$, the subset $\sigma_\delta(\mathcal{L})$ of the compression spectrum is unstable. It is, however, sensitive to changes at spatial infinity, and it is possible to find exponentially weighted spaces where spectral stability does hold.

6. Spectral stability in exponentially weighted spaces

In this section we prove Theorem 1.1. The key ingredient is to find a suitable exponentially weighted space in which the spectrum of the linearized operator, computed with respect to the new space, is stable. This is accomplished provided that certain conditions on the velocity hold.

6.1. **Exponentially weighted spaces.** It is well-known [20, 45] that the Fredholm borders of the compression spectrum $\sigma_\delta(\mathcal{L})$ may move when the eigenvalue problem is recast in an exponentially weighted space. We introduce, according to custom, the function spaces

$$H_a^m(\mathbb{R};\mathbb{C}) = \{v \,:\, e^{ax}v(x) \in H^m(\mathbb{R};\mathbb{C})\},$$

for $m \in \mathbb{Z}$, $m \geq 0$, and any $a \in \mathbb{R}$. The latter are Hilbert spaces endowed with the inner product (and norm),

$$\langle u, v\rangle_{H_a^m} := \langle e^{ax}u, e^{ax}v\rangle_{H^m}, \qquad \|v\|_{H_a^m}^2 := \|e^{ax}v\|_{H^m}^2 = \langle v, v\rangle_{H_a^m}.$$

As usual, we denote $L^2_a(\mathbb{R};\mathbb{C}) = H^0_a(\mathbb{R};\mathbb{C})$. Clearly, the norms $\|\cdot\|_{L^2_a}$ for different values of a are not equivalent. If we consider $\mathcal{L}$ as an operator acting on L^2_a,

$$\mathcal{L} : \mathcal{D}_a := H^2_a(\mathbb{R};\mathbb{C}) \subset L^2_a(\mathbb{R};\mathbb{C}) \to L^2_a(\mathbb{R};\mathbb{C}),$$

and compute its spectrum with respect to the new space for an appropriate value of a, then it is known [45] that the Fredholm borders move depending on the sign of a (if $a > 0$, for example, then the $\|\cdot\|_{L^2_a}$-norm penalizes perturbations at $-\infty$ while it tolerates exponentially growing perturbations at $+\infty$ at any rate less than $a > 0$, shifting, in this fashion, the Fredholm borders to the left), whereas the point spectrum is unmoved [20]. We shall prove that both the approximate spectrum $\sigma_\pi(\mathcal{L})$ and the point spectrum $\sigma_{\mathrm{pt}}(\mathcal{L})$ remain unmoved under appropriate choices of the weight $a \geq 0$ (see Proposition 6.1 below).

Finally, it is well-known that the analysis of the spectrum of the operator $\mathcal{L}$ on the space L^2_a is equivalent to that of a conjugated operator,

$$\mathcal{L}_a := e^{ax}\mathcal{L}e^{-ax} \,:\, \mathcal{D} = H^2(\mathbb{R};\mathbb{C}) \subset L^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R};\mathbb{C}),$$

acting on the original unweighted space. If φ is a traveling front and $\mathcal{L}$ is the associated linearized operator acting on L^2 defined in (3.5), then after simple calculations we find that, for any $a \in \mathbb{R}$, the associated conjugated operator $\mathcal{L}_a$ is given by

$$\begin{aligned} &\mathcal{L}_a : \mathcal{D}(\mathcal{L}_a) := H^2(\mathbb{R};\mathbb{C}) \subset L^2(\mathbb{R};\mathbb{C}) \to L^2(\mathbb{R} : \mathbb{C}),\\ &\mathcal{L}_a u = D(\varphi)u_{xx} + \Big(2D(\varphi)_x - 2aD(\varphi) + c\Big)u_x + \\ &\qquad + \Big(a^2 D(\varphi) - 2aD(\varphi)_x - ac + D(\varphi)_{xx} + f'(\varphi)\Big)u, \end{aligned} \tag{6.1}$$

for all $u \in H^2$. $\mathcal{L}_a$ is clearly a closed, densely defined operator in L^2.

6.2. **Calculation of the Fredholm curves.** In order to analyze how the Fredholm borders limiting σ_δ move under the influence of the weight function e^{ax}, let us consider the regularized conjugated operator $\mathcal{L}^\epsilon_a$ acting on L^2, for $0 < \epsilon \ll 1$, small (which results from substituting $D(\varphi)$ by $D^\epsilon(\varphi) = D(\varphi) + \epsilon$ in (6.1); see section 5.1). Thus, if for any $\lambda \in \mathbb{C}$ we define $\mathcal{J}^\epsilon_a(\lambda) := D^\epsilon(\varphi)^{-1}(\mathcal{L}^\epsilon_a - \lambda)$, then its explicit expression is

$$\mathcal{J}^\epsilon_a(\lambda)u = u_{xx} + D^\epsilon(\varphi)^{-1}b^\epsilon_{1,a}(x)u_x + D^\epsilon(\varphi)^{-1}b^\epsilon_{0,a}(x)u,$$

with

$$\begin{aligned} b^\epsilon_{1,a}(x) &:= 2D^\epsilon(\varphi)_x + c - 2aD^\epsilon(\varphi),\\ b^\epsilon_{0,a}(x) &:= a^2 D^\epsilon(\varphi) - 2aD^\epsilon(\varphi)_x - ac + D^\epsilon(\varphi)_{xx} + f'(\varphi). \end{aligned}$$

