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STABILITY OF SCALAR RADIATIVE SHOCK PROFILES*

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Abstract. This work establishes nonlinear orbital asymptotic stability of scalar radiative shock profiles, namely, traveling wave solutions to the simplified model system of radiating gas [K. Hamer, *Quart. J. Mech. Appl. Math.*, 24 (1971), pp. 155–168], consisting of a scalar conservation law coupled with an elliptic equation for the radiation flux. The method is based on the derivation of pointwise Green function bounds and the description of the linearized solution operator. A new feature in the present analysis is the construction of the resolvent kernel for the case of an eigenvalue system of equations of degenerate type. Nonlinear stability then follows in standard fashion by linear estimates derived from these pointwise bounds, combined with nonlinear-damping–type energy estimates.

Key words. hyperbolic-elliptic coupled systems, radiative shock, pointwise Green function bounds, Evans function

AMS subject classifications. 35B35, 34B27, 35M20, 76N15

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1. Introduction. The one-dimensional motion of a radiating gas (due to high-temperature effects) can be modeled by the compressible Euler equations coupled with an elliptic equation for the radiative flux term [8, 43]. The present work considers the following simplified *model system of a radiating gas*:

$$\begin{aligned} u_t + f(u)_x + Lq_x &= 0, \\ -q_{xx} + q + M(u)_x &= 0, \end{aligned} \tag{1.1}$$

consisting of a single regularized conservation law coupled with a scalar elliptic equation. In (1.1), $(x,t) \in \mathbb{R} \times [0,+\infty)$, u and q are scalar functions of (x,t), $L \in \mathbb{R}$ is a constant, and f, M are scalar functions of u. Typically, u and q represent velocity and heat flux of the gas, respectively. When the velocity flux is the Burgers flux function, $f(u) = \frac{1}{2}u^2$, and the coupling term $M(u) = b\,u$ is linear (b constant), this system constitutes a good approximation of the physical Euler system with radiation [8], and it has been extensively studied by Kawashima and Nishibata [16, 17, 18], Serre [41], and Ito [13], among others. For the details of such an approximation the reader may refer to [17, 19, 8].

Formally, one may express q in terms of u as $q = -\mathcal{K}M(u)_x$, where $\mathcal{K} = (1 - \partial_x^2)^{-1}$, so that system (1.1) represents some regularization of the hyperbolic (inviscid) associated conservation law for u. Thus, a fundamental assumption in the study of

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such systems is that

$$L\,\frac{dM}{du}(u) > 0, \tag{1.2}$$

for all u under consideration, conveying the right sign in the diffusion coming from the Chapman–Enskog expansion (see [40]).

We are interested in traveling wave solutions to system (1.1) of the form

$$(u,q)(x,t) = (U,Q)(x-st), \qquad (U,Q)(\pm\infty) = (u_\pm, 0), \tag{1.3}$$

where the triple (u_+, u_-, s) is a shock front of Lax type of the underlying scalar conservation law for the velocity,

$$u_t + f(u)_x = 0, \tag{1.4}$$

satisfying the Rankine–Hugoniot condition $f(u_+) - f(u_-) = s(u_+ - u_-)$ and the Lax entropy condition $\frac{df}{du}(u_+) < s < \frac{df}{du}(u_-)$. Moreover, we assume genuine nonlinearity of the conservation law (1.4), namely, that the velocity flux is strictly convex,

$$\frac{d^2 f}{du^2}(u) > 0, \tag{1.5}$$

for all u under consideration, for which the entropy condition reduces to $u_+ < u_-$. We refer to *weak* solutions of the form (1.3) to the system (1.1), under the Lax shock assumption for the scalar conservation law, as *radiative shock profiles*. The existence and regularity of traveling waves of this type under hypotheses (1.2) is known [16, 22], even for nonconvex velocity fluxes [22]. The existence of profiles for the Euler–Poisson system of radiating gases was addressed by Lin, Coulombel, and Goudon [26] (for discussion of numerical approximation of profiles, see [28, 27]).

According to custom, and without loss of generality, we can reduce the general shock profile (1.3) with $s \neq 0$ to the case of a stationary profile with $s = 0$ by introducing a convenient change of variable and relabeling the flux function f accordingly. Therefore, and after substitution, we consider a stationary radiative shock profile $(U,Q)(x)$ solution to (1.1), satisfying

$$\begin{aligned} f(U)' + L\,Q' &= 0, \\ -Q'' + Q + M(U)' &= 0 \end{aligned} \tag{1.6}$$

(here $'$ denotes differentiation with respect to x), connecting endpoints $(u_\pm, 0)$ at $\pm\infty$, that is,

$$\lim_{x\to\pm\infty} (U,Q)(x) = (u_\pm, 0).$$

Therefore, we summarize our main structural assumptions as follows:

$$f, M \in C^5 \qquad \text{(regularity)}, \tag{A0}$$

$$\frac{d^2 f}{du^2}(u) > 0 \qquad \text{(genuine nonlinearity)}, \tag{A1}$$

$$f(u_-) = f(u_+) \qquad \text{(Rankine–Hugoniot condition)}, \tag{A2}$$

$$u_+ < u_- \qquad \text{(Lax entropy condition)}, \tag{A3}$$

$$L\,\frac{dM}{du}(u) > 0 \qquad \text{(positive diffusion)}, \tag{A4}$$

where $u \in [u_+, u_-]$. For concreteness let us denote

$$a(x) := \frac{df}{du}(U(x)), \quad b(x) := \frac{dM}{du}(U(x)) \tag{1.7}$$

and assume (up to translation) that $a(0) = 0$. Besides the previous structural assumptions, we further suppose that

$$Lb(0) + \left(k + \frac{1}{2}\right) a'(0) > 0, \qquad k = 1, \ldots, 4. \tag{A5$_k$}$$

Remark 1.1. Under assumption (A4), the radiative shock profile is monotone, and, as shown later on, the spectral stability condition holds. Let us stress that, within the analysis of the linearized problem and of the nonlinear stability, we need (A4) to hold only at the end states $u_\pm$ and at the degenerating value $U(0)$.

Remark 1.2. Hypotheses (A5$_k$) are a set of additional technical assumptions inherited from the present stability analysis (see the establishment of H^k energy estimates in section 6, and of pointwise reduction bounds in Lemma 3.4) and are not necessarily sharp. It is worth mentioning, however, that assumptions (A5$_k$), with $k = 1, \ldots, 4$, are satisfied, for instance, for all profiles with small-amplitude $|u_- - u_+|$, in view of (1.2) and $|U'| = \mathcal{O}(|u_- - u_+|)$.

In the present paper, we establish the asymptotic stability of the shock profile (U, Q) under small initial perturbation. Nonlinear wave behavior for system (1.1) and its generalizations has been the subject of thorough research over the last decade. The well-posedness theory is the object of study in [21, 14, 15, 12, 2] for both the simplified model system and more general cases. The stability of constant states [41], rarefaction waves [19, 5], and asymptotic profiles [24, 4, 3] for the model system with Burgers flux has been addressed in the literature.

Regarding the asymptotic stability of radiative shock profiles, the problem has been previously studied by Kawashima and Nishibata [16] in the particular case of Burgers velocity flux and for linear $M(u) = b\,u$ (b constant), which is one of the few available stability results for *scalar* radiative shocks in the literature.[1] In [16], the authors establish asymptotic stability with basically the same rate of decay in L^2 and under fairly similar assumptions as we have here. Their method, though, relies on integrated coordinates and the L^1 contraction property, a technique which may not work for the system case, i.e., $\tilde{u} \in \mathbb{R}^n$, $n \geq 2$. In contrast, we provide techniques which may be extrapolated to systems, enable us to handle variable $\frac{dM}{du}(u)$, and provide a large-amplitude theory based on spectral stability assumptions in cases where linearized stability is not automatic (for the analysis in the system case, see [35]). These technical considerations are some of the main motivations for the present analysis.

The nonlinear asymptotic stability of traveling wave solutions to models in continuum mechanics, more specifically, of shock profiles under suitable regularizations of hyperbolic systems of conservation laws, has been the subject of intense research in recent years (see, e.g., [10, 47, 29, 30, 31, 44, 45, 46, 38, 36, 20]). The unifying methodological approach of these works consists of refined semigroup techniques and the establishment of sharp pointwise bounds on the Green function associated to the linearized operator around the wave, under the assumption of spectral stability.

[1] The other scalar result is the partial analysis of Serre [42] for the exact Rosenau model; in the case of systems, we mention the stability result of [25] for the full Euler radiating system under zero-mass perturbations, based on an adaptation of the classical energy method of Goodman [7] and Matsumura and Nishihara [33].

A key step in the analysis is the construction of the resolvent kernel, together with appropriate spectral bounds. The pointwise bounds on the Green function follow by the inverse Laplace transform (spectral resolution) formula [47, 30, 44]. The main novelty in the present case is the extension of the method to a situation in which the eigenvalue equations are written as a degenerate first-order ODE system (see the discussion in section 1.3). Such an extension, we hope, may serve as a blueprint for treating other model systems for which the resolvent equation becomes singular. This feature is also one of the main technical contributions of the present analysis.

1.1. Main results. In the spirit of [47, 29, 31, 32], we first consider solutions to (1.1) of the form $(u+U, q+Q)$, where u and q are perturbations, and study the linearized equations of (1.1) about the profile (U, Q), which read,

$$\begin{aligned} u_t + (a(x)u)_x + Lq_x &= 0, \\ -q_{xx} + q + (b(x)u)_x &= 0, \end{aligned} \tag{1.8}$$

with initial data $u(0) = u_0$ (functions a, b are defined in (1.7)). Hence, the Laplace transform applied to system (1.8) gives

$$\begin{aligned} \lambda u + (a(x)u)' + Lq' &= S, \\ -q'' + q + (b(x)u)' &= 0, \end{aligned} \tag{1.9}$$

where source S is the initial data u_0.

As is customary in related nonlinear wave stability analyses [1, 39, 47, 6, 29, 30, 44, 46], we start by studying the underlying spectral problem, namely, the homogeneous version of system (1.9):

$$\begin{aligned} (a(x)u)' &= -\lambda\, u - Lq', \\ q'' &= q + (b(x)u)'. \end{aligned} \tag{1.10}$$

An evident necessary condition for orbital stability is the absence of L^2 solutions to (1.10) for values of λ in $\{\mathrm{Re}\,\lambda \geq 0\}\backslash\{0\}$, where $\lambda = 0$ is the eigenvalue associated to translation invariance. This strong spectral stability can be expressed in terms of the *Evans function*, an analytic function playing a role for differential operators analogous to that played by the characteristic polynomial for finite-dimensional operators (see [1, 39, 6, 47, 29, 30, 45, 44, 46] and the references therein). The main property of the Evans function is that, on the resolvent set of a certain operator $\mathcal{L}$, its zeroes coincide in both location and multiplicity with the eigenvalues of $\mathcal{L}$.

In the present case, and due to the degenerate nature of system (1.10) (observe that $a(x)$ vanishes at $x = 0$), the number of decaying modes at $\pm\infty$, spanning possible eigenfunctions, depends on the region of space around the singularity (see Remark 3.1). Therefore, we define the following *stability criterion*, where the analytic functions $D_\pm(\lambda)$ (see their definition in (3.32)) denote the two Evans functions associated with the linearized operator about the profile in regions $x \gtrless 0$, correspondingly, where the analytic functions whose zeroes away from the essential spectrum agree in location and multiplicity with the eigenvalues of the linearized operator or solutions of (1.10):

(D) There exist no zeroes of $D_\pm(\cdot)$ in the nonstable half plane $\{\mathrm{Re}\,\lambda \geq 0\} \setminus \{0\}$.

Our main result is then as follows.

THEOREM 1.3. *Assuming* (A0)–(A5$_k$) *and the spectral stability condition* (D), *the Lax radiative shock profile* (U, Q) *is asymptotically orbitally stable. More precisely, the solution* $(\tilde{u}, \tilde{q})$ *of* (1.1) *with initial data* $\tilde{u}_0$ *satisfies*

$$|\tilde{u}(x,t) - U(x-\alpha(t))|_{L^p} \le C(1+t)^{-\frac{1}{2}(1-1/p)}|u_0|_{L^1\cap H^4},$$
$$|\tilde{u}(x,t) - U(x-\alpha(t))|_{H^4} \le C(1+t)^{-1/4}|u_0|_{L^1\cap H^4}$$

and

$$|\tilde{q}(x,t) - Q(x-\alpha(t))|_{W^{1,p}} \le C(1+t)^{-\frac{1}{2}(1-1/p)}|u_0|_{L^1\cap H^4},$$
$$|\tilde{q}(x,t) - Q(x-\alpha(t))|_{H^5} \le C(1+t)^{-1/4}|u_0|_{L^1\cap H^4}$$

for initial perturbations $u_0 := \tilde{u}_0 - U$ *that are sufficiently small in* $L^1 \cap H^4$, *for all* $p \ge 2$, *for some* $\alpha(t)$ *satisfying* $\alpha(0) = 0$ *and*

$$|\alpha(t)| \le C|u_0|_{L^1\cap H^4}, \qquad |\dot{\alpha}(t)| \le C(1+t)^{-1/2}|u_0|_{L^1\cap H^4},$$

where $\dot{}$ *denotes the derivative with respect to* t.

Remark 1.4. The time-decay rate of q is not optimal. In fact, it can be improved, as we observe that $|q(t)|_{L^2} \le C|u_x(t)|_{L^2}$, and $|u_x(t)|_{L^2}$ is expected to decay like $t^{-1/2}$; we omit the details of the proof.

The second result of this paper is the verification of the spectral stability condition (D) under particular circumstances.

PROPOSITION 1.5. *The spectral stability condition* (D) *holds under either of the following conditions:*

(i) b *is a constant, or*

(ii) $|u_+ - u_-|$ *is sufficiently small.*

Proof. See Appendix B. ∎

1.2. Discussions. Combining Theorem 1.3 and Proposition 1.5(i), we partially recover the results of [16] for the Burgers flux and linear $M(u) = b\,u$ (b constant) and, at the same time, we extend them to general convex flux f. Thanks to Proposition 1.5(ii), we also broaden the result to the case of nonlinear M in the small-shock case.

We note that the stability result of [16] was for all smooth shock profiles, for which the boundary (see [16, Thm. 1.5(ii)(a)]) is the condition $LM = 1 = -a'(0)$; that is, their result holds whenever $LM + a'(0) > 0$. By comparison, our results hold on the smaller set of waves for which $LM + (9/2)a'(0) > 0$; see Remark 1.2. By estimating high-frequency contributions explicitly, rather than by the simple energy estimates used here, we could, at the expense of further effort, reduce these conditions to the single condition

$$LM + 2a'(0) > 0 \tag{1.11}$$

used to prove Lemma 3.4. Elsewhere in the analysis, we need only $LM + a'(0) > 0$; however, at the moment we do not see how to remove (1.11) to recover the full result of [16] in the special case considered there. The interest of our technique, rather, is in its generality—particularly the possibility to extend to the system case—and in the additional information afforded by the pointwise description of behavior, which seems interesting in its own right.

Moreover, differently from [16], we also show convergence to the radiative shock (with same decay rates) without requiring any integrability of the antiderivative of the initial perturbation, paying the additional price of shifting the wave profile by a time-dependent amount α that is proved to be uniformly bounded in $(0, \infty)$.

Remark 1.6. Requiring the antiderivative V of the perturbation v to be bounded in L^1 is roughly equivalent to the condition that v have an L^1 first moment. Indeed, arranging as in [16] that $\int v = 0$, so that $V(x) := \int_{-\infty}^{x} v(y)dy = -\int_{x}^{+\infty} v(y)dy$, taking v odd, and positive for $x > 0$, we have

$$\int_0^{+\infty} |V(y)|dy = \int_0^{+\infty} |V(y)|dy = \int_0^{+\infty} yv(y)dy = \int_0^{+\infty} |y||v(y)|dy.$$

Thus integrability of the antiderivative V has to been considered as a localization assumption of the initial perturbation of the shock profile.

1.3. Abstract framework. Before beginning the analysis, we orient ourselves with a few simple observations framing the problem in a more standard way. Consider now the inhomogeneous version

$$\begin{aligned} u_t + (a(x)u)_x + Lq_x &= \varphi, \\ -q_{xx} + q + (b(x)u)_x &= \psi \end{aligned} \tag{1.12}$$

of (1.8), with initial data $u(x, 0) = u_0$. Defining the compact operator $\mathcal{K} := (-\partial_x^2 + 1)^{-1}$ and the bounded operator

$$\mathcal{J}\, u := -L\, \partial_x\, \mathcal{K}\, \partial_x(b(x)u),$$

we may rewrite this as a nonlocal equation

$$\begin{aligned} u_t + (a(x)u)_x + \mathcal{J} u &= \varphi - L\, \partial_x\, (\mathcal{K}\, \psi)\,, \\ u(x, 0) &= u_0(x) \end{aligned} \tag{1.13}$$

in u alone. The generator $\mathcal{L} := -(a(x)u)_x - \mathcal{J}u$ of (1.13) is a zero order perturbation of the generator $-a(x)u_x$ of a hyperbolic equation, so it generates a C^0 semigroup $e^{\mathcal{L}t}$ and an associated Green distribution $G(x, t; y) := e^{\mathcal{L}t}\delta_y(x)$. Moreover, $e^{\mathcal{L}t}$ and G may be expressed through the inverse Laplace transform formulae

$$\begin{aligned} e^{\mathcal{L}t} &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\lambda - \mathcal{L})^{-1} d\lambda, \\ G(x, t; y) &= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} G_\lambda(x, y) d\lambda \end{aligned} \tag{1.14}$$

for all $\gamma \geq \gamma_0$ (for some $\gamma_0 > 0$), where $G_\lambda(x, y) := (\lambda - \mathcal{L})^{-1}\delta_y(x)$ is the resolvent kernel of $\mathcal{L}$.