Like in section 5.1, we recast the spectral problem as a first order system of the form

$$W_x = \mathbb{A}^\epsilon_a(x,\lambda)W,$$

where

$$W = \begin{pmatrix} u \\ u_x \end{pmatrix} \in H^2(\mathbb{R};\mathbb{C}^2), \quad \mathbb{A}^\epsilon_a(x,\lambda) = \begin{pmatrix} 0 & 1 \\ D^\epsilon(\varphi)^{-1}(\lambda - b^\epsilon_{0,a}(x)) & -D^\epsilon(\varphi)^{-1}b^\epsilon_{1,a}(x) \end{pmatrix}.$$

Since,

$$\begin{aligned} \lim_{x\to\pm\infty} b^\epsilon_{0,a}(x) &= a^2 D^\epsilon(u_\pm) - ac + f'(u_\pm) =: b^{\epsilon,\pm}_{0,a},\\ \lim_{x\to\pm\infty} b^\epsilon_{1,a}(x) &= c - 2aD^\epsilon(u_\pm) =: b^{\epsilon,\pm}_{1,a}, \end{aligned}$$

the constant coefficients of the asymptotic systems can be written as

$$\mathbb{A}_a^{\epsilon,\pm}(\lambda) = \begin{pmatrix} 0 & 1 \\ D^\epsilon(u_\pm)^{-1}(\lambda - b_{0,a}^{\epsilon,\pm}) & -D^\epsilon(u_\pm)^{-1} b_{1,a}^{\epsilon,\pm} \end{pmatrix}.$$

Let us denote $\pi_a^{\epsilon,\pm}(\lambda, z) = \det(\mathbb{A}_a^{\epsilon,\pm}(\lambda) - zI)$, so that, for each $k \in \mathbb{R}$

$$\pi_a^{\epsilon,\pm}(\lambda, ik) = -k^2 + ik(cD^\epsilon(u_\pm)^{-1} - 2a) + a^2 - D^\epsilon(u_\pm)^{-1}(\lambda + ac - f'(u_\pm)).$$

Thus, the Fredholm borders, defined as the λ-roots of $\pi_a^{\epsilon,\pm}(\lambda, ik) = 0$, are given by

$$\lambda_a^{\epsilon,\pm}(k) := D^\epsilon(u_\pm)(a^2 - k^2) - ac + f'(u_\pm) + ik(c - 2aD^\epsilon(u_\pm)), \quad k \in \mathbb{R}.$$

Notice that

$$\max_{k\in\mathbb{R}} \operatorname{Re} \lambda_a^{\epsilon,\pm}(k) = D^\epsilon(u_\pm)a^2 - ac + f'(u_\pm);$$

therefore we denote the region of consistent splitting for each $a \in \mathbb{R}$ and $\epsilon \geq 0$ as

$$\Omega(a, \epsilon) = \left\{\lambda \in \mathbb{C} \,:\, \operatorname{Re} \lambda > \max\left\{D^\epsilon(u_+)a^2 - ac + f'(u_+),\; D^\epsilon(u_-)a^2 - ac + f'(u_-)\right\}\right\}.$$

By a similar argument as in section 5 (see Lemma 5.1), for each $\lambda \in \Omega(a, \epsilon)$ the coefficient matrices $\mathbb{A}_a^{\epsilon\pm}(\lambda)$ have no center eigenspace and $\dim S_a^{\epsilon,\pm}(\lambda) = \dim U_a^{\epsilon,\pm}(\lambda) = 1$, where $S_a^{\epsilon,\pm}(\lambda)$ and $U_a^{\epsilon,\pm}(\lambda)$ denote the stable and unstable eigenspaces of $\mathbb{A}_a^{\epsilon\pm}(\lambda)$, respectively.

Moreover, by taking the limit as $\epsilon \to 0^+$, we claim that

$$\sigma_\delta(\mathcal{L}_a) = \mathbb{C}\backslash\Omega(a), \tag{6.2}$$

where $\Omega(a) := \Omega(a, 0)$. Indeed, using the same arguments as in the proof of Lemma 5.2, we conclude that, for $\lambda \in \Omega(a, \epsilon)$, $\mathcal{L}_a^\epsilon - \lambda$ is Fredholm in L^2 with index zero. By keeping $a \in \mathbb{R}$ constant, one can verify that the operators $\mathcal{L}_a^\epsilon - \lambda$ converge in generalized sense to $\mathcal{L}_a - \lambda$ as $\epsilon \to 0^+$ (see Lemma 5.5; we omit the details as the proof is not only analogous, but the same). Furthermore, by the same arguments as in the proof of Theorem 5.8, for $0 < \epsilon \ll 1$ sufficiently small the Fredholm properties of $\mathcal{L}_a^\epsilon - \lambda$ and $\mathcal{L}_a - \lambda$ are the same.

Suppose that $\lambda \in \sigma_\delta(\mathcal{L}_a) \cap \Omega(a, \epsilon)$, with $0 < \epsilon \ll 1$, small. Then, by following the proof of Theorem 5.8, we find that $\operatorname{ind}(\mathcal{L}_a - \lambda) = \operatorname{ind}(\mathcal{L}_a^\epsilon - \lambda) = 0$, and at the same time, that $\operatorname{ind}(\mathcal{L}_a - \lambda) \neq 0$ because $\sigma_\delta(\mathcal{L}_a)$ is contained in the set of complex λ for which $\mathcal{L}_a - \lambda$ is semi-Fredholm with non-zero index. This is a contradiction, which yields $\sigma_\delta(\mathcal{L}_a) \subset \mathbb{C}\backslash\Omega(a, \epsilon)$ for all $0 < \epsilon \ll 1$ sufficiently small. By continuity, taking the limit as $\epsilon \to 0^+$ we obtain

$$\sigma_\delta(\mathcal{L}_a) \subset \mathbb{C}\backslash\Omega(a),$$

as claimed.

Henceforth, it suffices to choose $a \in \mathbb{R}$ appropriately in order to stabilize $\sigma_\delta(\mathcal{L}_a)$, and to show that the sets $\sigma_{\mathrm{pt}}(\mathcal{L})$ and $\sigma_\pi(\mathcal{L})$ remain stable under conjugation.

Proposition 6.1. *Let φ be a monotone front and $\mathcal{L}$ the associated linearized operator in L^2. Then, for any appropriate weight $a \geq 0$, we have:*

(a) $\sigma_{\mathrm{pt}}(\mathcal{L}) = \sigma_{\mathrm{pt}}(\mathcal{L}_a)$, *and,*
(b) $\sigma_\pi(\mathcal{L}_a) \subset \sigma_\pi(\mathcal{L})$.

(Here, both spectral sets, $\sigma_{\mathrm{pt}}(\cdot)$ and $\sigma_\pi(\cdot)$, are of course computed with respect to the space $L^2(\mathbb{R}; \mathbb{C})$.)

Proof. Suppose $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L})$. Then there exists $u \in \mathcal{D}(\mathcal{L}) = H^2$ such that $\mathcal{L}u = \lambda u$. For any appropriate weight $a \geq 0$, let us assume that

$$v(x) = e^{ax}u(x) \in H^2. \tag{6.3}$$

(This is equivalent to say that $u \in H^2_a$ for some apropriate $a \geq 0$.) Then

$$\mathcal{L}_a v = e^{ax}\mathcal{L}e^{-ax}v = e^{ax}\mathcal{L}u = \lambda e^{ax}u = \lambda v.$$

That is, $\lambda \in \sigma_{\mathrm{pt}}(\mathcal{L}_a)$. This shows (a).

Now suppose that $\lambda \in \sigma_\pi(\mathcal{L}_a)$, for some appropriate $a \geq 0$. Assuming that $(\mathcal{L} - \lambda)u = 0$, for $u \in H^2$, by the arguments above we have $(\mathcal{L}_a - \lambda)v = 0$ provided that $v = e^{ax}u \in H^2$. Since $\mathcal{L}_a - \lambda$ is injective, we have $v = 0$ a.e. and, therefore, $u = 0$ a.e. We conclude that $\mathcal{L} - \lambda$ is injective.

To show that $\mathcal{R}(\mathcal{L} - \lambda)$ is not closed we proceed by contradiction. Assume that $\mathcal{L} - \lambda$ has closed range. This means that $\gamma(\mathcal{L} - \lambda) > 0$ and $\|(\mathcal{L} - \lambda)u\|_{L^2} \geq \gamma_0 > 0$ for some uniform $\gamma_0 > 0$ and all $u \in \mathcal{D}(\mathcal{L}) = H^2$ with $\|u\|_{L^2} = 1$. Since $\mathcal{R}(\mathcal{L}_a - \lambda)$ is not closed, then there exists $v_n \in \mathcal{D}(\mathcal{L}_a) = H^2$ such that $\|v_n\|_{L^2} = 1$ and $g_n = (\mathcal{L}_a - \lambda)v_n \to 0$ in L^2. In view that they are localized functions and that the energy L^2-norm is invariant under translations, $v_n(\cdot) \to v_n(\cdot + y)$, we can assume that

$$\|v_n\|_{L^2(0,+\infty)} > 0, \qquad \text{for all } n \in \mathbb{N}.$$

Hence we define,

$$u_n(x) := \begin{cases} e^{-ax}v_n(x), & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $u_n \in H^2$ and $\|u_n\|_{L^2} > 0$ for all n. Moreover,

$$\begin{aligned} 0 \leq \|(\mathcal{L} - \lambda)u_n\|_{L^2} &= \|(\mathcal{L} - \lambda)u_n\|_{L^2(0,+\infty)} \\ &= \|e^{-ax}(\mathcal{L}_a - \lambda)v_n\|_{L^2(0,+\infty)} \\ &\leq \|(\mathcal{L}_a - \lambda)v_n\|_{L^2} \;\to\; 0, \end{aligned}$$

as $n \to +\infty$. Normalizing u_n, we obtain a sequence $u_n \in H^2$ with $\|u_n\|_{L^2} = 1$ such that $(\mathcal{L} - \lambda)u_n \to 0$ in L^2. This yields a contradiction with $\|(\mathcal{L} - \lambda)u\|_{L^2} \geq \gamma_0$ for all $\|u\|_{L^2} = 1$. Therefore, $\mathcal{R}(\mathcal{L} - \lambda)$ is not closed, and we conclude that $\lambda \in \sigma_\pi(\mathcal{L})$. This shows (b). ☐

6.3. **Choice of the weight $a \geq 0$: proof of Theorem 1.1.** In the Fisher-KPP case, $u_+ = 0$, $u_- = 1$, with $f'(0) > 0$ and $f'(1) < 0$. The fronts travel with any speed $c > c_* > 0$. Under these conditions we have

$$\Omega(a) = \{\lambda \in \mathbb{C} \,:\, \operatorname{Re}\lambda > \max\{f'(0) - ac,\, D(1)a^2 - ac + f'(1)\}\,\},$$

for any $a \in \mathbb{R}$. In order to have spectral stability we need to find $a \in \mathbb{R}$ such that $f'(0) - ac < 0$ and $p(a) := D(1)a^2 - ac + f'(1) < 0$, simultaneously. Since $p(0) = f'(1) < 0$ we have that $p(a) < 0$ for all $a \in [0, a_0)$, where a_0 is the first positive root of $p(a) = 0$, that is,

$$a_0(c) = (2D(1))^{-1}\big(c + \sqrt{c^2 - 4D(1)f'(1)}\,\big).$$

We need to find a such that $0 < f'(0)/c < a < a_0(c)$. Thus, it is necessary and sufficient that $a_0(c) > f'(0)/c$. (This imposes a condition on the speed c.) It is easy to verify that this is true provided that the front travels with speed c such that

$$c^2 > \frac{D(1)f'(0)^2}{f'(0) - f'(1)}. \tag{6.4}$$

If condition (6.4) holds then we can choose $a \in \mathbb{R}$ such that $0 < f'(0)/c < a < a_0(c)$, and, consequently,

$$\sigma_\delta(\mathcal{L}_a) \subset \mathbb{C} \backslash \Omega(a) \subset \{\lambda \in \mathbb{C} \, : \, \operatorname{Re} \lambda < 0\}.$$