Collecting information, we may write the solution of (1.12) using Duhamel's principle/variation of constants as

$$\begin{aligned} u(x, t) &= \int_{-\infty}^{+\infty} G(x, t; y) u_0(y) dy \\ &\quad + \int_0^t \int_{-\infty}^{+\infty} G(x, t - s; y)(\varphi - L\, \partial_x\, (\mathcal{K}\, \psi))(y, s)\, dy\, ds, \\ q(x, t) &= \mathcal{K}\big(\psi - \partial_x\, (b(x)u)\big)(x, t), \end{aligned}$$

where G is determined through (1.14).

That is, the solution of the linearized problem reduces to finding the Green kernel for the u-equation alone, which in turn amounts to solving the resolvent equation for $\mathcal{L}$ with delta-function data, or, equivalently, solving the differential equation (1.9) with source $S = \delta_y(x)$. We shall do this in standard fashion by writing (1.9) as a first-order system and solving appropriate jump conditions at y obtained by the requirement that G_λ be a distributional solution of the resolvent equations.

This procedure is greatly complicated by the circumstance that the resulting 3×3 first-order system, given by

$$(\Theta(x)W)_x = \mathbb{A}(x,\lambda)W, \qquad \text{where} \quad \Theta(x) := \begin{pmatrix} a(x) & 0 \\ 0 & I_2 \end{pmatrix},$$

is *singular* at the special point where $a(x)$ vanishes. However, in the end we find as usual that G_λ is uniquely determined by these criteria, not only for the values $\operatorname{Re}\lambda \geq \gamma_0 > 0$ guaranteed by C^0-semigroup theory/energy estimates, but, as in the usual nonsingular case [9], on the *set of consistent splitting* for the first-order system, which includes all of $\{\operatorname{Re}\lambda \geq 0\} \setminus \{0\}$. This has the implication that the essential spectrum of $\mathcal{L}$ is confined to $\{\operatorname{Re}\lambda < 0\} \cup \{0\}$.

Remark 1.7. The fact (obtained by energy-based resolvent estimates) that $\mathcal{L} - \lambda$ is coercive for $\operatorname{Re}\lambda \geq \gamma_0$ shows by elliptic theory that the resolvent is well defined and unique in the class of distributions for $\operatorname{Re}\lambda$ large, and thus the resolvent kernel may be determined by the usual construction using appropriate jump conditions. That is, from standard considerations, we see that the construction *must* work, despite the apparent wrong dimensions of decaying manifolds (which happen for any $\operatorname{Re}\lambda > 0$).

To deal with the singularity of the first-order system is the most delicate and novel part of the present analysis. It is our hope that the methods we use here may be of use also in other situations where the resolvent equation becomes singular, for example, in the closely related situation of relaxation systems discussed in [29, 32].

Plan of the paper. This work is structured as follows. Section 2 collects some of the properties of radiative profiles and contains a technical result which allows us to rigorously define the resolvent kernel near the singularity. The central section 3 is devoted to the construction of the resolvent kernel, based on the analysis of solutions to the eigenvalue equations both near and away from the singularity. Section 4 establishes the crucial low-frequency bounds for the resolvent kernel. The following section, 5, contains the desired pointwise bounds for the "low-frequency" Green function, based on the spectral resolution formulae. Section 6 establishes an auxiliary nonlinear damping energy estimate. Section 7 deals with the high-frequency region by establishing energy estimates on the solution operator directly. The final section, 8, blends all previous estimations into the proof of the main nonlinear stability result (Theorem (1.3)). We also include three appendices which contain a pointwise extension of the tracking lemma, the proof of spectral stability under linear coupling or small-amplitude assumptions, and the monotonicity of general scalar profiles, respectively.

2. Preliminaries.

2.1. Structure of profiles. Under definition (1.7), we may assume (thanks to translation invariance; see Remark C.5) that $a(x)$ vanishes exactly at one single point which we take as $x = 0$. Likewise, we know that the velocity profile is monotone decreasing (see [22, 23, 42] or Lemma C.4), that is, $U'(x) < 0$, which implies, in view of genuine nonlinearity (1.5), that

$$a'(x) < 0 \qquad \forall\, x \in \mathbb{R} \qquad \text{and} \qquad x\, a(x) < 0 \qquad \forall\, x \neq 0.$$

From the profile equations we obtain, after integration, that

$$LQ = f(u_\pm) - f(U) > 0$$

for all x, due to the Lax condition. Therefore, substitution of the profile equations (1.6) yields the relation

$$\big(a'(x) + L\, b(x)\big)U' = -LQ - a(x)U'',$$

which, evaluating at $x = 0$ and from monotonicity of the profile, implies that

$$a'(0) + L\, b(0) > 0. \tag{2.1}$$

Therefore, the last condition is a consequence of the existence result (see Theorem C.3), and it will be used throughout. Notice that condition (A5$_k$) with $k = 1$ implies condition (2.1).

Next, we show that the waves decay exponentially to their end states, a crucial fact in the forthcoming analysis.

LEMMA 2.1. *Assuming* (A0)–(A4), *a radiative shock profile* (U, Q) *of* (1.1) *satisfies*

$$\left|(d/dx)^k (U - u_\pm, Q)\right| \le C e^{-\eta|x|}, \quad k = 0, \dots, 4, \tag{2.2}$$

as $|x| \to +\infty$, *for some* $\eta > 0$.

Proof. As $|x| \to +\infty$, defining $a_\pm = a(\pm\infty)$ and $b_\pm = b(\pm\infty)$, we consider the asymptotic system of (1.6), that is, the constant coefficient linear system

$$\begin{aligned} a_\pm U' &= -LQ', \\ -Q'' + Q &= -b_\pm U', \end{aligned}$$

which, by substituting U' into the second equation, becomes

$$-Q'' - \frac{Lb_\pm}{a_\pm}\, Q' + Q = 0,$$

or equivalently,

$$\begin{pmatrix} Q \\ Q' \end{pmatrix}' = A_Q \begin{pmatrix} Q \\ Q' \end{pmatrix} \qquad \text{with } A_Q := \begin{pmatrix} 0 & 1 \\ 1 & -Lb_\pm/a_\pm \end{pmatrix},$$

which then gives the exponential decay estimate (2.2) for Q by the hyperbolicity of the matrix A_Q; that is, eigenvalues of A_Q are distinct and nonzero. Estimates for U follow immediately from those for Q and the relation

$$LQ = f(u_\pm) - f(U),$$

obtained by integrating the first equation of (1.6). ∎

2.2. Regularity of solutions near $x = 0$. In this section we establish some analytic properties of the solutions to system (1.10) near the singularity, which will be used during the construction of the resolvent kernel in the central section 3. Introducing the variable $p := b(x)u - q'$, system (1.10) takes the form of a first-order system, which reads

$$\begin{aligned} a(x)u' &= -\left(\lambda + a'(x) + Lb(x)\right)u + Lp, \\ q' &= b(x)u - p, \\ p' &= -q. \end{aligned} \tag{2.3}$$

For technical reasons which will be clear from the forthcoming analysis, in order to define the transmission conditions in the definition of the resolvent kernel (which is defined as solutions to the conservative form of system (2.3) in a distributional sense with appropriate jump conditions; see section 3.1), we need p and q to be regular across the singularity $x = 0$ (having finite limits at both sides) and u to have (at most) an integrable singularity at that point, namely, that $u \in L^1_{loc}$ near zero (away from zero it is bounded, so this is trivially true) and that it verifies $a(x)u \to 0$ as $x \to 0$. These properties are proved in the next technical lemma.

Lemma 2.2. *Given $\lambda \in \mathbb{C}$, set $\nu := (\operatorname{Re}\lambda + a'(0) + Lb(0))/|a'(0)|$. Under assumptions* (A0)–(A4) *and* $\operatorname{Re}\lambda > -Lb(0)$*, any solution of* (2.3) *verifies that*

1. *$|u(x)| \le C\,|x|^{\nu}$ for $x \sim 0$ and for some $C > 0$;*
2. *q is absolutely continuous and p is C^1 (for $x \sim 0$).*

In particular, $u \in L^1_{loc}$ (for $x \sim 0$) and $a(x)u(x) \to 0$ as $x \to 0$.

The proof will be done in two steps: (i) First, taking into account "elliptic regularity" in the equation for p,

$$-p'' + p = b(x)u, \tag{2.4}$$

we prove the L^1_{loc} bound for u close to zero and the subsequent regularity for p and q; and (ii) using such a bound, we then prove the pointwise control given in (i).

Alternatively, one can explicitly solve the above elliptic equation for p and directly obtain the pointwise result for u by plugging the relation into the Duhamel formula for u. Finally, such a control gives the L^1_{loc} property for u and all other regularity properties.

Proof of Lemma 2.2. Let us consider the case $x \ge 0$; the case $x \le 0$ is similar. Consider a fixed $x_0 > 0$, to be chosen afterwards, and let (u, q, p) be any solution of (2.3) emanating from that point. Therefore, from (2.4) we know that

$$p(x) = C_1 e^{-x} + C_2 e^{x} + \int_{x_0}^{x} g(x, y)u(y)dy \tag{2.5}$$

for a given (regular) kernel $g(x, y)$. Therefore there exists a constant C_{x_0} such that for any $\epsilon > 0$,

$$|p|_{L^\infty(\epsilon, x_0)} \le C_{x_0}(1 + |u|_{L^1(\epsilon, x_0)}).$$

Note that the constant C_{x_0} is uniform on ϵ, stays bounded as x_0 approaches zero, and depends only on the initial values $p(x_0)$, $q(x_0) = -p'(x_0)$. Moreover, the

Duhamel principle gives for any $x \in [\epsilon, x_0]$

$$
\begin{aligned}
u(x) &= u(x_0) \exp\left(-\int_{x_0}^{x} \frac{\lambda + a'(y) + L\,b(y)}{a(y)} dy\right) \\
&\quad + L \int_{x_0}^{x} \frac{1}{a(y)} \exp\left(-\int_{y}^{x} \frac{\lambda + a'(z) + L\,b(z)}{a(z)} dz\right) p(y) dy.
\end{aligned} \tag{2.6}
$$

From (1.5) we obtain

$$
\frac{\lambda + a'(x) + L\,b(x)}{a(x)} \sim \frac{\lambda + a'(0) + L\,b(0)}{a'(0)x} \quad \text{for } x \sim 0.
$$

Hence, for $x \sim 0$,

$$
\begin{aligned}
\exp\left(-\int_{x_0}^{x} \frac{\lambda + a'(y) + L\,b(y)}{a(y)} dy\right) &\sim \exp\left(-\int_{x_0}^{x} \frac{\lambda + a'(0) + L\,b(0)}{a'(0)y} dy\right) \\
&= \left|\frac{x}{x_0}\right|^{-\frac{\lambda + a'(0) + L\,b(0)}{a'(0)}}.
\end{aligned} \tag{2.7}
$$

Hence the first term of (2.6) is integrable in $[0, x_0]$ provided $\operatorname{Re}\lambda > -L\,b(0)$, noting that $a'(0) < 0$ (our argument applies for $\lambda \neq -L\,b(0) - a'(0)$; for $\lambda = -L\,b(0) - a'(0)$, all functions in the integrals above are indeed bounded at zero, and the proof of the lemma is even simpler). Thus, for a constant C_{x_0} as above,

$$
\begin{aligned}
|u|_{L^1(\epsilon,x_0)} &\le u(x_0)C_{x_0} + C_{x_0}(1 + |u|_{L^1(\epsilon,x_0)}) \\
&\quad \times \int_{\epsilon}^{x_0} \int_{x_0}^{x} \frac{1}{|a(y)|} \exp\left(-\int_{y}^{x} \frac{\operatorname{Re}\lambda + a'(z) + L\,b(z)}{a(z)} dz\right) dy\, dx.
\end{aligned} \tag{2.8}
$$

Now we use again (2.7) to estimate the integral term in (2.8) as follows:

$$
\begin{aligned}
&\int_{x_0}^{x} \frac{1}{|a(y)|} \exp\left(-\int_{y}^{x} \frac{\operatorname{Re}\lambda + a'(z) + L\,b(z)}{a(z)} dz\right) dy \\
&\quad \sim \int_{x_0}^{x} \frac{1}{|a'(0)y|} \left|\frac{x}{y}\right|^{\nu} dy = \frac{|a'(0)|\, x^{\nu}}{\operatorname{Re}\lambda + a'(0) + L\,b(0)} \left(x^{-\nu} - x_0^{-\nu}\right) \\
&\quad = -\frac{1}{\operatorname{Re}\lambda + a'(0) + L\,b(0)} \left(1 - \left(\frac{x}{x_0}\right)^{\nu}\right).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|u|_{L^1(\epsilon,x_0)} &\le u(x_0)C_{x_0} + C_{x_0}(1 + |u|_{L^1(\epsilon,x_0)})x_0 \\
&\quad + C_{x_0}(1 + |u|_{L^1(\epsilon,x_0)})x_0^{-\nu} \frac{1}{\operatorname{Re}\lambda + L\,b(0)} x_0^{-\frac{\operatorname{Re}\lambda + L\,b(0)}{a'(0)}} \\
&= u(x_0)C_{x_0} + C_{x_0}(1 + |u|_{L^1(\epsilon,x_0)})x_0.
\end{aligned}
$$

Finally, for a sufficiently small, but *fixed*, $x_0 > 0$, from the above relation we conclude

$$
|u(x)|_{L^1(\epsilon,x_0)} \le C_{x_0}
$$

uniformly in ϵ, namely, $u \in L^1(0, \epsilon_0)$ for $\epsilon_0 > 0$. At this point, part 2 of the lemma is an easy consequence of expressions (2.5), $(2.3)_2$ and $(2.3)_3$.

Once we have obtained the L^1_{loc} property of u at zero, we know in particular that $|p|_{L^\infty(0,x_0)}$ is bounded. Hence we can repeat all estimates on the integral terms of (2.6) to obtain part 1 of the lemma. Finally,

$$\lim_{x\to 0} a(x)u(x) = 0$$

is again a consequence of $\operatorname{Re}\lambda > -Lb(0)$. $\square$

Remark 2.3. From condition (2.1) it is clear that, for $\operatorname{Re}\lambda < 0$, but sufficiently close to zero, $u(x)$ is not blowing up for $x \to 0$ but vanishes in that limit, regardless of the shock strength (the negative term $a'(0)$ approaches zero as the strength of the shock tends to zero).

3. Construction of the resolvent kernel.

3.1. Outline. Let us now construct the resolvent kernel for $\mathcal{L}$, or equivalently, the solution of (2.3) with delta-function source in the u component. The novelty in the present case is the extension of this standard method to a situation in which the spectral problem can be written only as a *degenerate* first-order ODE. Unlike the real viscosity and relaxation cases [29, 30, 31, 32] (where the operator $\mathcal{L}$, although degenerate, yields a nondegenerate first-order ODE in an appropriate reduced space), here we deal with the resolvent system for the unknown $W := (u, q, p)^\top$,

$$(\Theta(x)W)' = \mathbb{A}(x,\lambda)W, \tag{3.1}$$

where

$$\Theta(x) := \begin{pmatrix} a(x) & 0 \\ 0 & I_2 \end{pmatrix}, \qquad \mathbb{A}(x,\lambda) := \begin{pmatrix} -(\lambda + L\,b(x)) & 0 & L \\ b(x) & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

that degenerates at $x = 0$.

To construct the resolvent kernel $\mathcal{G}_\lambda = \mathcal{G}_\lambda(x, y)$, we solve

$$\partial_x (\Theta(x)\,\mathcal{G}_\lambda) - \mathbb{A}(x,\lambda)\,\mathcal{G}_\lambda = \delta_y(x)\,I \tag{3.2}$$

in the distributional sense, so that

$$\partial_x (\Theta(x)\,\mathcal{G}_\lambda) - \mathbb{A}(x,\lambda)\,\mathcal{G}_\lambda = 0$$

for all $x \neq y$ with appropriate jump conditions (to be determined) at $x = y$. The first element in the first row of the matrix-valued function $\mathcal{G}_\lambda$ is the resolvent kernel G_λ of $\mathcal{L}$ that we seek.

3.2. Asymptotic behavior. First, we study the asymptotic behavior of solutions to the spectral system

$$\begin{aligned} a(x)u' &= -(\lambda + a'(x) + L\,b(x))u + Lp, \\ q' &= b(x)u - p, \\ p' &= -q, \end{aligned} \tag{3.3}$$

away from the singularity at $x = 0$, and for values of $\lambda \neq 0$, $\operatorname{Re}\lambda \geq 0$. We pay special attention to the small frequency regime, $\lambda \sim 0$. Denote the limits of the coefficients as

$$a_\pm := \lim_{x\to\pm\infty} a(x) = \frac{df}{du}(u_\pm), \qquad b_\pm := \lim_{x\to\pm\infty} b(x) = \frac{dM}{du}(u_\pm).$$

From the structure of the wave we already have that $a_+ < 0 < a_-$. The asymptotic system can be written as

$$\Theta_\pm W' = \mathbb{A}_\pm(\lambda) W, \tag{3.4}$$

where

$$\Theta_\pm := \begin{pmatrix} a_\pm & 0 \\ 0 & I_2 \end{pmatrix}, \qquad \mathbb{A}_\pm(\lambda) := \begin{pmatrix} -(\lambda + L\, b_\pm) & 0 & L \\ b_\pm & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}.$$

To determine the dimensions of the stable/unstable eigenspaces, let $\lambda \in \mathbb{R}^+$, $\lambda \to +\infty$. The characteristic polynomial reads

$$\pi_\pm(\mu) := |\mu\, I - \Theta_\pm^{-1}\mathbb{A}_\pm(\lambda)| = \mu^3 + a_\pm^{-1}(\lambda + Lb_\pm)\mu^2 - \mu - a_\pm^{-1}\lambda,$$

for which

$$\frac{d\pi_\pm}{d\mu} = 3\mu^2 + 2a_\pm^{-1}(\lambda + Lb_\pm)\mu - 1$$

has one negative and one positive zero, regardless of the sign of $a_\pm$, for each $\lambda \gg 1$; they are local extrema of $\pi_\pm$. Since $\pi_\pm \to \pm\infty$ as $\mu \to \pm\infty$, $\pi_\pm(0) = -a_\pm^{-1}\lambda$ has the opposite sign with respect to $a_\pm$ and

$$\pi_\pm(-a_\pm\,\lambda) = a_\pm\left(a_\pm^2 + \frac{1}{a_\pm^4}\right)\lambda^3 + o(\lambda^3), \qquad \lambda \to \infty,$$

so that π_-/π_+ is positive/negative at some negative/positive value of μ, we get two positive zeroes and one negative zero for π_+, and two negative zeroes and one positive zero for π_-, whenever $\lambda \in \mathbb{R}^+$, $\lambda \gg 1$.

We readily conclude that for each $\operatorname{Re}\lambda > 0$, there exist two unstable eigenvalues $\mu_1^+(\lambda)$ and $\mu_2^+(\lambda)$ with $\operatorname{Re}\mu > 0$, and one stable eigenvalue $\mu_3^+(\lambda)$ with $\operatorname{Re}\mu < 0$. The stable $S^+(\lambda)$ and unstable $U^+(\lambda)$ manifolds (solutions which decay (resp., grow) at $+\infty$) have, thus, dimensions

$$\dim U^+(\lambda) = 2, \qquad \dim S^+(\lambda) = 1$$

in $\operatorname{Re}\lambda > 0$. Likewise, there exist two unstable eigenvalues $\mu_1^-(\lambda), \mu_2^-(\lambda)$ with $\operatorname{Re}\mu < 0$, and one stable eigenvalue $\mu_3^-(\lambda)$ with $\operatorname{Re}\mu > 0$, so that the stable (solutions which grow at $-\infty$) and unstable (solutions which decay at $-\infty$) manifolds have dimensions

$$\dim U^-(\lambda) = 1, \qquad \dim S^-(\lambda) = 2. \tag{3.5}$$

Remark 3.1. Notice that, unlike customary situations in the Evans function literature [1, 47, 6, 29, 30, 39], here the dimensions of the stable (resp., unstable) manifolds S^+ and S^- (resp., U^+ and U^-) *do not agree*. Under these considerations, we look at the dispersion relation

$$\pi_\pm(i\xi) = -i\xi^3 - a_\pm^{-1}(\lambda + Lb_\pm)\xi^2 - i\xi - a_\pm^{-1}\,\lambda = 0.$$

For each $\xi \in \mathbb{R}$, the λ-roots of last equation define algebraic curves

$$\lambda_\pm(\xi) = -i\, a_\pm\, \xi - \frac{L\, b_\pm\, \xi^2}{1 + \xi^2}, \quad \xi \in \mathbb{R},$$

touching the origin at $\xi = 0$. Denote Λ as the open connected subset of $\mathbb{C}$ bounded on the left by the two curves $\lambda_\pm(\xi)$, $\xi \in \mathbb{R}$. Since $L\, b_\pm > 0$ by assumption (A4), the set $\{\mathrm{Re}\,\lambda \geq 0, \lambda \neq 0\}$ is properly contained in Λ. By connectedness the dimensions of $U^\pm(\lambda)$ and $S^\pm(\lambda)$ do not change in $\lambda \in \Lambda$. We define Λ as the set of *(not so) consistent splitting* [1], in which the matrices $\Theta_\pm^{-1}\mathbb{A}_\pm(\lambda)$ remain hyperbolic, with not necessarily agreeing dimensions of stable (resp., unstable) manifolds.

In the low-frequency regime $\lambda \sim 0$, we notice, by taking $\lambda = 0$, that the eigenvalues behave like those of $\Theta_\pm^{-1}\mathbb{A}_\pm(0)$. If we define

$$\begin{aligned}
\theta_1^+ &:= \frac{1}{2}\left(-a_+^{-1}Lb_+ + \sqrt{a_+^{-2}L^2b_+^2 + 4}\right),\\
\theta_1^- &:= \frac{1}{2}\left(a_-^{-1}Lb_+ + \sqrt{a_-^{-2}L^2b_-^2 + 4}\right),\\
\theta_3^+ &:= \frac{1}{2}\left(a_+^{-1}Lb_+ + \sqrt{a_+^{-2}L^2b_+^2 + 4}\right),\\
\theta_3^- &:= \frac{1}{2}\left(-a_-^{-1}Lb_+ + \sqrt{a_-^{-2}L^2b_-^2 + 4}\right)
\end{aligned}$$

as the decay/growth rates for the fast modes (notice that $\theta_j^\pm > 0$, $j = 1, 3$), then the latter are given by

$$\begin{aligned}
&\mu_2^\pm(0) = 0,\\
&\mu_1^-(0) = -\theta_1^- < 0 < \theta_1^+ = \mu_1^+(0),\\
&\mu_3^+(0) = -\theta_3^+ < 0 < \theta_3^- = \mu_3^-(0)\ .
\end{aligned}$$

The associated eigenvectors are given by

$$V_j^\pm = \begin{pmatrix} b_\pm^{-1}(1 - \mu_j^\pm(0)^2) \\ -\mu_j^\pm(0) \\ 1 \end{pmatrix}.$$

Since the highest order coefficient of $\pi_\pm$ as a polynomial in μ is different from zero, then $\lambda = 0$ is a regular point from whence, by standard algebraic curves theory, there exist convergent series in powers of λ for the eigenvalues. For low frequency, the eigenvalues of $\Theta_\pm^{-1}\mathbb{A}_\pm(\lambda)$ have analytic expansions of the form

$$\begin{aligned}
\mu_2^\pm(\lambda) &= -\frac{\lambda}{a_\pm} + \mathcal{O}(|\lambda|^2),\\
\mu_1^\pm(\lambda) &= \pm\theta_1^\pm + \mathcal{O}(|\lambda|),\\
\mu_3^\pm(\lambda) &= \mp\theta_3^\pm + \mathcal{O}(|\lambda|),
\end{aligned} \tag{3.6}$$

corresponding to a slow varying mode and two fast modes, respectively, for low frequencies. By inspection, the associated eigenvectors can be chosen as

$$V_j^\pm = \begin{pmatrix} b_\pm^{-1}(1 - \mu_j^\pm(\lambda)^2) \\ -\mu_j^\pm(\lambda) \\ 1 \end{pmatrix}. \tag{3.7}$$

Notice, in particular, that for this choice of bases, there hold, for $\lambda \sim 0$,

$$V_2^\pm(\lambda) = \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}(\lambda) \\ \mathcal{O}(1) \end{pmatrix}, \qquad V_j^\pm(\lambda) = \mathcal{O}(1), \quad j = 1, 3.$$

LEMMA 3.2. *Under the same assumptions as in Theorem* 1.3*, for each* $\lambda \in \Lambda$*, the spectral system* (3.4) *associated to the limiting, constant coefficients asymptotic behavior of* (3.3) *has a basis of solutions*

$$e^{\mu_j^\pm(\lambda)x} V_j^\pm(\lambda), \quad x \gtrless 0, \ j = 1, 2, 3. \tag{3.8}$$

Moreover, for $|\lambda| \sim 0$*, we can find analytic representations for* $\mu_j^\pm$ *and* $V_j^\pm$*, which consist of two slow modes,*

$$\mu_2^\pm(\lambda) = -a_\pm^{-1}\lambda + \mathcal{O}(\lambda^2),$$

and four fast modes,

$$\mu_1^\pm(\lambda) = \pm\theta_1^\pm + \mathcal{O}(\lambda), \qquad \mu_3^\pm(\lambda) = \mp\theta_3^\pm + \mathcal{O}(\lambda),$$

with associated eigenvectors (3.7)*.*

Proof. The proof is immediate by directly plugging (3.8) into (3.3) and using the previous computations (3.6), (3.7). ◻

In view of the structure of the asymptotic systems, we are able to conclude that for each initial condition $x_0 > 0$, the solutions to (3.3) in $x \geq x_0$ are spanned by two growing modes $\{\psi_1^+(x,\lambda), \psi_2^+(x,\lambda)\}$ and one decaying mode $\{\phi_3^+(x,\lambda)\}$ as $x \to +\infty$, whereas for each initial condition $x_0 < 0$, the solutions to (3.3) are spanned in $x < x_0$ by two growing modes $\{\psi_1^-(x,\lambda), \psi_2^-(x,\lambda)\}$ and one decaying mode $\{\phi_3^-(x,\lambda)\}$ as $x \to -\infty$.

We rely on the conjugation lemma of [34] to link such modes to those of the limiting constant coefficient system (3.4).

LEMMA 3.3. *Under the same assumptions as in Theorem* 1.3*, for* $|\lambda|$ *sufficiently small, there exist growing* $\psi_j^\pm(x,\lambda)$*,* $j = 1, 2$*, and decaying solutions* $\phi_3^\pm(x,\lambda)$*, in* $x \gtrless 0$*, of class* C^1 *in* x *and analytic in* λ*, satisfying*

$$\psi_j^\pm(x,\lambda) = e^{\mu_j^\pm(\lambda)\,x}\, V_j^\pm(\lambda)(I + \mathcal{O}(e^{-\eta|x|})), \quad j = 1, 2,$$
$$\phi_3^\pm(x,\lambda) = e^{\mu_3^\pm(\lambda)\,x}\, V_3^\pm(\lambda)(I + \mathcal{O}(e^{-\eta|x|})),$$

where $\eta > 0$ *is the decay rate of the traveling wave, and* $\mu_j^\pm$ *and* $V_j^\pm$ *are as in Lemma* 3.2*.*

Proof. This is a direct application of the conjugation lemma of [34] (see also the related gap lemma in [6, 47, 29, 30]). ◻

As a corollary, and in order to sum up the observations of this section, we make note that for $\lambda \sim 0$, the solutions to (3.3) in $x \geq x_0 > 0$ are spanned by

$$\psi_1^+(x,\lambda) = e^{(\theta_1^+ + \mathcal{O}(|\lambda|))x}\, V_1^+(\lambda)(I + \mathcal{O}(e^{-\eta|x|})) \qquad \text{(fast growing)}, \tag{3.9}$$

$$\psi_2^+(x,\lambda) = e^{(-\lambda/a_+ + \mathcal{O}(|\lambda|^2))x}\, V_2^+(\lambda)(I + \mathcal{O}(e^{-\eta|x|})) \ \text{(slowly growing)}, \tag{3.10}$$

$$\phi_3^+(x,\lambda) = e^{(-\theta_3^+ + \mathcal{O}(|\lambda|))x}\, V_3^+(\lambda)(I + \mathcal{O}(e^{-\eta|x|})) \qquad \text{(fast decaying)}. \tag{3.11}$$

Likewise, all the solutions for $x \leq x_0 < 0$ comprise the modes

$$\psi_1^-(x,\lambda) = e^{(-\theta_1^- + \mathcal{O}(|\lambda|))x} V_1^-(\lambda)(I + \mathcal{O}(e^{-\eta|x|})) \quad \text{(fast growing)}, \tag{3.12}$$

$$\psi_2^-(x,\lambda) = e^{(-\lambda/a_- + \mathcal{O}(|\lambda|^2))x} V_2^-(\lambda)(I + \mathcal{O}(e^{-\eta|x|})) \text{ (slowly growing)}, \tag{3.13}$$

$$\phi_3^-(x,\lambda) = e^{(\theta_3^- + \mathcal{O}(|\lambda|))x} V_3^-(\lambda)\,(I + \mathcal{O}(e^{-\eta|x|})) \quad \text{(fast decaying)}. \tag{3.14}$$

The analytic coefficients $V_j^{\pm}(\lambda)$ are given by (3.7).

3.3. Solutions near $x \sim 0$. Our goal now is to analyze system (3.3) close to the singularity $x = 0$. For concreteness, let us restrict the analysis to the case $x > 0$. We introduce a "stretched" variable ξ as follows: Fix $\epsilon_0 > 0$ and let

$$\xi = \int_{\epsilon_0}^{x} \frac{dz}{a(z)},$$

so that $\xi(\epsilon_0) = 0$, and $\xi \to +\infty$ as $x \to 0^+$. Under this change of variables, we get

$$u' = \frac{du}{dx} = \frac{1}{a(x)}\frac{du}{d\xi} = \frac{1}{a(x)}\dot{u}$$

after denoting $\dot{} = d/d\xi$. In the stretched variables, system (2.3) becomes

$$\dot{W} = \tilde{\mathbb{A}}(\xi,\lambda)W, \qquad \text{where} \quad \tilde{\mathbb{A}}(\xi,\lambda) := \begin{pmatrix} -\omega & 0 & L \\ \tilde{a}\,\tilde{b} & 0 & -\tilde{a} \\ 0 & -\tilde{a} & 0 \end{pmatrix},$$

and functions $\omega, \tilde{a}, \tilde{b}$ are defined by

$$\omega(\xi) := \lambda + a'(x(\xi)) + L\,b(x(\xi)), \qquad \tilde{a}(\xi) := a(x(\xi)), \qquad \tilde{b}(\xi) := b(x(\xi)).$$

Note that from (2.1), for small frequencies $\lambda \sim 0$, and choosing $0 < \epsilon_0 \ll 1$ sufficiently small, we have the uniform bound

$$\operatorname{Re}\omega(\xi) \sim \operatorname{Re}\omega(0) = \eta := \operatorname{Re}\lambda + a'(0) + L\,b(0) > 0$$

for all $\xi \in [0, +\infty)$. In addition, we have

$$\omega_\xi = \tilde{a}(\xi)(a''(x(\xi)) + Lb'(x(\xi))) = \mathcal{O}(|\tilde{a}(\xi)|).$$

Next, we apply the transformation $Z := \mathbb{L}W$, where

$$\mathbb{L} := \begin{pmatrix} 1 & 0 & -L/\omega \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathbb{R} := \mathbb{L}^{-1} = \begin{pmatrix} 1 & 0 & L/\omega \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since

$$|\dot{\mathbb{L}}\mathbb{R}| = |\mathbb{L}\dot{\mathbb{R}}| = \mathcal{O}(|\tilde{a}|) \qquad \text{and} \qquad \dot{\mathbb{L}} = \begin{pmatrix} 0 & 0 & L\omega_\xi/\omega^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \tilde{a}\,\mathcal{O}(1),$$

we obtain a block-diagonal system at leading order of the form

$$\dot{Z} = \begin{pmatrix} -\omega & 0 \\ 0 & 0 \end{pmatrix} Z + \tilde{a}\,\Theta(\xi)Z, \tag{3.15}$$

where

$$\Theta = \begin{pmatrix} 0 & L/\omega & L(a'' + L\, b')/\omega^2 \\ \tilde{b} & 0 & -1 + L\, \tilde{b}/\omega \\ 0 & -1 & 0 \end{pmatrix}$$

is uniformly bounded. The blocks $-\omega I$ and 0 are clearly spectrally separated, and the error is of order $\mathcal{O}(|\tilde{a}(\xi)|) \to 0$ as $\xi \to +\infty$. System (3.15) has the form (A.3) of Appendix A (block diagonal at leading order) and satisfies the hypotheses of the pointwise reduction lemma (see Proposition A.1). In our case, there is no dependence on a parameter ϵ, $M_2 = -\omega I$, $M_1 \equiv 0$, and the pointwise error is $\delta(\xi) = \tilde{a}(\xi)$, with constant spectral gap η.

Hence, there exist analytic transformations $\Phi_j(\xi, \lambda)$, $j = 1, 2$, satisfying the pointwise bound (A.4), for which the graphs $\{(Z_1, \Phi_2(Z_1))\}$, $\{(\Phi_1(Z_2), Z_2)\}$ are invariant under the flow of (3.15). We now take a closer look at the pointwise error (A.4).