Finally, to apply Proposition 6.1 we only have to verify that such $a \in \mathbb{R}$ is an appropriate weight in the sense that condition (6.3) holds. By inspection of the spectral equation $\mathcal{L}u = \lambda u$, written as a first order system, it is easy to check that any solution $u \in H^2$ decays at order $O(e^{-c|x|/\delta_0})$ at the non-degenerate side as $x \to -\infty$, where $D(\varphi) \geq \delta_0 > 0$ for all $x \in (-\infty, -x_0]$, with $x_0 > 0$ fixed. Moreover, u_x and u_{xx} also decay at the same rate as $x \to -\infty$. This follows by standard ODE estimates on the first order system linearized around hyperbolic rest points (as the diffusion coefficient does not vanish in the limit). Thus,

$$\int_{-\infty}^{-x_0} e^{2ax} |u(x)|^2 \, dx \leq C \int_{-\infty}^{-x_0} e^{2(a + c/\delta_0)x} \, dx < +\infty,$$

because $a + c/\delta_0 > f'(0)/c + c/\delta_0 > 0$. Thus, $v \in L^2(-\infty, -x_0)$. Likewise, it can be verified that $v_x, v_{xx} \in L^2(-\infty, x_0)$. Therefore, $v \in H^2(-\infty, x_0)$.

On the degenerate side, as $x \to +\infty$, we have precise information on the decaying behavior of solutions to resolvent type equations provided by Lemma A.1. If u is an H^2-solution of $\mathcal{L}u = \lambda u$ then for $x > x_0 \gg 1$, u can be written in the form (A.1) with $\zeta \in H^2(x_0, +\infty)$. It is easy to verify that for all $a > f'(0)/c > 0$ we have $\zeta e^{ax} \in H^2(x_0, +\infty)$, thanks to the fast decay of ζ (see Appendix A). Therefore, $v = e^{ax}u \in H^2(x_0, +\infty)$. For $x_0 \in \mathbb{R}$ fixed it is obvious that $v = e^{ax}u \in H^2(-x_0, x_0)$ whenever $u \in H^2$. Hence, we conclude that $v = e^{ax}u \in H^2(\mathbb{R}; \mathbb{C})$, and condition (6.3) holds.

Consequently, we can apply Proposition 6.1 (a), and together with Theorem 4.5, we obtain $\sigma_{\rm pt}(\mathcal{L}_a) \subset (-\infty, 0]$. Likewise, from Proposition 6.1 (b) and Lemma 4.7 we conclude that $\sigma_\pi(\mathcal{L}_a) \subset (-\infty, 0]$.

Thus, we have proved that for any diffusion-degenerate monotone Fisher-KPP traveling front φ, traveling with speed

$$c > \max \left\{ c_*, \; \frac{f'(0)\sqrt{D(1)}}{\sqrt{f'(0) - f'(1)}} \right\} > 0, \tag{6.5}$$

we can choose $a \in \mathbb{R}$ satisfying

$$0 < \frac{f'(0)}{c} < a < (2D(1))^{-1} \big(c + \sqrt{c^2 - 4D(1)f'(1)} \, \big),$$

such that the front is L^2_a-spectrally stable; more precisely,

$$\sigma(\mathcal{L})_{|L^2_a} = \sigma(\mathcal{L}_a)_{|L^2} \subset \{\operatorname{Re} \lambda \leq 0\}.$$

This finishes the proof of the main Theorem 1.1

Remark 6.2. The theorem guarantees that all traveling fronts with speed satisfying condition (6.5) are spectrally stable in an appropriate weighted space L^2_a. The condition depends on the choice of the reaction f and the density-dependent diffusion D under consideration. For example, in the particular case of the diffusion-degenerate Fisher-KPP equation,

$$u_t = (\alpha u u_x)_x + u(1-u), \tag{6.6}$$

with $\alpha > 0$, that is, the special case of (1.1) with $D(u) = \alpha u$ and $f(u) = u(1-u)$, it is known (see, e.g., [2, 13, 14]) that the threshold speed is given by $c_* = \sqrt{\alpha/2}$. Thus, in this case,

$$\frac{f'(0)\sqrt{D(1)}}{\sqrt{f'(0)-f'(1)}} = \frac{\sqrt{\alpha}}{\sqrt{2}} = c_*,$$

and, by Theorem 1.1, all traveling waves for equation (6.6) with speed $c > c_*$ are L^2_a-spectrally stable for some $a \geq 0$.

There might be choices of D and f for which the maximum in (6.5) is not c_* and, therefore, the waves traveling with speed $c_* < c < f'(0)\sqrt{D(1)}/\sqrt{f'(0)-f'(1)}$ are spectrally unstable in any weighted space. This case is associated to the presence of absolute instabilities and the location of the leftmost limit of the Fredholm borders as a varies (see [20], section 3.1, for further information). For example, Sanchez-Garduño and Maini [40] calculate, for the choices $D(u) = u + \varepsilon u^2$ and $f(u) = u(1-u)$, with $\varepsilon > 0$ small, that the threshold velocity is $c_* = (1+\varepsilon/5)/\sqrt{2}$. Therefore

$$\max\left\{ c_*,\ \frac{f'(0)\sqrt{D(1)}}{\sqrt{f'(0)-f'(1)}} \right\} = \max\left\{ \frac{1}{\sqrt{2}}(1+\varepsilon/5)\,,\ \frac{\sqrt{1+\varepsilon}}{\sqrt{2}} \right\} = \frac{\sqrt{1+\varepsilon}}{\sqrt{2}},$$

for all small $\varepsilon > 0$. Thus, stability holds for all fronts traveling with speed $c > \sqrt{1+\varepsilon}/\sqrt{2}$, whereas those which are slower are spectrally unstable in any weighted space. This case covers the normalized Shigesada function (1.3) mentioned in the introduction, with $D(u) = \varepsilon(u^2 + bu)$, and $b = 1/\varepsilon > 0$ large.

7. Discussion

In this paper we have shown that the spectrum of the linearized differential operator around any diffusion-degenerate Fisher-KPP traveling front with speed satisfying condition (6.5) is located in the stable complex half plane, $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$, when it is computed with respect to an appropriate exponentially weighted L^2-space. In other words, if the front satisfies (6.5) then we can always find a local energy space under which spectral stability holds. A few remarks, however, are in order. First, it is important to notice that our main result does not imply the existence of a "spectral gap", that is, that $\sigma \subset \{\operatorname{Re}\lambda \leq -\omega < 0\} \cup \{0\}$, for some $\omega > 0$. This is a limitation of the technique used. Moreover, we have not proved that there is accumulation of the continuous spectrum near $\operatorname{Re}\lambda = 0$ either. The seasoned reader might rightfully ask why the parabolic regularization technique (see section 5) is not used to locate the whole Weyl's essential spectrum. The answer to that question is precisely that, due to the degeneracy, it might happen that the reduced minimum modulus, $\gamma(\mathcal{L}^\epsilon - \lambda)$, tends to zero as $\epsilon \to 0^+$, as it does for points in $\sigma_\pi(\mathcal{L})$. Therefore, one must sort out the points for which $\mathcal{L} - \lambda$ is a closed range operator. As a consequence, the set which must be controlled with the use of energy estimates, namely $\sigma_\pi \cup \sigma_{\rm pt}$, is much larger and not necessarily composed of isolated points with finite multiplicity only, like in the standard approach for strictly parabolic problems. The existence of a spectral gap is an important issue to be resolved prior to studying the nonlinear (orbital) stability, inasmuch as it is well-known that the existence of a spectral gap simplifies the nonlinear study with the use of standard exponentially decaying semigroup tools (see, e.g., [17, 20, 38, 45] and the references therein).

In addition, we conjecture that the ideas introduced here to handle the degeneracy of the diffusion can be applied to other circumstances. For instance, it is clear that other reaction functions might as well be taken into account, such as the bistable (or Nagumo) type [28, 35]; this case will be addressed in a companion paper [25]. Another possible application is the case of traveling fronts with doubly-degenerate diffusions, which arise naturally in bacterial aggregation models [24], and whose existence has been already studied (cf. [26, 27]). We believe that the analysis presented here can be applied to those situations as well, by taking care of the points in the argumentation where monotonicity of $D = D(\cdot)$ should be dropped, and by extending the methods to more orders of degeneracy. Finally, the stability of traveling fronts for systems with degenerate diffusion tensors is an important open problem whose investigation could profit from some of the ideas developed in this paper.

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Appendix A. Proof of Lemma 4.1

In this section, we verify that for fixed $\lambda \in \mathbb{C}$ and $g \in L^2$, if $u \in H^2$ is a solution to the resolvent equation $(\mathcal{L} - \lambda)u = g$ then

$$w(x) = \exp\left(\frac{c}{2}\int_{x_0}^{x} \frac{ds}{D(\varphi(s))}\right) u(x) = e^{-\theta(x)}u,$$

belongs to H^2 as well. Here $\mathcal{L}$ is the linearized operator around a traveling front φ for the Fisher-KPP equation (1.1), traveling with speed $c > c_* > 0$, and $x_0 \in \mathbb{R}$ is fixed but arbitrary.

For Fisher-KPP diffusion-degenerate fronts one has $u_+ = 0$, $u_- = 1$, with $f'(0) > 0$, $f'(1) < 0$ and they are monotone decreasing with $\varphi_x < 0$. These fronts are diffusion-degenerate as $x \to +\infty$ in view that $D(u_+) = D(0) = 0$.

On the non-degenerate side, as $x \to -\infty$, notice that

$$\int_{x_0}^{x} D(\varphi(s))^{-1}\, ds \leq 0,$$

for all $x \leq x_0$. Therefore,

$$1 \geq \Theta(x) := \exp\left(\frac{c}{2}\int_{x_0}^{x} \frac{ds}{D(\varphi(s))}\right),$$

This yields $|w(x)| \leq \Theta(x)|u(x)| \leq |u(x)|$ for $x \in (-\infty, x_0)$ and, consequently, $w \in L^2(-\infty, x_0)$. Now, for $x_0 \in \mathbb{R}$ fixed, let

$$\delta_0 := \inf_{x\in(-\infty,x_0]} D(\varphi(x)) > 0.$$

Then for all $x < x_0$ we have

$$\Theta(x) \leq \exp\Big(-\frac{c}{2\delta_0}|x - x_0|\Big) \to 0, \qquad \text{as } x \to -\infty.$$

Also from monotonicity of $D = D(\cdot)$ we have

$$0 < \frac{c}{2}\frac{\Theta(x)}{D(1)} \leq \Theta_x(x) = \frac{c}{2}\frac{\Theta(x)}{D(\varphi(x))} \leq \frac{c}{2\delta_0}\Theta(x) \to 0,$$

as $x \to -\infty$. We conclude that $w_x = \Theta_x u + \Theta u_x \in L^2(-\infty, x_0)$. Analogously, it can be easily verified that $w_{xx} \in L^2(-\infty, x_0)$ (we omit the details).