LEMMA 3.4. *For the stretched system and for low-frequency $\lambda \sim 0$, there holds*

$$|\Phi_j(\xi, \lambda)| \le C\, \tilde{a}(\xi), \quad j = 1, 2, \tag{3.16}$$

provided that

$$Lb(0) + 2a'(0) > 0. \tag{3.17}$$

Proof. From Proposition A.1, there holds the pointwise bound (A.4), namely,

$$|\Phi_j(\xi, \lambda)| \le C \int_0^{\xi} e^{-\eta(\xi - y)} \tilde{a}(y)\, dy,$$

which in terms of the original variables looks like

$$|\tilde{\Phi}_j(x, \lambda)| := |\Phi_j(\xi(x), \lambda)| \le C \int_{\epsilon_0}^{x} \exp\left(\eta \int_x^{\tilde{x}} \frac{dz}{a(z)} \right) d\tilde{x}.$$

Since for z small, $a(z) \sim a'(0)z$, we get

$$\begin{aligned} |\tilde{\Phi}_j(x, \lambda)| &\lesssim \int_{\epsilon_0}^{x} \exp\left(\eta \int_x^{\tilde{x}} \frac{dz}{a'(0)z} \right) d\tilde{x} = \frac{C\, a'(0)}{\eta + a'(0)} \big(x - \epsilon_0 (x/\epsilon_0)^{\eta/|a'(0)|} \big) \\ &\le \frac{C\, a'(0)x}{\eta + a'(0)} \sim \frac{C\, a(x)}{\eta + a'(0)}, \end{aligned}$$

in view of $0 < x < \epsilon_0$, and as long as $a'(0) + \eta > 0$. Since $\eta = \operatorname{Re}\lambda + a'(0) + Lb(0)$, condition (3.17) implies (3.16) for small λ. □

Remark 3.5. Notice that (3.17) is a stronger condition than (2.1), which is inherited by the existence result of [22] or Theorem C.3. Notably, this new condition (3.17) holds if we assume (A5$_k$) with $k = 2$.

In view of the pointwise error bound (3.16) of order $\mathcal{O}(a)$ and by the pointwise reduction lemma (see Proposition A.1 and Remark A.2), we can separate the flow into slow and fast coordinates. Indeed, after proper transformations we separate the flows on the reduced manifolds of the form

$$\begin{aligned} \dot{Z}_1 &= -\omega\, Z_1 + \mathcal{O}(\tilde{a}) Z_1, \\ \dot{Z}_2 &= \mathcal{O}(\tilde{a}) Z_2. \end{aligned} \tag{3.18}$$

Observe that the Z_1 modes decay to zero as $\xi \to +\infty$, in view of

$$e^{-\int_0^\xi \omega(z)\,dz} \lesssim e^{-(\mathrm{Re}\,\lambda + \frac{1}{2}\eta)\xi} \to 0,$$

as $\xi \to +\infty$. These fast decaying modes correspond to fast decaying to zero solutions when $x \to 0^+$ in the original u-variable. The Z_2 modes comprise slow dynamics of the flow as $x \to 0^+$.

PROPOSITION 3.6. *Under assumptions* (A0)–(A4) *and* (A5$_k$) *with* $k = 2$*, there exists* $0 < \epsilon_0 \ll 1$ *sufficiently small such that, in the small frequency regime* $\lambda \sim 0$*, the solutions to the spectral system* (3.3) *in* $(-\epsilon_0, 0) \cup (0, \epsilon_0)$ *are spanned by fast modes*

$$w_2^\pm(x,\lambda) = \begin{pmatrix} u_2^\pm \\ q_2^\pm \\ p_2^\pm \end{pmatrix} = \begin{pmatrix} Z_1(x) \\ 0 \\ 0 \end{pmatrix} (1 + \mathcal{O}(a(x))), \qquad \pm\epsilon_0 \gtrless x \gtrless 0,$$

where Z_1 *is the mode of* (3.18)*, decaying to zero as* $x \to 0^\pm$*, and slowly varying modes*

$$z_j^\pm(x,\lambda) = \begin{pmatrix} u_j^\pm \\ q_j^\pm \\ p_j^\pm \end{pmatrix}, \qquad \pm\epsilon_0 \gtrless x \gtrless 0, \quad j = 1, 3,$$

with bounded limits as $x \to 0^\pm$*. Moreover, the fast modes defined above decay as*

$$u_2^\pm \sim |x|^\nu \to 0, \qquad \begin{pmatrix} q_2^\pm \\ p_2^\pm \end{pmatrix} \sim \mathcal{O}(|x|^\nu a(x)) \to 0, \tag{3.19}$$

as $x \to 0^\pm$*, where* $\nu := (\mathrm{Re}\,\lambda + a'(0) + Lb(0))/|a'(0)|$.

Proof. This is a direct consequence of applying our pointwise tracking lemma (Lemma 3.4) to the reduced system (3.18). The claimed estimate (3.19) for u follows in the same way as done in Lemma 2.2. ∎

3.4. Decaying modes. We next derive explicit representation formulae for the resolvent kernel $\mathcal{G}_\lambda(x,y)$ using the classical construction in terms of decaying solutions of the homogeneous spectral problem, matched across the singularity by appropriate jump conditions at $x = y$. The novelty of our approach circumvents the inconsistency between the number of decaying modes at $\pm\infty$. In this section we describe how to construct all decaying solutions at each side of the singularity with matching dimensions.

Choose $\epsilon_0 > 0$ small enough so that the representations of the solutions of Proposition 3.6 hold. We are going to construct two decaying modes W_j^+, $j = 1, 2$, at $+\infty$, and one decaying mode W_3^- at $-\infty$. For that purpose, we choose the decaying mode at $-\infty$ as

$$W_3^-(x,\lambda) := \begin{cases} \phi_3^-(x,\lambda), & x < -\epsilon_0, \\ (\gamma_1 z_1^- + \gamma_3 z_3^- + \gamma_2 w_2^-)(x,\lambda), & -\epsilon_0 < x < 0, \end{cases} \tag{3.20}$$

where the coefficients $\gamma_j = \gamma_j(\lambda)$ are analytic in λ and such that W_3^- is of class C^1 in all $x < 0$.

To select the decaying modes at $+\infty$, consider

$$W_2^+(x,\lambda) := \begin{cases} 0, & x > 0, \\ w_2^-(x,\lambda), & -\epsilon_0 < x < 0, \\ (\kappa_1 \psi_1^- + \kappa_2 \psi_2^- + \kappa_3 \phi_3^-)(x,\lambda), & x < -\epsilon_0, \end{cases} \tag{3.21}$$

where w_2^- is the vanishing at $x = 0$ solution in (3.6) (the solution is, thus, continuous at $x = 0$), and the coefficients $\kappa_j = \kappa_j(\lambda)$ are analytic in λ, and such that the matching is of class C^1 a.e. in x.

Finally, we define

$$W_1^+(x,\lambda) := \begin{cases} \phi_3^+(x,\lambda), & x > \epsilon_0, \\ (\alpha_1 z_1^+ + \alpha_3 z_3^+ + \alpha_2 w_2^+)(x,\lambda), & 0 < x < \epsilon_0, \\ (\beta_1 z_1^- + \beta_3 z_3^- + \beta_2 w_2^-)(x,\lambda), & -\epsilon_0 < x < 0 \\ (\delta_1 \psi_1^- + \delta_2 \psi_2^- + \delta_3 \phi_3^-)(x,\lambda), & x < -\epsilon_0 \end{cases} \tag{3.22}$$

as the other decaying mode at $+\infty$, with analytic coefficients α_j, β_j, δ_j in λ, and W_1^+ is of class C^1 a.e. in x.

Remark 3.7. A similar definition of two decaying modes W_3^-, W_2^- at $-\infty$ and one decaying mode W_1^+ at $+\infty$, on the positive side of the singularity, is clearly available. See Figure 3.1.

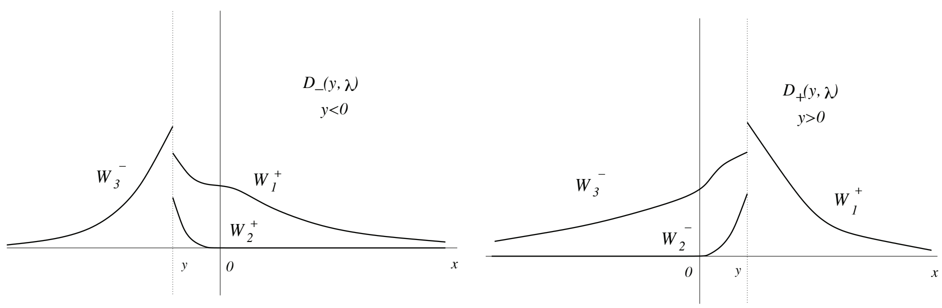

FIG. 3.1. *Two Evans functions: Representation of the decaying modes at $\pm\infty$ for $y \gtrless 0$. Left picture: This considers the case $y < 0$. The sole decaying mode W_3^- at $-\infty$ is represented as the fast decaying solution, and there are decaying modes at $+\infty$: the exponentially fast decaying solution W_1^+, and the constructed mode W_2^+, which is identically zero for $x > 0$ and matched across the singularity to the solution which decays to zero as $x \to 0^-$ in the region $(y, 0)$. This provides a full set of decaying modes for $y < 0$. Right picture: A symmetric construction for the $y > 0$ case is depicted.*

3.5. Two Evans functions. We first define two related Evans functions,

$$D_\pm(y,\lambda) := \det(W_1^+\ W_2^\mp\ W_3^-)(y,\lambda) \qquad \text{for } y \gtrless 0,$$

where $W_j^\pm = (u_j^\pm, q_j^\pm, p_j^\pm)^\top$ is defined as above (see (3.20)–(3.22)).

We first observe the following simple properties of $D_\pm$.

LEMMA 3.8. *For λ sufficiently small, we have*

$$D_\pm(y,\lambda) = -a(y)^{-1}\lambda[u] \det \begin{pmatrix} q_1^+ & q_2^\mp \\ p_1^+ & p_2^\mp \end{pmatrix}_{|\lambda=0} + \mathcal{O}(|\lambda|^2), \tag{3.23}$$

where $[u] = u_+ - u_-$.

Proof. Let us consider (3.23) for D_-. By our choice, at $\lambda = 0$, we can take

$$W_1^+(x,0) = W_3^-(x,0) = \bar{W}'(x), \tag{3.24}$$

where $\bar{W}$ is the shock profile. By Leibnitz's rule, we first compute

$$\partial_\lambda D_-(y,0) = \det\Big(\partial_\lambda W_1^+, W_2^+, W_3^-\Big)_{|\lambda=0} + \det\Big(W_1^+, \partial_\lambda W_2^+, W_3^-\Big)_{|\lambda=0} + \det\Big(W_1^+, W_2^+, \partial_\lambda W_3^-\Big)_{|\lambda=0},$$

where, by using (3.24), the second term on the right-hand side vanishes and the first and third terms can be grouped together, yielding

$$\partial_\lambda D_-(y,0) = \det\Big(W_1^+, W_2^+, \partial_\lambda W_3^- - \partial_\lambda W_1^+\Big)_{|\lambda=0}. \tag{3.25}$$

Since $W_j^\pm(\cdot,\lambda)$ satisfies (3.1), $\partial_\lambda W_1^+(x,0) = (\partial_\lambda u_1^+, \partial_\lambda q_1^+, \partial_\lambda p_1^+)$ satisfies

$$\Theta(\partial_\lambda W_1^+)' = \mathbb{A}(x,0)\partial_\lambda W_1^+(x,0) + \partial_\lambda \mathbb{A}(x,0) W_1^+(x,0),$$

which directly gives

$$(a\,\partial_\lambda u_1^+)' = -L(\partial_\lambda q_1^+)' - \bar{u}'. \tag{3.26}$$

Likewise, $\partial_\lambda W_3^-(x,0) = (\partial_\lambda u_3^-, \partial_\lambda q_3^-, \partial_\lambda p_3^-)$ satisfies

$$(a\,\partial_\lambda u_3^-)' = -L(\partial_\lambda q_3^-)' - \bar{u}'. \tag{3.27}$$

Integrating (3.26) and (3.27) from $+\infty$ and $-\infty$, respectively, with use of boundary conditions $\partial_\lambda W_1^+(+\infty,0) = \partial_\lambda W_3^-(-\infty,0) = 0$, we obtain

$$a\,\partial_\lambda u_1^+ = -L\partial_\lambda q_1^+ - \bar{u} + u_+ \qquad \text{and} \qquad a\,\partial_\lambda u_3^- = -L\partial_\lambda q_3^- - \bar{u} + u_-.$$

Thus

$$a(\partial_\lambda u_3^- - \partial_\lambda u_1^+) = -L(\partial_\lambda q_3^- - \partial_\lambda q_1^+) - [u]. \tag{3.28}$$

Meanwhile, since W_j^+, $j = 1,2$, satisfy (3.1), and thus $(au)' = -Lq'$ with $W_j^+(+\infty,\lambda) = 0$, we integrate the latter equation, yielding

$$au_j^+ = -Lq_j^+ \qquad \text{for } j = 1,2. \tag{3.29}$$

Using estimates (3.29) and (3.28), we can now compute the λ-derivative (3.25) of $D_\pm$ at $\lambda = 0$ as

$$\begin{aligned}\partial_\lambda D_-(y,0) &= \det\begin{pmatrix} u_1^+ & u_2^+ & \partial_\lambda u_3^- - \partial_\lambda u_1^+ \\ q_1^+ & q_2^+ & \partial_\lambda q_3^- - \partial_\lambda q_1^+ \\ p_1^+ & p_2^+ & \partial_\lambda p_3^- - \partial_\lambda p_1^+ \end{pmatrix} \\ &= \det\begin{pmatrix} u_1^+ & u_2^+ & \partial_\lambda u_3^- - \partial_\lambda u_1^+ \\ 0 & 0 & -[u]/L \\ p_1^+ & p_2^+ & \partial_\lambda p_3^- - \partial_\lambda p_1^+ \end{pmatrix} \\ &= L^{-1}[u]\det\begin{pmatrix} u_1^+ & u_2^+ \\ p_1^+ & p_2^+ \end{pmatrix}.\end{aligned} \tag{3.30}$$

Applying again relation (3.28), we obtain (3.23).

Similarly, for D_+ we obtain

$$\partial_\lambda D_+(y,0) = -L^{-1}[u] \det \begin{pmatrix} u_1^+ & u_2^- \\ p_1^+ & p_2^- \end{pmatrix} \tag{3.31}$$

from which the conclusion follows. ◻

Since there are two different Evans functions for $y \gtrless 0$, we need to be sure that if one vanishes to order one at $\lambda = 0$ (part of the stability criterion), then the other does too. Such a property—the content of the following lemma—guarantees that pole terms are the same on $y < 0$ and $y > 0$.

LEMMA 3.9. *Defining the Evans functions*

$$D_\pm(\lambda) := D_\pm(\pm 1, \lambda), \tag{3.32}$$

we then have $D_+(\lambda) = mD_-(\lambda) + \mathcal{O}(|\lambda|^2)$*, where* m *is some nonzero factor.*

Proof. Since $W_1^+(x) = \bar{W}'$ is a nonvanishing, bounded solution of the ODE (3.1), we must have $W_1^+(1) = m_1 W_1^+(-1)$ for some m_1 nonzero. Meanwhile, Proposition 3.6 gives

$$\begin{pmatrix} u_2^\pm \\ p_2^\pm \end{pmatrix} = \begin{pmatrix} |x|^\nu \\ 0 \end{pmatrix} + \mathcal{O}(|x|^\nu a(x)),$$

as $x \to 0$, where $\nu = (a'(0) + Lb(0))/|a'(0)|$. Thus, smoothness of a near zero guarantees the existence of ϵ_1, ϵ_2 near zero such that

$$\begin{pmatrix} u_2^+ \\ p_2^+ \end{pmatrix}_{x=-\epsilon_1} = \begin{pmatrix} u_2^- \\ p_2^- \end{pmatrix}_{x=\epsilon_2}.$$

This, together with the fact that $W_2^\pm$ are solutions of the ODE (3.1), yields

$$\begin{pmatrix} u_2^+ \\ p_2^+ \end{pmatrix}_{x=-1} = m_2 \begin{pmatrix} u_2^- \\ p_2^- \end{pmatrix}_{x=1}$$

for some m_2 nonzero. Putting these estimates into (3.30) and (3.31) and using analyticity of $D_\pm$ in λ near zero, we easily obtain the conclusion. ◻

Either one of these Evans functions evidently has the property for $\lambda > 0$ and $\lambda \neq 0$ that it vanishes at λ if and only if λ is an L^2 eigenvalue of the linearized operator $\mathcal{L}$. Though we did not explicitly verify it, the interested reader may verify through our construction that both Evans functions have the usual property of analyticity in λ as well.

4. Resolvent kernel bounds in low-frequency regions. In this section, we shall derive pointwise bounds on the resolvent kernel $G_\lambda(x,y)$ in low-frequency regimes, that is, $|\lambda| \to 0$. For definiteness, throughout this section, we consider only the case $y < 0$. The case $y > 0$ is completely analogous by symmetry.