On the degenerate side, as $x \to +\infty$, however, observe that $\int_{x_0}^x D(\varphi(s))^{-1}\, ds \geq 0$ for all $x \geq x_0$ and, thus, the analysis of the decay at $+\infty$ of solutions to resolvent type equations is more delicate. The details are provided by the following lemma.

Lemma A.1. *Suppose that φ is a Fisher-KPP diffusion-degenerate front. Then, for any fixed $\lambda \in \mathbb{C}$ and $g \in L^2(\mathbb{R};\mathbb{C})$, any H^2-solution u to the resolvent equation $(\mathcal{L} - \lambda)u = g$ can be written as*

$$u(x) = C \exp\left(-\frac{c}{2}\int_{x_0}^x \frac{ds}{D(\varphi(s))}\right)\zeta(x), \tag{A.1}$$

for $x > x_0$, $x_0 \in \mathbb{R}$ fixed, $x_0 \gg 1$ sufficiently large, with some constant $C \in \mathbb{C}$, and where $\zeta = \zeta(x) \in H^2(x_0, +\infty)$ decays to zero as $x \to +\infty$ like

$$\zeta(x) \sim e^{f'(0)x/2c} \exp\left(-\frac{c^2}{2D'(0)f'(0)} e^{f'(0)x/c}\right). \tag{A.2}$$

In view of Lemma A.1, for $x_0 \gg 1$ fixed we have $w(x) = C\zeta(x) \in H^2(x_0, +\infty)$. Therefore we conclude that $w \in H^2(\mathbb{R};\mathbb{C})$ as claimed. This finishes the proof of Lemma 4.1. We are left to prove Lemma A.1.

Proof of Lemma A.1. Consider the change of variables $u = \varphi_x v$. Upon substitution into the resolvent equation $(\mathcal{L} - \lambda)u = g$ and using $\mathcal{L}\varphi_x = 0$, we obtain

$$v_{xx} + \rho(x)v_x - \frac{\lambda}{D(\varphi)}v = \widetilde{g},$$

where,

$$\tilde{g} = \frac{g}{D(\varphi)\varphi_x}, \qquad \rho(x) = \frac{2(D(\varphi)\varphi_x)_x}{D(\varphi)\varphi_x} + \frac{c}{D(\varphi)}.$$

Now, let us define

$$v(x) = \exp\left(-\frac{1}{2}\int_{x_0}^x \rho(s)\, ds\right) z(x).$$

Substituting we obtain the second order equation

$$z_{xx} - F(x,\lambda)z = h, \tag{A.3}$$

with

$$F(x,\lambda) = \frac{\lambda}{D(\varphi)} + \frac{1}{2}\rho_x + \frac{1}{4}\rho^2, \quad h(x) = \frac{g(x)}{D(\varphi)\varphi_x}\exp\left(\frac{1}{2}\int_{x_0}^x \rho(s)\, ds\right).$$

Now, it is to be noticed that

$$\begin{aligned}\exp\left(-\frac{1}{2}\int_{x_0}^{x}\rho(s)\,ds\right) &= \frac{|(D(\varphi)\varphi_x)(x_0)|}{|D(\varphi)\varphi_x|}\exp\left(-\frac{c}{2}\int_{x_0}^{x}\frac{ds}{D(\varphi(s))}\,ds\right)\\ &= -\frac{C_0}{D(\varphi)\varphi_x}\exp\left(-\frac{c}{2}\int_{x_0}^{x}\frac{ds}{D(\varphi(s))}\,ds\right),\end{aligned}$$

where $C_0 = |(D(\varphi)\varphi_x)(x_0)| > 0$, and in view of monotonicity, $\varphi_x < 0$. Let us now define

$$\widetilde{\Theta}(x) := \exp\left(-\frac{c}{2}\int_{x_0}^{x}\frac{ds}{D(\varphi(s))}\,ds\right).$$

Observe that $\widetilde{\Theta} \to 0$ as $x \to +\infty$, and also that

$$u = \varphi_x v = \varphi_x \exp\left(-\frac{1}{2}\int_{x_0}^{x}\rho(s)\,ds\right) z(x) = -C_0\frac{\widetilde{\Theta}(x)}{D(\varphi)} z(x) =: C_0\zeta(x)\widetilde{\Theta}(x).$$

Here $\zeta = -z/D(\varphi)$ and z is the decaying solution to equation (A.3). Since by definition $w(x) = \widetilde{\Theta}(x)^{-1}u(x)$, we arrive at

$$w(x) = C_0\zeta(x).$$

Thus, the goal is to show that ζ decays as $x \to +\infty$ fast enough, so that $\zeta \in L^2(x_0, +\infty)$.