We solve (3.2) with the jump conditions at $x = y$:

$$[\mathcal{G}_\lambda(.,y)] = \begin{pmatrix} a(y)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.1}$$

Meanwhile, we can write $\mathcal{G}_\lambda(x,y)$ in terms of decaying solutions at $\pm\infty$ as follows:

$$\mathcal{G}_\lambda(x,y) = \begin{cases} W_1^+(x,\lambda)C_1^+(y,\lambda) + W_2^+(x,\lambda)C_2^+(y,\lambda), & x > y, \\ -W_3^-(x,\lambda)C_3^-(y,\lambda), & x < y, \end{cases} \tag{4.2}$$

where $C_j^{\pm} = (C_{jk}^{\pm})_{k=1,2,3}$ are row vectors. We compute the coefficients $C_j^{\pm}$ by means of the transmission conditions (4.1) at y. Therefore, solving by Cramer's rule the system

$$\begin{pmatrix} W_1^+ & W_2^+ & W_3^- \end{pmatrix} \begin{pmatrix} C_1^+ \\ C_2^+ \\ C_3^- \end{pmatrix} \Bigg|_{(y,\lambda)} = \begin{pmatrix} a(y)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

we readily obtain

$$\begin{pmatrix} C_1^+ \\ C_2^+ \\ C_3^- \end{pmatrix} = D_-(y,\lambda)^{-1} \begin{pmatrix} W_1^+ & W_2^+ & W_3^- \end{pmatrix}^{adj} \Big|_{(y,\lambda)} \begin{pmatrix} a(y)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where M^{adj} denotes the adjugate matrix of a matrix M. For example,

$$C_{11}^+(y,\lambda) = a(y)^{-1} D_-(y,\lambda)^{-1} \begin{vmatrix} q_2^+ & q_3^- \\ p_2^+ & p_3^- \end{vmatrix} (y,\lambda), \tag{4.3}$$

$$C_{21}^+(y,\lambda) = a(y)^{-1} D_-(y,\lambda)^{-1} \begin{vmatrix} q_3^- & q_1^+ \\ p_3^- & p_1^+ \end{vmatrix} (y,\lambda), \tag{4.4}$$

$$C_{31}^-(y,\lambda) = a(y)^{-1} D_-(y,\lambda)^{-1} \begin{vmatrix} q_1^+ & q_2^+ \\ p_1^+ & p_2^+ \end{vmatrix} (y,\lambda). \tag{4.5}$$

Here, note that these are the only coefficients that are possibly singular as y nears zero because of singularity in the first column of the jump-condition matrix (4.1).

We then easily obtain the following.

LEMMA 4.1. *For y near zero, we have*

$$\begin{aligned} C_1^+(y,\lambda) &= \frac{1}{\lambda}[u]^{-1}(1,\ -L,\ 0) + \mathcal{O}(1), \\ C_3^-(y,\lambda) &= -\frac{1}{\lambda}[u]^{-1}(1,\ -L,\ 0) + \mathcal{O}(1), \end{aligned} \tag{4.6}$$

and

$$C_2^+(y,\lambda) = a(y)^{-1}|y|^{-\nu}\mathcal{O}(1), \tag{4.7}$$

where ν is defined as in Proposition 3.6 *and $\mathcal{O}(1)$ is a uniformly bounded function, possibly depending on y and λ.*

Proof. It suffices to estimate $C_{j1}^{\pm}$ when the singularity plays a role. Recalling (3.23) and (4.5), we can estimate $C_{31}^-(y,\lambda)$ as

$$C_{31}^-(y,\lambda) = -\frac{1}{\lambda[u]} \det \begin{pmatrix} q_1^+ & q_2^+ \\ p_1^+ & p_2^+ \end{pmatrix}^{-1}_{|\lambda=0} \left[\begin{vmatrix} q_1^+ & q_2^+ \\ p_1^+ & p_2^+ \end{vmatrix} (y,0) + \mathcal{O}(\lambda) \right] = -\frac{1}{\lambda[u]} + \mathcal{O}(1),$$

where $\mathcal{O}(1)$ is uniformly bounded since $a(y)D_-(y,\lambda)$ and normal modes $W_j^{\pm}$ are all bounded uniformly in y near zero. This yields the bound for C_{31}^- as claimed. The bound for C_{11}^+ follows similarly, noting that $W_3^- \equiv W_1^+$ at $\lambda = 0$.

For the estimate on C_2^+, we first observe that in view of (3.23) and the estimate (3.19) on u_2^+,

$$D_-(y,\lambda) \geq c\,\lambda\,|y|^{\nu}$$

for some $c > 0$. This, together with the fact that $W_3^- \equiv W_1^+$ at $\lambda = 0$, yields the estimate for C_2^+ as claimed. ∎

PROPOSITION 4.2 (resolvent kernel bounds as $|y| \to 0$). *Assume* (A0)–(A5$_k$). *For y near zero, there hold*

$$\mathcal{G}_\lambda(x,y) = \lambda^{-1}[u]^{-1}\bar{W}'(1,\ -L,\ 0) + \mathcal{O}(e^{-\eta|x|}) \tag{4.8}$$

for $y < 0 < x$, and

$$\mathcal{G}_\lambda(x,y) = \lambda^{-1}[u]^{-1}\bar{W}'(1,\ -L,\ 0) + \mathcal{O}(1)\left(1 + \frac{|x|^\nu}{a(y)|y|^\nu}\right) \tag{4.9}$$

for $y < x < 0$, and

$$\mathcal{G}_\lambda(x,y) = \lambda^{-1}[u]^{-1}\bar{W}'(1,\ -L,\ 0) + \mathcal{O}(e^{-\eta|x|})$$

for $x < y < 0$, for some $\eta > 0$. Similar bounds can be obtained for the case $y > 0$.

Proof. For the case $y < 0 < x$, using (4.6) and recalling that $W_1^+(x) = \bar{W}' + \mathcal{O}(\lambda)e^{-\eta|x|}$ and $W_2^+(x) \equiv 0$, we have

$$\mathcal{G}_\lambda(x,y) = W_1^+(x)\, C_1^+(y) = \left(\bar{W}' + \mathcal{O}(\lambda)e^{-\eta|x|}\right)\left(\frac{1}{\lambda[u]}(1,\ -L,\ 0) + \mathcal{O}(1)\right),$$

yielding (4.8). In the second case $y < x < 0$, from the formula (4.2) projected on the first component, we have

$$C_1^+(y,\lambda)u_1^+(x,\lambda) + C_2^+(y,\lambda)u_2^+(x,\lambda),$$

where the first term contributes $\lambda^{-1}[u]^{-1}\bar{W}' + \mathcal{O}(1)$ as in the first case, and the second term is estimated by (4.7) and (3.19).

Finally, we estimate the last case $x < y < 0$ in the same way as in the first case, noting that y is still near zero and $W_3^-(x) = \bar{W}' + \mathcal{O}(\lambda)e^{-\eta|x|}$. ∎

Next, we derive pointwise bounds of $G_\lambda(x,y)$ in regions $|y| \to +\infty$. Note, however, that representations (4.2) and the above estimates fail to be useful in the $y \to -\infty$ limit, since to get an estimate of the form

$$|G_\lambda(x,y)| \le Ce^{-\eta|x-y|},$$

we actually need precise decay rates, which are unavailable from W_j^+ in the $y \to -\infty$ regime. Thus, we need to express the (+)-bases in terms of the growing modes ψ_j^- at $-\infty$, and the sole decaying mode ϕ_3^- where ψ_j^-, ϕ_3^- are defined as in Lemma 3.3. Expressing such solutions in the basis for $y < 0$, away from zero, there exist *analytic* coefficients $\alpha_{jk} := \alpha_{jk}(\lambda)$ such that

$$\begin{aligned} W_1^+(x,\lambda) &= \alpha_{11}(\lambda)\psi_1^-(x,\lambda) + \alpha_{12}(\lambda)\psi_2^-(x,\lambda) + \alpha_{13}(\lambda)\phi_3^-(x,\lambda), \\ W_2^+(x,\lambda) &= \alpha_{21}(\lambda)\psi_1^-(x,\lambda) + \alpha_{22}(\lambda)\psi_2^-(x,\lambda) + \alpha_{23}(\lambda)\phi_3^-(x,\lambda). \end{aligned} \tag{4.10}$$

At $\lambda = 0$, we choose $W_1^+(\cdot,0) \equiv \phi_3^-(\cdot,0) \equiv \bar{W}'$. Thus,

$$\alpha_{11}(0) = \alpha_{12}(0) = 0. \tag{4.11}$$

Furthermore, still as $\lambda = 0$, ψ_2^- is a (nearly constant) bounded solution and has the form $(b_-^{-1}, 0, 1)^\top$ as x near zero. Observe also that W_2^+ is the solution converging to zero in form of $|x|^\nu(1, a(x), a(x))^\top$ as $x \to 0^-$. Thus, we can choose

$$\alpha_{22}(0) = 0. \tag{4.12}$$

To express the coefficients C_j^+ in terms of the uniformly decaying/growing modes at $-\infty$, with a slight abuse of notation we write

$$\psi_j^- = (u_j^-, q_j^-, p_j^-)^\top, \quad j = 1, 2,$$

and define the 2×2 minors,

$$\Omega_{ij}^\pm(y,\lambda) := \begin{vmatrix} q_i^\pm & q_j^\pm \\ p_i^\pm & p_j^\pm \end{vmatrix} = -\Omega_{ji}^\pm(y,\lambda),$$

and the analytic minors,

$$\hat{d}_{12}(\lambda) := \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}, \qquad \hat{d}_{23}(\lambda) := \begin{vmatrix} \alpha_{12} & \alpha_{13} \\ \alpha_{22} & \alpha_{23} \end{vmatrix}, \qquad \hat{d}_{13}(\lambda) := \begin{vmatrix} \alpha_{11} & \alpha_{13} \\ \alpha_{21} & \alpha_{23} \end{vmatrix},$$

where, by (4.11) and (4.12), we note

$$\hat{d}_{12}(0) = \hat{d}_{23}(0) = 0. \tag{4.13}$$

By elementary column operations we notice that

$$\begin{vmatrix} q_2^+ & q_3^- \\ p_2^+ & p_3^- \end{vmatrix} = \alpha_{21}\Omega_{13}^- + \alpha_{22}\Omega_{23}^-, \tag{4.14}$$

$$\begin{vmatrix} q_3^- & q_1^+ \\ p_3^- & p_1^+ \end{vmatrix} = -\alpha_{11}\Omega_{13}^- - \alpha_{12}\Omega_{23}^-, \tag{4.15}$$

$$\begin{vmatrix} q_1^+ & q_2^+ \\ p_1^+ & p_2^+ \end{vmatrix} = \Omega_{12}^+ = \hat{d}_{12}\Omega_{12}^- + \hat{d}_{13}\Omega_{13}^- + \hat{d}_{23}\Omega_{23}^-. \tag{4.16}$$

LEMMA 4.3. *The minor in* (4.15) *can be improved by*

$$\begin{vmatrix} q_3^- & q_1^+ \\ p_3^- & p_1^+ \end{vmatrix} = \lambda(\hat{\alpha}_{11}\Omega_{31}^- + \hat{\alpha}_{12}\Omega_{32}^-) \tag{4.17}$$

for some coefficients $\hat{\alpha}_{ij}$.

Proof. The estimate is clear, due to the fact that at $\lambda = 0$, we can take $W_1^+(x,0) = W_3^-(x,0) = \bar{W}'(x)$. ∎

We also have the following crucial cancellation for $x > y$:

$$\begin{aligned} &W_1^+ C_{11}^+ + W_2^+ C_{21}^+ \\ &\quad = a^{-1} D_-^{-1}\Big((\alpha_{21}\Omega_{13}^- + \alpha_{22}\Omega_{23}^-)(\alpha_{11}\psi_1^- + \alpha_{12}\psi_2^- + \alpha_{13}\phi_3^-) \\ &\qquad\qquad - (\alpha_{11}\Omega_{13}^- + \alpha_{12}\Omega_{23}^-)(\alpha_{21}\psi_1^- + \alpha_{22}\psi_2^- + \alpha_{23}\phi_3^-)\Big) \\ &\quad = a^{-1} D_-^{-1}\left(\hat{d}_{12}\Omega_{23}^-\psi_1^- - \hat{d}_{12}\Omega_{13}^-\psi_2^- - (\hat{d}_{13}\Omega_{13}^- + \hat{d}_{23}\Omega_{23}^-)\phi_3^-\right), \end{aligned} \tag{4.18}$$

where Ω_{ij}^- are functions in y, and $\phi_1^-, \phi_2^-, \psi_3^-$ are in x, noting that $\Omega_{13}^-\psi_1^-$ and $\Omega_{23}^-\psi_2^-$ are canceled out.

We recall here that μ_j^-, $j = 1, 2, 3$, are three eigenvalues satisfying

$$\mu_1^- \le -c_0 < 0, \quad \mu_2^- = -\lambda/a_- + \mathcal{O}(\lambda^2), \quad \mu_3^- \ge c_0 > 0$$

for some $c_0 > 0$.

LEMMA 4.4. *For $y<0$ away from zero, we have*

$$C_1^+(y,\lambda) = \lambda^{-1}[u]^{-1}e^{-\mu_2^- y}(1,\ -L,\ 0) + \mathcal{O}(e^{-\mu_1^- y} + e^{-\mu_2^- y}), \tag{4.19}$$

$$C_2^+(y,\lambda) = \mathcal{O}(e^{-\mu_1^- y} + e^{-\mu_2^- y}), \tag{4.20}$$

$$C_3^-(y,\lambda) = -\lambda^{-1}[u]^{-1}e^{-\mu_2^- y}(1,\ -L,\ 0) + \mathcal{O}(e^{-\mu_3^- y}). \tag{4.21}$$

Proof. First, by using (4.10) and the estimates in the previous sections on normal modes ψ_j^-, ϕ_3^-, we readily obtain

$$|D_-(y,\lambda)| = |\det(W_1^+, W_2^+, W_3^-)| = \mathcal{O}(\lambda)e^{\mu_1^- y}e^{\mu_2^- y}e^{\mu_3^- y}$$

and

$$|\Omega_{12}^-| = \mathcal{O}(e^{\mu_1^- y}e^{\mu_2^- y}), \quad |\Omega_{13}^-| = \mathcal{O}(e^{\mu_1^- y}e^{\mu_3^- y}), \quad |\Omega_{23}^-| = \mathcal{O}(e^{\mu_2^- y}e^{\mu_3^- y}).$$

Using (4.14) and noting that $\alpha_{22}(0)=0$, we estimate

$$C_{11}^+(y,\lambda) = -a^{-1}D_-^{-1}\Big(\alpha_{21}\Omega_{13}^- + \alpha_{22}\Omega_{23}^-\Big) = \lambda^{-1}\mathcal{O}(e^{-\mu_2^- y}) + \mathcal{O}(e^{-\mu_2^- y} + e^{-\mu_1^- y}),$$

which gives (4.19) for λ small, by observing that the coefficient in the Laurent expansions at order λ^{-1} is $[u]^{-1}(1,\ -L,\ 0)$ (see the proof of Lemma 4.1).

Next, by (4.17), we estimate

$$C_{21}^+(y,\lambda) = a^{-1}D_-^{-1}\,\lambda(\hat\alpha_{11}\Omega_{31}^- + \hat\alpha_{12}\Omega_{32}^-) = \mathcal{O}(e^{-\mu_1^- y} + e^{-\mu_2^- y}).$$

Finally, we can estimate

$$\begin{aligned} C_{31}^-(y,\lambda) &= a^{-1}D_-^{-1}\Big(\hat d_{12}\Omega_{12}^- + \hat d_{13}\Omega_{13}^- + \hat d_{23}\Omega_{23}^-\Big) \\ &= \lambda^{-1}\mathcal{O}(e^{-\mu_2^- y}) + \mathcal{O}(e^{-\mu_1^- y} + e^{-\mu_2^- y} + e^{-\mu_3^- y}) \end{aligned}$$

in the same way as done for C_{11}^+, yielding the estimate as claimed; note that the constraint (4.13) on $\hat d(0)$ shows that only slow-growing mode ψ_2^- appears in the λ^{-1} term. The appearance of the fast-growing term is due to $D_-^{-1}\Omega_{12}^-$, a new feature as compared to the estimate of C_{11}^+. ∎

PROPOSITION 4.5 (resolvent kernel bounds as $|y|\to+\infty$). *Under* (A0)–(A5$_k$), *for $|y|$ large, there hold*

$$\begin{aligned} \mathcal{G}_\lambda(x,y) =& \lambda^{-1}[u]^{-1}e^{-\mu_2^- y}\bar W'(1,\ -L,\ 0) \\ &+ \mathcal{O}((e^{-\mu_2^- y} + e^{-\mu_1^- y})e^{\mu_3^+ x}) \end{aligned} \tag{4.22}$$

for $y<0<x$, and

$$\begin{aligned} \mathcal{G}_\lambda(x,y) =& \lambda^{-1}[u]^{-1}e^{-\mu_2^- y}\bar W'(1,\ -L,\ 0) \\ &+ \mathcal{O}(e^{\mu_1^- (x-y)}) + \mathcal{O}(e^{\mu_2^- (x-y)}) + \mathcal{O}(e^{-\mu_2^- y}e^{\mu_3^- x}) \end{aligned} \tag{4.23}$$

for $y<x<0$, and

$$\begin{aligned} \mathcal{G}_\lambda(x,y) =& -\lambda^{-1}[u]^{-1}e^{-\mu_2^- y}\bar W'(1,\ -L,\ 0) \\ &+ \mathcal{O}(e^{-\mu_2^- y}e^{\mu_3^- x}) + \mathcal{O}(e^{\mu_3^- (x-y)}) \end{aligned} \tag{4.24}$$

for $x<y<0$. Similar bounds can be obtained for the case $y>0$.