First we observe that, substituting the profile equation (3.2), we may write

$$\rho = \frac{2(D(\varphi)\varphi_x)_x}{D(\varphi)\varphi_x} + \frac{c}{D(\varphi)} = -\frac{c}{D(\varphi)} - \frac{2f(\varphi)}{D(\varphi)\varphi_x}.$$

By Lemma 2.3, the wave decays as $\varphi = O(e^{-f'(0)x/c}) \to 0$ as $x \to +\infty$. Making Taylor expansions near $\varphi = 0$ of the form

$$D(\varphi) = \varphi(D'(0) + O(\varphi)), \qquad D(\varphi)^{-1} = \varphi^{-1}D'(0)^{-1} + O(1), \tag{A.4}$$

we find that

$$\rho(x) = \frac{c}{D'(0)}\frac{1}{\varphi(x)} + O(1) \to +\infty, \qquad \text{as } x \to +\infty.$$

Moreover,

$$\frac{1}{4}\rho(x)^2 = \frac{c^2}{4D'(0)^2}\frac{1}{\varphi(x)^2} \to +\infty, \qquad \frac{\lambda}{D(\varphi)} = \frac{\lambda}{D'(0)}\frac{1}{\varphi(x)} + O(1),$$

as $x \to +\infty$. Notice that $\lambda/D(\varphi)$ diverges at order $O(1/|\varphi|)$ for λ fixed. Upon differentiation of the expression for ρ and substitution of the Taylor expansions around φ, one can show that

$$\rho_x = \frac{3f'(0)}{D'(0)}\frac{1}{\varphi} + O(1), \qquad \varphi \sim 0^+.$$

Thus, we reckon that

$$F(x,\lambda) = \frac{\lambda}{D(\varphi)} + \frac{1}{2}\rho_x + \frac{1}{4}\rho^2 = \frac{c^2}{4D'(0)^2}\varphi^{-2} + \Big(\frac{\lambda}{D'(0)} - \frac{3f'(0)}{2D'(0)}\Big)\varphi^{-1} + O(1),$$

for $\varphi \sim 0$ as $x \to +\infty$. The leading term is $\varphi^{-2}c^2/(4D'(0)^2) > 0$, and since $\varphi = O(e^{-f'(0)x/c})$, then $F(x,\lambda)$ diverges at order $O(e^{2f'(0)x/c})$ as $x \to +\infty$. For fixed $\lambda \in \mathbb{C}$, the leading term does not depend on λ and has a definite sign. Therefore, by taking real and imaginary parts, we can assume, without loss of generality, that z decays as the real solution to the equation

$$z_{xx} - F(x)z = h, \tag{A.5}$$

with

$$F(x) := \frac{c^2}{4D'(0)^2} e^{2f'(0)x/c}.$$

We now apply the following theorem by Coppel (cf. [6], pg. 122):

Theorem A.2 (Coppel)**.** *Let $F(x) > 0$, $F \in C^2$, be such that*

$$\int_{x_0}^{x} |F^{-3/2}F''| \, dx < +\infty.$$

Then the homogeneous equation

$$z_{xx} - F(x)z = 0, \tag{A.6}$$

has a fundamental system of solutions satisfying

$$\begin{aligned} z &\sim F(x)^{-1/4} \exp\left(\pm \int_{x_0}^{x} F(s)^{1/2} \, ds \right), \\ z_x &\sim F(x)^{1/4} \exp\left(\pm \int_{x_0}^{x} F(s)^{1/2} \, ds \right). \end{aligned} \tag{A.7}$$

In order to simplify the notation, let us define

$$\beta = \frac{2f'(0)}{c} > 0,$$

so that $\varphi = O(e^{-\beta x/2})$, as $x \to +\infty$. Since $F > 0$ diverges at order $O(e^{\beta x})$, then one can verify that $F^{-3/2}F'' \sim e^{-\beta x/2}$ is integrable in $(x_0, +\infty)$, if $x_0 \gg 1$ is chosen sufficiently large. Let $z_1(x)$ and $z_2(x)$ be two linearly independent solutions in $[x_0, +\infty)$ to the homogeneous equation (A.6), decaying and diverging at $+\infty$, respectively. Then, by Coppel's theorem, $z_1(x)$ behaves like

$$\begin{aligned} z_1(x) &\sim F(x)^{-1/4} \exp\left(-\int_{x_0}^{x} F(s)^{1/2} \, ds \right) \\ &= \big(\frac{2D'(0)}{c}\big)^{1/2} e^{-\beta x/4} \exp\left(-\frac{c}{2D'(0)} \int_{x_0}^{x} e^{\beta s/2} \, ds \right) \\ &\leq \big(\frac{2D'(0)}{c}\big)^{1/2} e^{-\beta x/4} \exp\left(-\frac{c}{\beta D'(0)} e^{\beta x/2} \right), \end{aligned}$$

that is,

$$z_1(x) \sim e^{-\beta x/4} \exp\left(-\frac{c}{\beta D'(0)} e^{\beta x/2} \right) \to 0, \tag{A.8}$$

as $x \to +\infty$. Likewise, it is easy to check that

$$\partial_x z_1(x) \sim e^{\beta x/4} \exp\left(-\frac{c}{\beta D'(0)} e^{\beta x/2} \right) \to 0, \tag{A.9}$$

as $x \to +\infty$. Upon normalization, any decaying solution z to the non-homogeneous equation (A.5) can be written as

$$z(x) = \left(\alpha_1 - \int_{x_0}^{x} z_2(s)h(s) \, ds \right) z_1(x),$$

for some constant α_1, where $z_2(x)$ satisfies

$$z_2 \sim F(x)^{-1/4} \exp\left(\int_{x_0}^{x} F(s)^{1/2}\, ds\right), \qquad x \to +\infty.$$

Since

$$h(s) = \frac{g(s)}{(D(\varphi)\varphi_x)(s)} \exp\left(\frac{1}{2}\int_{x_0}^{s} \rho(\xi)\, d\xi\right) = -C_0 \frac{g(s)}{(D(\varphi)\varphi_x)(s)^2} \widetilde{\Theta}(s),$$

we need to determine how $\widetilde{\Theta}(x)$ decays to zero at $+\infty$. Substitute the expansion (A.4) to estimate

$$\int_{x_0}^{x} \frac{ds}{D(\varphi(s))} = \frac{2}{\beta}\frac{1}{D'(0)} e^{\beta(x-x_0)/2} + \hat{C}(x - x_0),$$

for $x > x_0 \gg 1$, sufficiently large. This yields,

$$\widetilde{\Theta}(x) = \exp\left(-\frac{c}{\beta D'(0)} e^{\beta(x-x_0)/2} - \hat{C}(x-x_0)\right) \leq C_1 \exp\left(-\frac{c}{\beta D'(0)} e^{\beta x/2}\right) \to 0,$$

as $x \to +\infty$, for some uniform $C_1 > 0$. Let us denote

$$\widetilde{z}(x) := z_1(x) \int_{x_0}^{x} z_2(s) h(s)\, ds,$$

so that $z(x) = \alpha_1 z_1(x) - \widetilde{z}(x)$. By substitution of $F(x)^{-1/4} = O(e^{-\beta x/4})$, together with

$$\begin{aligned}
|h(s)| &= C_0 \frac{|g(s)|}{(D(\varphi)\varphi_x)(s)^2} \widetilde{\Theta}(s) \\
&\leq C_0 C_1 |g(s)| O(e^{2\beta s}) \exp\left(-\frac{c}{\beta D'(0)} e^{\beta s/2}\right) \\
&\leq C_2 |g(s)| e^{2\beta s} \exp\left(-\frac{c}{\beta D'(0)} e^{\beta s/2}\right),
\end{aligned}$$

and with the asymptotic behavior of $z_2(s)$, we observe that, for $x_0 \gg 1$ sufficiently large,

$$\begin{aligned}
\int_{x_0}^{x} z_2(s) h(s)\, ds &\sim \int_{x_0}^{x} F(s)^{-1/4} \exp\left(\int_{x_0}^{s} F(y)^{1/2}\, dy\right) ds \\
&\leq C_2 \Big(\frac{2D'(0)}{c}\Big)^{1/2} \int_{x_0}^{x} \left\{ e^{-\beta s/4} \exp\left(\frac{c}{\beta D'(0)} e^{\beta(s-x_0)/2}\right) |g(s)| \times \right. \\
&\qquad\qquad \left. \times e^{2\beta s} \exp\left(-\frac{c}{\beta D'(0)} e^{\beta s/2}\right) \right\} ds \\
&= C_3 \int_{x_0}^{x} |g(s)| e^{7\beta s/4} \exp\left(-C_4 e^{\beta s/2}\right) ds,
\end{aligned}$$

where

$$C_4 = \frac{c}{\beta D'(0)} \big(1 - e^{-\beta x_0/2}\big) > 0.$$

Since the function $\omega(s) = e^{7\beta s/4} \exp(-C_4 e^{\beta s/2})$ clearly belongs to $L^2(x_0, +\infty)$, we notice that the term

$$\begin{aligned}
\int_{x_0}^{x} z_2(s) h(s)\, ds \sim \hat{C} \int_{x_0}^{x} \omega(s) |g(s)|\, ds &\leq \hat{C} \|\omega\|_{L^2(x_0,+\infty)} \|g\|_{L^2(x_0,+\infty)} \\
&\leq \overline{C} \|g\|_{L^2} = O(\|g\|_{L^2}),
\end{aligned}$$

is bounded as $x \to +\infty$. This shows, together with the decay of $z_1(x)$, (A.8), that

$$|\widetilde{z}(x)| \leq C_g e^{-\beta x/4} \exp\left(-\frac{c}{\beta D'(0)} e^{\beta x/2}\right), \tag{A.10}$$

with $C_g = O(\|g\|_{L^2})$. Therefore, the decaying solution to (A.5) (and, consequently, the decaying solution to (A.3)), behaves like

$$z(x) \sim e^{-\beta x/4} \exp\left(-\frac{c}{\beta D'(0)} e^{\beta x/2}\right), \qquad x \to +\infty.$$

(It can be easily verified with the same arguments that the term of the from $z_2(x) \int^x h(s) z_1(s)\, ds$ is actually divergent as $x \to +\infty$. We omit the details.) This yields,

$$\begin{aligned}\zeta(x) = -\frac{z(x)}{D(\varphi(x))} &\sim e^{\beta x/4} \exp\left(-\frac{c}{\beta D'(0)} e^{\beta x/2}\right) \\ &\sim e^{f'(0)x/2c} \exp\left(-\frac{c^2}{2D'(0)f'(0)} e^{f'(0)x/c}\right),\end{aligned}$$

when $x \to +\infty$, as claimed.

It can be shown, using the decay rate (A.7) for the derivatives of solutions to the homogeneous equation, $\partial_x z_1$, that $\zeta_x \in L^2(x_0, +\infty)$, with a decay of the form $e^{kx} \exp(-\bar{C} e^{f'(0)x/c})$, with $k > 0$, $\bar{C} > 0$. A similar procedure leads to $\zeta_{xx} \in L^2(x_0, +\infty)$. The details are omitted as the proof is analogous (the rapidly decaying term $\exp(-\bar{C} e^{f'(0)x/c})$ controls the possibly blowing up terms of form e^{kx}, so that derivatives remain in L^2). This concludes the proof. $\square$

Remark A.3. It is to be noticed that $D(\varphi)\zeta e^{-\theta}$ decays as $x \to +\infty$, inasmuch as

$$\begin{aligned}D(\varphi)|\zeta| e^{-\theta} &= \frac{|z(x)|}{\widetilde{\Theta}(x)} \\ &\sim e^{-f'(0)x/2c} \exp\left(-\frac{c^2}{2D'(0)f'(0)} e^{f'(0)x/c}\right) \exp\left(\frac{c^2}{2D'(0)f'(0)} e^{f'(0)x/c}\right) \\ &= O(1)\, e^{-f'(0)x/2c}.\end{aligned} \tag{A.11}$$

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