Proof. For the first case $y < 0 < x$, since $W_2^+ \equiv 0$ on $(0, +\infty)$, we have

$$\begin{aligned}\mathcal{G}_\lambda(x,y) &= W_1^+(x)\,C_1^+(y,\lambda) = \Big(\bar W'(x) + \mathcal{O}(\lambda e^{\mu_3^+ x})\Big) C_1^+(y,\lambda)\\ &= \Big(\bar W'(x) + \mathcal{O}(\lambda e^{\mu_3^+ x})\Big)\\ &\quad\times \Big(\lambda^{-1}[u]^{-1} e^{-\mu_2^- y}(1,\ -L,\ 0) + \mathcal{O}(e^{-\mu_1^- y} + e^{-\mu_2^- y})\Big),\end{aligned}$$

giving the estimate (4.22).

For the third case $x < y < 0$, we have

$$\begin{aligned}\mathcal{G}_\lambda(x,y) &= -W_3^-(x)\,C_3^-(y,\lambda) = -\Big(\bar W'(x) + \mathcal{O}(\lambda e^{\mu_3^- x})\Big) C_3^-(y,\lambda)\\ &= \Big(\bar W'(x) + \mathcal{O}(\lambda e^{\mu_3^- x})\Big)\\ &\quad\times \Big(-\lambda^{-1}[u]^{-1} e^{-\mu_2^- y}(1,\ -L,\ 0) + \mathcal{O}(e^{-\mu_1^- y} + e^{-\mu_2^- y} + e^{-\mu_3^- y})\Big),\end{aligned}$$

proving the estimate (4.24).

Finally, for the second case $y < x < 0$, we have

$$\mathcal{G}_\lambda(x,y) = W_1^+(x,\lambda)\,C_1^+(y,\lambda) + W_2^+(x,\lambda)\,C_2^+(y,\lambda).$$

In this case, besides the fact that the λ^{-1} term comes from the expression $W_1^+ C_1^+$ as above, there is a crucial cancellation as computed in (4.18), which proves (4.23), using the crucial constraint (4.13) (by our choice of the bases) to eliminate fast modes in the λ^{-1} term. ◻

5. Pointwise bounds and low-frequency estimates. In this section, using the previous pointwise bounds (Propositions 4.2 and 4.5) for the resolvent kernel in low-frequency regions, we derive pointwise bounds for the "low-frequency" Green function:

$$G^I(x,t;y) := \frac{1}{2\pi i}\int_{\Gamma\bigcap\{|\lambda|\le r\}} e^{\lambda t}\mathcal{G}_\lambda(x,y)\,d\lambda, \tag{5.1}$$

where Γ is any contour near zero, but away from the essential spectrum, and $r > 0$ is a sufficiently small constant such that all previous computations on G_λ hold.

PROPOSITION 5.1. *Assuming* (A0)–(A5$_k$) *and defining the effective diffusion* $L\,b_\pm$ *(see* [30]*), the low-frequency Green distribution* $G^I(x,t;y)$ *associated with the linearized evolution equations may be decomposed as*

$$G^I(x,t;y) = E + \widetilde G^I + R,$$

where, for $y < 0$,

$$E(x,t;y) := \bar U_x(x)[u]^{-1} e(y,t),$$

$$e(y,t) := \left(\operatorname{errfn}\left(\frac{y + a_- t}{\sqrt{4\,L\,b_-\,t}}\right) - \operatorname{errfn}\left(\frac{y - a_- t}{\sqrt{4\,L\,b_-\,t}}\right)\right),$$

$$|\partial_x^\kappa \partial_y^\beta \widetilde G^I(x,t;y)| \le C_1\, t^{-(|\beta|+|\kappa|)/2 - 1/2} e^{-(x-y-a_-t)^2/C_2\,t},$$

$$R(x,t;y) = \mathcal{O}(e^{-\eta(|x-y|+t)}) + \mathcal{O}(e^{-\eta t})\chi(x,y)\left[1 + \frac{1}{a(y)}(x/y)^{\nu}\right]$$

for some η, C_1, $C_2 > 0$, *where* $\beta, \kappa = 0, 1$, *and* $\nu = \frac{Lb(0)+a'(0)}{|a'(0)|}$ *and*

$$\chi(x,y) = \begin{cases} 1, & -1 < y < x < 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Symmetric bounds hold for $y \geq 0$.

Proof. Having resolvent kernel estimates in Propositions 4.2 and 4.5, we can now follow the previous analyses of [47, 29, 30]. Indeed, the claimed bound for E precisely comes from the term $\lambda^{-1}[u]^{-1}e^{-\mu_2^- y}\bar{U}_x$, where $\mu_2^- = -\lambda/a_- + \mathcal{O}(\lambda^2)$. Likewise, estimates of $\widetilde{G}^I$ are due to bounds in Proposition 4.5 for y away from zero and those in Proposition 4.2 for y near zero but x away from zero. The singularity occurs only in the case $-1 < y < x < 0$, as reported in Proposition 4.2. In this case, using estimate (4.9) and moving the contour Γ in (5.1) into the stable half plane $\{\operatorname{Re}\lambda < 0\}$, we have

$$\int_\Gamma e^{\lambda t}\left(1 + \frac{|x|^\nu}{a(y)|y|^\nu}\right)d\lambda = \mathcal{O}(e^{-\eta t})\left(1 + \frac{|x|^\nu}{a(y)|y|^\nu}\right)$$

which precisely contributes to the second term in $R(x,t;y)$. The first term is, as usual, the fast decaying term. ∎

With the above pointwise estimates on the (low-frequency) Green function, we have the following from [29, 30].

LEMMA 5.2 (see [29, 30]). *Assuming* (A0)–(A5$_k$), $\widetilde{G}^I$ *satisfies*

$$\Big|\int_{-\infty}^{+\infty}\partial_y^\beta \widetilde{G}^I(\cdot,t;y)f(y)dy\Big|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1/q-1/p)-|\beta|/2}|f|_{L^q} \tag{5.2}$$

for all $t \geq 0$, *for some* $C > 0$, *and for any* $1 \leq q \leq p$.

We recall the following fact from [45].

LEMMA 5.3 (see [45]). *The kernel* e *satisfies*

$$|e_y(\cdot,t)|_{L^p}, |e_t(\cdot,t)|_{L^p}, \leq Ct^{-\frac{1}{2}(1-1/p)},$$
$$|e_{yt}(\cdot,t)|_{L^p} \leq Ct^{-\frac{1}{2}(1-1/p)-1/2}$$

for all $t > 0$, *for some* $C > 0$, *and for any* $p \geq 1$.

Finally, we have the following estimate on the R term.

LEMMA 5.4. *Under* (A0)–(A5$_k$), $R(x,t;y)$ *satisfies*

$$\Big|\int_{-\infty}^{+\infty}R(\cdot,t;y)f(y)dy\Big|_{L^p} \leq Ce^{-\eta t}(|f|_{L^p} + |f|_{L^\infty})$$

for all $t \geq 0$, *for some* $C, \eta > 0$, *and for any* $1 \leq p \leq \infty$.

Proof. The estimate clearly holds for the fast decaying term $e^{-\eta(|x-y|+t)}$ in R, whereas, to estimate the second term, first notice that it is only nonzero precisely when $-1 < y < x < 0$ or $0 < x < y < 1$. Thus, for instance, when $-1 < x < 0$, we estimate

$$\Big|\int_{-\infty}^{+\infty}\chi(x,y)\left[1 + \frac{1}{a(y)}(x/y)^\nu\right]f(y)dy\Big| = \Big|\int_{-1}^{x}\left[1 + \frac{1}{a(y)}(x/y)^\nu\right]f(y)dy\Big|$$
$$\leq C|f|_{L^\infty}\left[1 + \int_{-1}^{x}\frac{1}{|a(y)|}(x/y)^\nu dy\right] \leq C|f|_{L^\infty},$$

where the last integral is bounded by the fact that $a(x) \sim x$ as $|x| \to 0$. From this, we easily obtain

$$\Big| \int_{-\infty}^{+\infty} e^{-\eta t}\chi(x,y)\left[1+\frac{1}{a(y)}(x/y)^{\nu}\right] f(y)dy\Big|_{L^p(-1,0)} \le Ce^{-\eta t}|f|_{L^\infty},$$

which proves the lemma. □

Remark 5.5. We note here that the singular term $a^{-1}(y)(x/y)^{\nu}$ appearing in (4.9) and (5.1) contributes to the time-exponentially decaying term. Note that this part agrees with the resolvent kernel for the scalar convected-damped equation $u_t + au_x = -Lgu$, for which we can find explicitly the Green function as a convected time-exponentially decaying delta function similar to terms appearing in the relaxation or real viscosity case.

6. Nonlinear damping estimate. In this section, we establish an auxiliary damping energy estimate. We consider the eventual nonlinear perturbation equations for variables (u, q),

$$\begin{aligned} &u_t + (\hat{a}(u)u)_x + Lq_x = \dot{\alpha}(U_x + u_x),\\ &-q_{xx} + q + (\hat{b}(u)\,u)_x = 0, \end{aligned} \tag{6.1}$$

where α represents the shock location and

$$\begin{aligned} \hat{a}(u) &:= \frac{df}{du}(U+u) = \frac{df}{du}(U) + \mathcal{O}(|u|) = a(x) + \mathcal{O}(|u|),\\ \hat{b}(u) &:= \frac{dM}{du}(U+u) = \frac{dM}{du}(U) + \mathcal{O}(|u|) = b(x) + \mathcal{O}(|u|) \end{aligned}$$

(see subsequent section 8). In view of

$$\hat{a}_x = a'(x) + \mathcal{O}(|u| + |u_x|),$$

under assumptions (A4) and (A5$_k$) we get that, for all $|u|_\infty$ and $|u_x|_\infty$ sufficiently small, there holds

$$L\hat{b} + \left(k + \frac{1}{2}\right)\hat{a}_x > 0 \tag{6.2}$$

for all $k = 1, 2, 3, 4$ and all $x \sim 0$. We are going to profit from (6.2) in proving the following.

PROPOSITION 6.1. *Assume* (A0)–(A5$_k$). *So long as* $|u|_{W^{2,\infty}}$ *and* $|\dot{\alpha}|$ *remain sufficiently small, we obtain*

$$|u|^2_{H^k}(t) \le Ce^{-\eta t}|u|^2_{H^k}(0) + C\int_0^t e^{-\eta(t-s)}(|u|^2_{L^2} + |\dot{\alpha}|^2)(s)\,ds, \qquad \eta > 0, \tag{6.3}$$

for $k = 1, \dots, 4$.

Proof. Let us work for the case $\dot{\alpha} \equiv 0$. The general case will be seen as a straightforward extension. For our convenience, we denote the ϕ-weighted norm as

$$|f|_{H^k_\phi} := \sum_{i=0}^{k} \langle \phi\, \partial_x^i f, \partial_x^i f\rangle^{1/2}_{L^2}$$

for nonnegative functions ϕ. Now, taking the inner product of q against the second equation in (6.1) and applying integration by parts, we obtain

$$|q_x|^2_{L^2} + |q|^2_{L^2} = \langle \hat{b}u, q_x\rangle \le \frac{1}{2}|q_x|^2_{L^2} + C|u|^2_{L^2}.$$

In fact, we also can easily get for $k \ge 1$,

$$|q|_{H^k_\phi} \le C|u|_{H^{k-1}_\phi}. \tag{6.4}$$

Likewise, taking the inner product of u against the first equation in (6.1) and integrating by parts, we get

$$\frac{1}{2}\frac{d}{dt}|u|^2_{L^2} = -\frac{1}{2}\int \hat{a}_x|u|^2\,dx - \langle Lq_x, u\rangle,$$

which, together with (6.4) and the Hölder inequality, gives

$$\frac{d}{dt}|u|^2_{L^2} \le C|u|^2_{L^2}. \tag{6.5}$$

In order to establish estimates for derivatives, for each $k \ge 1$ and $\phi \ge 0$ to be determined later, we compute

$$\frac{1}{2}\frac{d}{dt}\langle \partial_x^k u, \phi\,\partial_x^k u\rangle = \langle \partial_x^k u_t, \phi\,\partial_x^k u\rangle = -\langle L\partial_x^{k+1}q + \partial_x^{k+1}(\hat{a}u), \phi\,\partial_x^k u\rangle, \tag{6.6}$$

where, using the equation for q, we estimate

$$\begin{aligned}
-\langle L\partial_x^{k+1}q, \phi\,\partial_x^k u\rangle &= -\langle L\partial_x^k(\hat{b}u) + L\partial_x^{k-1}q, \phi\,\partial_x^k u\rangle\\
&\le -\langle L\,\hat{b}\,\phi\,\partial_x^k u, \partial_x^k u\rangle + \epsilon|\partial_x^k u|^2_{L^2_\phi} + C_\epsilon\left[|u|^2_{H^{k-1}_\phi} + |q|^2_{H^{k-1}_\phi}\right]\\
&\le -\frac{\eta}{2}|\partial_x^k u|^2_{L^2_\phi} + C|u|^2_{H^{k-1}_\phi}
\end{aligned}$$

and

$$\begin{aligned}
-\langle \partial_x^{k+1}(\hat{a}u), \phi\,\partial_x^k u\rangle &= -\langle \hat{a}\partial_x^{k+1}u + (k+1)\hat{a}_x\partial_x^k u + L.O.T., \phi\,\partial_x^k u\rangle\\
&= \left\langle \left(\frac{1}{2}(\hat{a}\phi)_x - (k+1)\hat{a}_x\phi\right)\partial_x^k u, \partial_x^k u\right\rangle - L.O.T., \phi\,\partial_x^k u\\
&\le \left\langle \left(\frac{1}{2}(\hat{a}\phi)_x - (k+1)\hat{a}_x\phi\right)\partial_x^k u, \partial_x^k u\right\rangle + \epsilon|\partial_x^k u|^2_{L^2_\phi} + C_\epsilon|u|^2_{H^{k-1}_\phi},
\end{aligned}$$

where L.O.T. denotes lower orders of derivatives of u in expansion of $\partial_x^{k+1}(\hat{a}u)$. By choosing $\phi := |\hat{a}|^{2k+1}$, we observe that

$$\frac{1}{2}(\hat{a}\phi)_x - (k+1)\hat{a}_x\phi \equiv 0,$$

and thus

$$-\langle \partial_x^{k+1}(\hat{a}u), \phi\,\partial_x^k u\rangle \le \epsilon|\partial_x^k u|^2_{L^2_\phi} + C_\epsilon|u|^2_{H^{k-1}_\phi}$$

for any positive number ϵ. Taking ϵ small enough and putting the above estimates together into (6.6), we have just obtained

$$\frac{d}{dt}\langle \partial_x^k u, |\hat{a}|^{2k+1}\partial_x^k u\rangle \le -\eta_1\langle \partial_x^k u, |\hat{a}|^{2k+1}\partial_x^k u\rangle + C|u|^2_{H^{k-1}} \tag{6.7}$$

for each $k \ge 1$ and some small $\theta_1 > 0$.

In addition, by choosing $\phi \equiv 1$ in (6.6), we obtain

(6.8)
$$\frac{1}{2}\frac{d}{dt}\langle \partial_x^k u, \partial_x^k u\rangle = -\left\langle \left(L\hat{b} + \left(k+\frac{1}{2}\right)\hat{a}_x\right)\partial_x^k u, \partial_x^k u\right\rangle + \epsilon|\partial_x^k u|_{L^2}^2 + C_\epsilon |u|_{H^{k-1}}^2$$

for any $\epsilon > 0$. By assumption (6.2), there exist η_2 sufficiently small and $M > 0$ sufficiently large such that

$$M\theta_1|\hat{a}|^{2k+1} + \left(L\hat{b} + \left(k+\frac{1}{2}\right)\hat{a}_x\right) \ge \eta_2 > 0 \tag{6.9}$$

for all $x \in \mathbb{R}$ (by taking M large enough away from zero; for $x \sim 0$ the bound follows from (6.2)). Therefore, by adding (6.8) with M times (6.7), using (6.9), and taking $\epsilon = \eta_2/2$ in (6.8), we obtain

$$\frac{d}{dt}\langle (1+M|\hat{a}|^{2k+1})\partial_x^k u, \partial_x^k u\rangle \le -\frac{\eta_2}{2}|\partial_x^k u|_{L^2}^2 + C|u|_{H^{k-1}}^2. \tag{6.10}$$

Now, for $\delta > 0$, let us define

$$\mathcal{E}(t) := \sum_{i=0}^{k} \delta^i \langle (1+M|a|^{2k+1})\partial_x^k u, \partial_x^k u\rangle.$$

Observe that $\mathcal{E}(t) \sim |u|_{H^k}^2$. We then use (6.5) and (6.10) for $k = 1, \dots, 4$ and take δ sufficiently small to derive

$$\frac{d}{dt}\mathcal{E}(t) \le -\eta_3\,\mathcal{E}(t) + C|u|_{L^2}^2(t)$$

for some $\eta_3 > 0$, from which (6.3) follows by the standard Gronwall's inequality. ∎

7. High-frequency estimate. In this section, we estimate the high-frequency part of the solution operator $e^{\mathcal{L}t}$ (see (1.14)),

$$\mathcal{S}_2(t) = \frac{1}{2\pi i}\int_{-\gamma_1 - i\infty}^{-\gamma_1 + i\infty} \chi_{\{|\mathrm{Im}\,\lambda| \ge \gamma_2\}} e^{\lambda t}(\lambda - \mathcal{L})^{-1} d\lambda \tag{7.1}$$

for small constants $\gamma_1, \gamma_2 > 0$ (here χ_I is the characteristic function of the set I).

Proposition 7.1 (high-frequency estimate). *Under assumptions* (A0)–(A5$_k$)*, we obtain*

$$|\partial_x^\kappa \mathcal{S}_2(t)(\varphi - L\,\partial_x(\mathcal{K}\psi))|_{L^2} \le Ce^{-\eta_1 t}\Big(|\psi|_{H^{\kappa+2}} + |\varphi|_{H^{\kappa+2}}\Big), \qquad \kappa = 0, 1, \tag{7.2}$$

for some $\eta_1 > 0$*, where* $\mathcal{K} = (-\partial_x^2 + 1)^{-1}$ *and* L *is a constant (see* (1.13)*).*

Our first step in proving (7.2) is to estimate the solution of the resolvent system

$$\begin{aligned} \lambda u + (a(x)\,u)_x + Lq_x &= \varphi, \\ -q_{xx} + q + (b(x)\,u)_x &= \psi, \end{aligned}$$

where $a(x) = \frac{df}{du}(U(x))$ and $b(x) = \frac{dM}{du}(U)$ as before.

Proposition 7.2 (high-frequency bounds). *Assuming* (A0)–(A5$_k$)*, for some* R, C *sufficiently large and* $\gamma > 0$ *sufficiently small, we obtain*

$$|(\lambda - \mathcal{L})^{-1}(\varphi - L\partial_x(\mathcal{K}\psi))|_{H^1} \le C\Big(|\varphi|_{H^1}^2 + |\psi|_{L^2}^2\Big),$$
$$|(\lambda - \mathcal{L})^{-1}(\varphi - L\partial_x(\mathcal{K}\psi))|_{L^2} \le \frac{C}{|\lambda|^{1/2}}\Big(|\varphi|_{H^1}^2 + |\psi|_{L^2}^2\Big)$$

for all $|\lambda| \ge R$ *and* $\mathrm{Re}\,\lambda \ge -\gamma$.

Proof. A Laplace transformed version of the nonlinear energy estimates (6.5) and (6.10) in section 6 with $k = 1$ (see [46, pp. 272–273, proof of Proposition 4.7] for further details) yields

$$\Big(\operatorname{Re}\lambda + \frac{\gamma_1}{2}\Big)|u|^2_{H^1} \le C\Big(|u|^2_{L^2} + |\varphi|^2_{H^1} + |\psi|^2_{L^2}\Big). \tag{7.3}$$

On the other hand, taking the imaginary part of the L^2 inner product of U against $\lambda u = \mathcal{L}u + \partial_x L\mathcal{K}h + f$ and applying the Young's inequality, we also obtain the standard estimate

$$\begin{aligned} |\operatorname{Im}\lambda||u|^2_{L^2} &\le |\langle \mathcal{L}u, u\rangle| + |\langle L\mathcal{K}\psi, u_x\rangle| + |\langle \varphi, u\rangle| \\ &\le C\Big(|u|^2_{H^1} + |\psi|^2_{L^2} + |\varphi|^2_{L^2}\Big), \end{aligned} \tag{7.4}$$

noting the fact that $\mathcal{L}$ is a bounded operator from H^1 to L^2 and $\mathcal{K}$ is bounded from L^2 to H^1.

Therefore, taking $\gamma = \gamma_1/4$, we obtain from (7.3) and (7.4),

$$|\lambda||u|^2_{L^2} + |u|^2_{H^1} \le C\Big(|u|^2_{L^2} + |\psi|^2_{L^2} + |\varphi|^2_{H^1}\Big)$$

for any $\operatorname{Re}\lambda \ge -\gamma$. Now take R sufficiently large such that $|u|^2_{L^2}$ on the right-hand side of the above can be absorbed into the left-hand side for $|\lambda| \ge R$, thus yielding

$$|\lambda||u|^2_{L^2} + |u|^2_{H^1} \le C\Big(|\psi|^2_{L^2} + |\varphi|^2_{H^1}\Big)$$

for some large $C > 0$, which gives the result as claimed. ☐

Next, we have the following.

PROPOSITION 7.3 (mid-frequency bounds). *Assuming* (A0)–(A5$_k$), *we obtain*

$$|(\lambda - \mathcal{L})^{-1}\varphi|_{L^2} \le C\,|\varphi|_{H^1} \qquad \textit{for } R^{-1} \le |\lambda| \le R \textit{ and } \operatorname{Re}\lambda \ge -\gamma$$

for any R and $C = C(R)$ sufficiently large and $\gamma = \gamma(R) > 0$ sufficiently small.

Proof. The proof is immediate by compactness of the set of frequencies under consideration together with the fact that the resolvent $(\lambda - \mathcal{L})^{-1}$ is analytic with respect to H^1 in λ; see, for instance, [45]. ☐

With Propositions 7.2 and 7.3 in hand, we are now ready to give the following proof.

Proof of Proposition 7.1. The proof starts with the following resolvent identity, using analyticity on the resolvent set $\rho(\mathcal{L})$ of the resolvent $(\lambda - \mathcal{L})^{-1}$ for all $\varphi \in \mathcal{D}(\mathcal{L})$:

$$(\lambda - \mathcal{L})^{-1}\varphi = \lambda^{-1}(\lambda - \mathcal{L})^{-1}\mathcal{L}\varphi + \lambda^{-1}\varphi.$$

Using this identity and (7.1), we estimate

$$\begin{aligned} \mathcal{S}_2(t)\varphi &= \frac{1}{2\pi i}\int_{-\gamma_1 - i\infty}^{-\gamma_1 + i\infty} \chi_{\{|\operatorname{Im}\lambda| \ge \gamma_2\}} e^{\lambda t}\lambda^{-1}(\lambda - \mathcal{L})^{-1}\mathcal{L}\,\varphi\,d\lambda \\ &\quad + \frac{1}{2\pi i}\int_{-\gamma_1 - i\infty}^{-\gamma_1 + i\infty} \chi_{\{|\operatorname{Im}\lambda| \ge \gamma_2\}} e^{\lambda t}\lambda^{-1}\varphi\,d\lambda \\ &=: S_1 + S_2, \end{aligned}$$

where, by Propositions 7.1 and 7.3, we have

$$\begin{aligned}|S_1|_{L^2} &\le C\int_{-\gamma_1-i\infty}^{-\gamma_1+i\infty}|\lambda|^{-1}e^{\mathrm{Re}\,\lambda t}|(\lambda-\mathcal{L})^{-1}\mathcal{L}\varphi|_{L^2}|d\lambda|\\ &\le Ce^{-\gamma_1 t}\int_{-\gamma_1-i\infty}^{-\gamma_1+i\infty}|\lambda|^{-3/2}|\mathcal{L}\varphi|_{H^1}|d\lambda|\\ &\le Ce^{-\gamma_1 t}|\varphi|_{H^2}\end{aligned}$$

and

$$\begin{aligned}|S_2|_{L^2} &\le \frac{1}{2\pi}\Big|\varphi\int_{-\gamma_1-i\infty}^{-\gamma_1+i\infty}\lambda^{-1}e^{\lambda t}d\lambda\Big|_{L^2}+\frac{1}{2\pi}\Big|\varphi\int_{-\gamma_1-ir}^{-\gamma_1+ir}\lambda^{-1}e^{\lambda t}d\lambda\Big|_{L^2}\\ &\le Ce^{-\gamma_1 t}|\varphi|_{L^2}\end{aligned}$$

by direct computations, noting that the integral in λ in the first term is identically zero. This completes the proof of the bound for the term involving φ as stated in the proposition. The estimate involving ψ follows by observing that $L\,\partial_x\mathcal{K}$ is bounded from H^s to H^s. Derivative bounds can be obtained similarly. ∎

Remark 7.4. We note that in our treatment of the high-frequency terms by energy estimates (as also done in [20, 36]), we are ignoring the pointwise contribution there, which would also be convected time-decaying delta functions. To see these features, a simple exercise is to to apply the Fourier transform to the linearized equations about a constant state.

8. Nonlinear analysis. In this section, we shall prove the main nonlinear stability theorem. Following [11, 30], define the nonlinear perturbation

$$\begin{pmatrix}u\\ q\end{pmatrix}(x,t) := \begin{pmatrix}\tilde u\\ \tilde q\end{pmatrix}(x+\alpha(t),t)-\begin{pmatrix}U\\ Q\end{pmatrix}(x), \tag{8.1}$$

where the shock location $\alpha(t)$ is to be determined later.

Plugging (8.1) into (1.1), we obtain the perturbation equation

$$\begin{aligned}u_t+(a(x)\,u)_x+L\,q_x &= N_1(u)_x+\dot\alpha(t)\,(u_x+U_x),\\ -q_{xx}+q+(b(x)\,u)_x &= N_2(u)_x,\end{aligned}$$

where $N_j(u)=O(|u|^2)$ so long as u stays uniformly bounded.

We decompose the Green function as

$$G(x,t;y)=G^I(x,t;y)+G^{II}(x,t;y), \tag{8.2}$$

where $G^I(x,t;y)$ is the low-frequency part. We further define, as in Proposition 5.1,

$$\widetilde G^I(x,t;y)=G^I(x,t;y)-E(x,t;y)-R(x,t;y)$$

and

$$\widetilde G^{II}(x,t;y)=G^{II}(x,t;y)+R(x,t;y).$$

Then, we immediately obtain the following from Lemmas 5.2 and 5.4 and Proposition 7.1.

LEMMA 8.1. *There holds*

$$\Big| \int_{-\infty}^{+\infty} \partial_y^\beta \widetilde{G}^I(\cdot,t;y)f(y)dy\Big|_{L^p} \le C(1+t)^{-\frac{1}{2}(1/q-1/p)-|\beta|/2}|f|_{L^q} \tag{8.3}$$

for all $1 \le q \le p,\ \beta = 0,1,$ *and*

$$\Big| \int_{-\infty}^{+\infty} \widetilde{G}^{II}(x,t;y)f(y)dy\Big|_{L^p} \le Ce^{-\eta t}|f|_{H^3} \tag{8.4}$$

for all $2 \le p \le \infty$.

Proof. Bound (8.3) is precisely the estimate (5.2) in Lemma 5.2, recalled here for our convenience. Inequality (8.4) is a straightforward combination of Lemma 5.4 and Proposition 7.1, followed by a use of the interpolation inequality between L^2 and L^∞ and an application of the standard Sobolev imbedding. ∎

We next show the following, which we have by Duhamel's principle.

LEMMA 8.2. *There hold the reduced integral representations*

$$\begin{aligned} u(x,t) =& \int_{-\infty}^{+\infty} (\widetilde{G}^I + \widetilde{G}^{II})(x,t;y)u_0(y)dy \\ & - \int_0^t \int_{-\infty}^{+\infty} \widetilde{G}^I_y(x,t-s;y)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y N_2(u) + \dot\alpha\, u\Big)(y,s)\,dy\,ds \\ & + \int_0^t \int_{-\infty}^{+\infty} \widetilde{G}^{II}(x,t-s;y)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y N_2(u) + \dot\alpha\, u\Big)_y(y,s)\,dy\,ds, \\ q(x,t) =& (\mathcal{K}\partial_x)(N_2(u) - b\,u)(x,t) \end{aligned} \tag{8.5}$$

and

$$\begin{aligned} \alpha(t) =& - \int_{-\infty}^{+\infty} e(y,t)u_0(y)dy \\ & + \int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y N_2(u) + \dot\alpha\, u\Big)(y,s)\,dy\,ds, \end{aligned} \tag{8.6}$$

$$\begin{aligned} \dot\alpha(t) =& - \int_{-\infty}^{+\infty} e_t(y,t)u_0(y)dy \\ & + \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y N_2(u) + \dot\alpha\, u\Big)(y,s)\,dy\,ds. \end{aligned} \tag{8.7}$$

Proof. By Duhamel's principle and the fact that

$$\int_{-\infty}^{+\infty} G(x,t;y)U'(y)dy = e^{\mathcal{L}t}U'(x) = U'(x),$$

we obtain

$$\begin{aligned} u(x,t) =& \int_{-\infty}^{+\infty} G(x,t;y)u_0(y)dy \\ & + \int_0^t \int_{-\infty}^{+\infty} G(x,t-s;y)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y N_2(u) + \dot\alpha\, u\Big)_y(y,s)\,dy\,ds \\ & + \alpha(t)\,U'. \end{aligned}$$

Thus, by defining the *instantaneous shock location*

$$\alpha(t) = -\int_{-\infty}^{+\infty} e(y,t)u_0(y)dy + \int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y N_2(u) + \dot\alpha\, u\Big)(y,s)\,dy\,ds$$

and using the Green function decomposition (8.2), we easily obtain the integral representation as claimed in the lemma. ▯

With these preparations, we are now ready to prove the main theorem, following the standard stability analysis of [31, 44, 45].

Proof of Theorem 1.3. Define

$$\zeta(t) := \sup_{0\le s\le t,\, 2\le p\le\infty} \Big[|u(s)|_{L^p}(1+s)^{\frac12(1-1/p)} + |\alpha(s)| + |\dot\alpha(s)|(1+s)^{\frac12}\Big].$$

We shall prove here that for all $t \ge 0$ for which a solution exists with $\zeta(t)$ uniformly bounded by some fixed, sufficiently small constant, there holds

$$\zeta(t) \le C(|u_0|_{L^1\cap H^s} + \zeta(t)^2). \tag{8.8}$$

This bound, together with continuity of $\zeta(t)$, implies that

$$\zeta(t) \le 2C|u_0|_{L^1\cap H^s} \tag{8.9}$$

for $t \ge 0$, provided that $|u_0|_{L^1\cap H^s} < 1/4C^2$. This would complete the proof of the bounds as claimed in the theorem, and thus give the main theorem.

By standard short-time theory/local well-posedness in H^s and the standard principle of continuation, there exists a solution $u \in H^s$ on the open time-interval for which $|u|_{H^s}$ remains bounded, and on this interval $\zeta(t)$ is well defined and continuous. Now, let $[0,T)$ be the maximal interval on which $|u|_{H^s}$ remains strictly bounded by some fixed, sufficiently small constant $\delta > 0$. By Proposition 6.1 and the Sobolev embedding inequality $|u|_{W^{2,\infty}} \le C|u|_{H^s}$, $s \ge 3$, we have

$$\begin{aligned} |u(t)|_{H^s}^2 &\le Ce^{-\eta t}|u_0|_{H^s}^2 + C\int_0^t e^{-\eta(t-\tau)}\Big(|u(\tau)|_{L^2}^2 + |\dot\alpha(\tau)|^2\Big)d\tau \\ &\le C(|u_0|_{H^s}^2 + \zeta(t)^2)(1+t)^{-1/2}, \end{aligned} \tag{8.10}$$

and so the solution continues so long as ζ remains small, with bound (8.9), yielding existence and the claimed bounds.

Thus, it remains to prove claim (8.8). First, by representation (8.5) for u, for any $2 \le p \le \infty$, we obtain

$$\begin{aligned} |u|_{L^p}(t) \le& \Big|\int_{-\infty}^{+\infty} (\widetilde G^I + \widetilde G^{II})(x,t;y)u_0(y)dy\Big|_{L^p} \\ &+ \int_0^t \Big|\int_{-\infty}^{+\infty} \widetilde G^I_y(x,t-s;y)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y K N_2(u) + \dot\alpha\, u\Big)(y,s)\,dy\Big|_{L^p} ds \\ &+ \int_0^t \Big|\int_{-\infty}^{+\infty} \widetilde G^{II}(x,t-s;y)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y N_2(u) + \dot\alpha\, u\Big)_y(y,s)\,dy\Big|_{L^p} ds \\ =& I_1 + I_2 + I_3, \end{aligned}$$

where estimates (8.3) and (8.4) yield

$$\begin{aligned}
I_1 &= \Big| \int_{-\infty}^{+\infty} (\widetilde{G}^I + \widetilde{G}^{II})(x,t;y) u_0(y) dy \Big|_{L^p} \\
&\le C(1+t)^{-\frac{1}{2}(1-1/p)} |u_0|_{L^1} + C e^{-\eta t} |u_0|_{H^3} \\
&\le C(1+t)^{-\frac{1}{2}(1-1/p)} |u_0|_{L^1 \cap H^3},
\end{aligned}$$

and, noting that $L\mathcal{K}\partial_y$ is bounded from L^2 to L^2, yield

$$\begin{aligned}
I_2 &= \int_0^t \Big| \int_{-\infty}^{+\infty} \widetilde{G}^I_y(x,t-s;y) \Big(N_1(u) - L\mathcal{K}\partial_y N_2(u) + \dot{\alpha}\, u \Big)(y,s)\, dy \Big|_{L^p} ds \\
&\le C \int_0^t (t-s)^{-\frac{1}{2}(1/2-1/p)-\frac{1}{2}} (|u|_{L^\infty} + |\dot{\alpha}|) |u|_{L^2}(s) ds \\
&\le C\zeta(t)^2 \int_0^t (t-s)^{-\frac{1}{2}(1/2-1/p)-\frac{1}{2}} (1+s)^{-\frac{3}{4}} ds \\
&\le C\zeta(t)^2 (1+t)^{-\frac{1}{2}(1-1/p)},
\end{aligned}$$

and, together with (8.10) and $s \ge 4$, yield

$$\begin{aligned}
I_3 &= \int_0^t \Big| \int_{-\infty}^{+\infty} \widetilde{G}^{II}(x,t-s;y) \Big(N_1(u) - L\mathcal{K}\partial_y N_2(u) + \dot{\alpha}\, u \Big)_y (y,s)\, dy \Big|_{L^p} ds \\
&\le C \int_0^t e^{-\eta(t-s)} |N_1(u) - L\mathcal{K}\partial_y N_2(u) + \dot{\alpha}\, u|_{H^4}(s) ds \\
&\le C \int_0^t e^{-\eta(t-s)} (|u|_{H^s} + |\dot{\alpha}|) |u|_{H^s}(s) ds \\
&\le C(|u_0|^2_{H^s} + \zeta(t)^2) \int_0^t e^{-\eta(t-s)} (1+s)^{-1} ds \\
&\le C(|u_0|^2_{H^s} + \zeta(t)^2)(1+t)^{-1}.
\end{aligned}$$

Thus, we have proved

$$|u(t)|_{L^p} (1+t)^{\frac{1}{2}(1-1/p)} \le C(|u_0|_{L^1 \cap H^s} + \zeta(t)^2).$$

Similarly, using representations (8.6) and (8.7) and the estimates in Lemma 5.3 on the kernel $e(y,t)$, we can estimate (see, e.g., [31, 45])

$$|\dot{\alpha}(t)|(1+t)^{1/2} + |\alpha(t)| \le C(|u_0|_{L^1} + \zeta(t)^2).$$

This completes the proof of claim (8.8), and thus the result for u as claimed. To prove the result for q, we observe that $\mathcal{K}\partial_x$ is bounded from $L^p \to W^{1,p}$ for all $1 \le p \le \infty$, and thus from representation (8.5) for q, we estimate

$$\begin{aligned}
|q|_{W^{1,p}}(t) &\le C(|N_2(u)|_{L^p} + |u|_{L^p})(t) \\
&\le C|u|_{L^p}(t) \le C|u_0|_{L^1 \cap H^s} (1+t)^{-\frac{1}{2}(1-\frac{1}{p})}
\end{aligned}$$

and

$$|q|_{H^{s+1}}(t) \le C|u|_{H^s}(t) \le C|u_0|_{L^1 \cap H^s} (1+t)^{-\frac{1}{4}},$$

which complete the proof of the main theorem. ∎

Appendix A. Pointwise reduction lemma. Let us consider a system of equations of the form

$$W_x = \mathbb{A}^\epsilon(x,\lambda)W, \tag{A.1}$$

for which the coefficient $\mathbb{A}^\epsilon$ does not exhibit uniform exponential decay to its asymptotic limits, but instead is *slowly varying* (uniformly on an ϵ-neighborhood $\mathcal{V}$, where $\epsilon > 0$ is a parameter). This case occurs in different contexts for rescaled equations, such as (3.15) in the present analysis.

In this situation, it frequently occurs that not only $\mathbb{A}^\epsilon$ but also certain of its invariant eigenspaces are slowly varying with x, i.e., there exist matrices

$$\mathbb{L}^\epsilon = \begin{pmatrix} L_1^\epsilon \\ L_2^\epsilon \end{pmatrix}(x), \quad \mathbb{R}^\epsilon = \begin{pmatrix} R_1^\epsilon & R_2^\epsilon \end{pmatrix}(x)$$

for which $\mathbb{L}^\epsilon\mathbb{R}^\epsilon(x) \equiv I$ and $|\mathbb{L}\mathbb{R}'| = |\mathbb{L}'\mathbb{R}| \le C\delta^\epsilon(x)$, uniformly in ϵ, where the pointwise error bound $\delta^\epsilon = \delta^\epsilon(x)$ is small, relative to

$$\mathbb{M}^\epsilon := \mathbb{L}^\epsilon\mathbb{A}^\epsilon\mathbb{R}^\epsilon(x) = \begin{pmatrix} M_1^\epsilon & 0 \\ 0 & M_2^\epsilon \end{pmatrix}(x), \tag{A.2}$$

and $'$, as usual, denotes $\partial/\partial x$. In this case, making the change of coordinates $W^\epsilon = \mathbb{R}^\epsilon Z$, we may reduce (A.1) to the approximately block-diagonal equation

$$Z^{\epsilon\prime} = \mathbb{M}^\epsilon Z^\epsilon + \delta^\epsilon\Theta^\epsilon Z^\epsilon, \tag{A.3}$$

where $\mathbb{M}^\epsilon$ is as in (A.2), $\Theta^\epsilon(x)$ is a uniformly bounded matrix, and $\delta^\epsilon(x)$ is (relatively) small. Assume that such a procedure has been successfully carried out, and, moreover, that there exists an approximate *uniform spectral gap in numerical range*, in the strong sense that

$$\min\sigma(\operatorname{Re} M_1^\epsilon) - \max\sigma(\operatorname{Re} M_2^\epsilon) \ge \eta^\epsilon(x) \qquad \forall x, \tag{A}$$

with pointwise gap $\eta^\epsilon(x) > \eta_0 > 0$ uniformly bounded in x and in ϵ; here and elsewhere $\operatorname{Re} N := \frac{1}{2}(N + N^*)$ denotes the "real," or symmetric, part of an operator N. Then, the following *pointwise reduction lemma* holds, a refinement of the reduction lemma of [30] (see the related "tracking lemma" given in varying degrees of generality in [6, 29, 37, 47, 44]).

PROPOSITION A.1. *Consider a system* (A.3) *under the gap assumption* (A), *with* Θ^ϵ *uniformly bounded in* $\epsilon \in \mathcal{V}$ *and for all* x. *If, for all* $\epsilon \in \mathcal{V}$, $\sup_{x\in\mathbb{R}}(\delta^\epsilon/\eta^\epsilon)$ *is sufficiently small (i.e., the ratio of pointwise gap* $\eta^\epsilon(x)$ *and pointwise error bound* $\delta^\epsilon(x)$ *is uniformly small), then there exist (unique) linear transformations* $\Phi_1^\epsilon(x,\lambda)$ *and* $\Phi_2^\epsilon(x,\lambda)$ *possessing the same regularity with respect to the various parameters* ϵ, x, λ *as do coefficients* $\mathbb{M}^\epsilon$ *and* $\delta^\epsilon(x)\Theta^\epsilon(x)$, *for which the graphs* $\{(Z_1, \Phi_2^\epsilon(Z_1))\}$ *and* $\{(\Phi_1^\epsilon(Z_2), Z_2)\}$ *are invariant under the flow of* (A.3), *and satisfying*

$$\sup_{\mathbb{R}} |\Phi_j^\epsilon| \le C \sup_{\mathbb{R}}(\delta^\epsilon/\eta^\epsilon).$$

Moreover, we have the pointwise bounds

$$|\Phi_2^\epsilon(x)| \le C\int_{-\infty}^x e^{-\int_y^x \eta^\epsilon(z)dz}\delta^\epsilon(y)dy. \tag{A.4}$$

Symmetric bounds hold for Φ_1^ϵ.

Proof. By a change of independent coordinates, we may arrange that $\eta^\epsilon(x) \equiv$ constant, whereupon the first assertion reduces to the conclusion of the tracking/reduction lemma of [30]. Recall that this conclusion was obtained by seeking Φ_2^ϵ as the solution of a fixed-point equation

$$\Phi_2^\epsilon(x) = \mathcal{T}\Phi_2^\epsilon(x) := \int_{-\infty}^x \mathcal{F}^{y\to x}\delta^\epsilon(y)Q(\Phi_2^\epsilon)(y)dy.$$

Observe that in the present context we have allowed δ^ϵ to vary with x, but otherwise follow the proof of [30] word for word to obtain the conclusion (see Appendix C of [30], proof of Proposition 3.9). Here, $Q(\Phi_2^\epsilon) = \mathcal{O}(1 + |\Phi_2^\epsilon|^2)$ by construction, and $|\mathcal{F}^{y\to x}| \le Ce^{-\eta(x-y)}$. Thus, using only the fact that $|\Phi_2^\epsilon|$ is bounded, we obtain the bound (A.4) as claimed, in the new coordinates for which η^ϵ is constant. Switching back to the old coordinates, we have instead $|\mathcal{F}^{y\to x}| \le Ce^{-\int_y^x \eta^\epsilon(z)dz}$, yielding the result in the general case. ∎

Remark A.2. From Proposition A.1, we obtain reduced flows

$$\begin{cases} Z_1^{\epsilon\prime} = M_1^\epsilon Z_1^\epsilon + \delta^\epsilon(\Theta_{11} + \Theta_{12}^\epsilon\Phi_2^\epsilon)Z_1^\epsilon, \\ Z_2^{\epsilon\prime} = M_2^\epsilon Z_2^\epsilon + \delta^\epsilon(\Theta_{22} + \Theta_{21}^\epsilon\Phi_1^\epsilon)Z_2^\epsilon \end{cases}$$

on the two invariant manifolds described.

Appendix B. Spectral stability. Consider the eigenvalue system (1.10). Integrating the equations, we find the zero-mass conditions for u and q,

$$\int_{\mathbb{R}} u\,dx = 0, \qquad \int_{\mathbb{R}} q\,dx = 0,$$

which allows us to recast system (1.10) in terms of the integrated coordinates, which we denote, again, as u and q. The result is

$$\begin{aligned} \lambda u + a(x)\,u' + Lq' &= 0, \\ -q'' + q + b(x)\,u' &= 0. \end{aligned} \tag{B.1}$$

The following proposition is the main result of this section.

PROPOSITION B.1. *Let (u, q) be a bounded solution of* (B.1) *corresponding to a complex number $\lambda \neq 0$. Then* $\operatorname{Re}\lambda < 0$, *provided that at least one of the following conditions holds:*

(i) *b is a constant;*

(ii) *$|u_+ - u_-|$ is sufficiently small.*

Proof. In any case, we can assume $b > 0$ by redefining q by $-q$ if necessary, still preserving the condition $Lb > 0$. Taking the real part of the inner product of the first equation against $b\,u$ and using integration by parts, we obtain

$$\begin{aligned} \operatorname{Re}\lambda|b^{1/2}u|_{L^2}^2 &= -\operatorname{Re}\langle a\,b\,u', u\rangle - \operatorname{Re}\langle Lq', b\,u\rangle \\ &= \operatorname{Re}\Big(\langle (a\,b)'u, u\rangle + \langle Lq, (b\,u)'\rangle\Big) \\ &= \operatorname{Re}\Big(\langle (a\,b)'u, u\rangle + \langle Lq, q'' - q + b'\,u\rangle\Big) \\ &= \operatorname{Re}\Big(\langle a'\,b\,u, u\rangle - \langle L\,q', q'\rangle - \langle Lq, q\rangle + \langle a\,b'\,u, u\rangle + \langle L\,b'\,q, u\rangle\Big) \\ &\le \langle a'\,b\,u, u\rangle - \frac{L}{2}|q|_{H^1}^2 + C\langle(|a| + |b'|)|b'|u, u\rangle, \end{aligned}$$

which proves the proposition in the first case, noting $a' = \frac{d^2 f}{du^2}(U)\, U' < 0$ (by monotonicity of the profile) and $b \geq \theta > 0$. For the second case, observe that $|a| + |b'|$ is now sufficiently small and $|b'|$ and $|a'|$ have the same order of "smallness," that is, of order $\mathcal{O}(|U'|) = \mathcal{O}(|u_+ - u_-|)$. Thus, the last term on the right-hand side of the above estimate can be absorbed into the first term, yielding the result for this second case as well. ◻

Appendix C. Monotonicity of profiles under nonlinear coupling. In this appendix we show that radiative scalar shock profiles for general nonlinear coupling are monotone, a feature which plays a key role in our stability analysis. Although the existence of profiles for nonlinear coupling is already addressed in [23], and the monotonicity for the linear coupling case is discussed in [42, 23], for completeness (and convenience of the reader) we closely review the (scalar) existence proof of [22] and extend it to the nonlinear coupling case, a procedure which leads to monotonicity in a very simple way.

The main observation of this section is precisely that, thanks to assumptions (A0) and (A4), the mapping $u \mapsto LM(u)$ is a diffeomorphism on its range [23], which can be regarded as the identity along the arguments of the proof leading to the existence result of [22]. Since LM is monotone increasing in $[u_+, u_-]$, setting $M_\pm := M(u_\pm)$, there exists an inverse function $H : [LM_+, LM_-] \to [u_+, u_-]$ such that

$$y = LM(u) \iff u = H(y)$$

for each $u \in [u_+, u_-]$ and with derivative

$$\frac{dH}{dy} = \left(L\, \frac{dM}{du}(H(y)) \right)^{-1} \, > \, 0.$$

Consider once again the stationary profile equations (1.6) (after appropriate flux normalizations), with $(U, Q)(\pm\infty) = (u_\pm, 0)$. Integration of the equation for Q leads to $\int_{\mathbb{R}} Q = -[M] = M_- - M_+$. Let us introduce the variable Z as

$$Z := -L \int_{-\infty}^{x} Q(\xi)\, d\xi + LM_-$$

such that $Z' = -LQ$ and $Z \to LM_\pm$ as $x \to +\infty$. In terms of the new variable Z, the profile equations are

$$\begin{aligned} Z'' &= f(U)', \\ Z' - Z''' &= LM(u)'. \end{aligned}$$

Integrating these equations, and using the asymptotic limits for Z, we arrive at the system

$$\begin{aligned} Z' &= f(U) - f(u_\pm), \\ Z - Z'' &= LM(u). \end{aligned} \tag{C.1}$$

We can thus rewrite the ODE for Z as

$$Z' = F(H(Z - Z'')), \tag{C.2}$$

where $F(u) := f(u) - f(u_\pm)$. In view of strict convexity of f, the function F is strictly decreasing in the interval $[u_+, u_*]$ and strictly increasing in $[u_*, u_-]$, with $F(u_\pm) = 0$

and $F(u_*) = -m < 0$. Hence, F is invertible in those intervals with corresponding inverse functions $h_\pm$, and we look at the solutions to two ODEs, namely,

$$\begin{gathered} Z'' = Z - LM(h_\pm(Z')), \\ Z(\pm\infty) = L\,M_\pm, \quad Z'(\pm\infty) = 0, \end{gathered} \tag{C.3}$$

in their corresponding intervals of existence. Observe that the derivatives of the functions $h_\pm$ are given by $h'_\pm = 1/f'(h_\pm(\cdot))$, with $f'(u) \neq 0$ in $[u_+, u_*) \cup (u_*, u_-]$. Note that $h_+ : [-m, 0] \to [u_+, u_*]$ and $h_- : [-m, 0] \to [u_*, u_-]$, and that h_+ (h_-) is monotonically decreasing (increasing) on its domain of definition.

Following [22] closely, we shall exhibit the existence of a Z-profile solution to (C.2) between the states $LM_- > LM_+$, for which the velocity profile follows by $U = H(Z - Z'')$ (see (C.1)). In what follows we indicate only the differences with the proofs in section 2 of [22] and pay particular attention to the monotonicity properties of Z, which lead to the monotonicity of U in Lemma C.4.

The following proposition is an extension of Propositions 2.2 and 2.3 in [22] to the variable G' case.

PROPOSITION C.1 (see [22]). (i) *Denote by $Z_+ = Z_+(x)$ the (unique up to translations) maximal solution to*

$$Z'' = Z - LM(h_+(Z')),$$

with conditions $Z(+\infty) = LM_+$ and $Z'(+\infty) = 0$. Then Z_+ is monotone decreasing, Z'_+ is monotone increasing, and Z_+ is not globally defined; that is, there exists a point that we can take without loss of generality as $x = 0$ (because of translation invariance) such that

$$Z_+(0) - Z''_+(0) = LM(u_*), \qquad Z'_+(0) = -m < 0.$$

(ii) *Denote by $Z_- = Z_-(x)$ the (unique up to translations) maximal solution to*

$$Z'' = Z - LM(h_-(Z')),$$

with conditions $Z(-\infty) = LM_-$ and $Z'(-\infty) = 0$. Then Z_- and Z'_- are monotone decreasing, and Z_- is not globally defined; that is, there exists a point that we can take without loss of generality as $x = 0$ (because of translation invariance) such that

$$Z_-(0) - Z''_-(0) = LM(u_*), \qquad Z'_-(0) = -m < 0.$$

Proof. We focus on part (i) of the proposition. The second part is analogous. Rewrite the equation for Z_+ as $X' = J_+(X)$ with $X = (Z, Z')^\top$ and

$$J_+(X) = \begin{pmatrix} Z' \\ Z - LM(h_+(Z')) \end{pmatrix},$$

for which

$$\nabla J_{+|(LM_+,0)} = \begin{pmatrix} 0 & 1 \\ 1 & \dfrac{-L}{f'(u_+)}\dfrac{dM}{du}(u_+) \end{pmatrix},$$

in view of $h_+(0) = u_+$, and therefore, the starting point $(LM_+, 0)$ of the trajectory is a saddle point. We focus on the stable manifold, as we need Z to be decreasing.

Follow the trajectory that exits from $(LM_+, 0)$ in the lower half plane of the phase field (Z, Z'). We claim that Z is strictly monotone decreasing and Z' is strictly monotone increasing. Suppose, by contradiction, that Z attains a local maximum at $x_0 \in \mathbb{R}$. Then $Z'(x_0) = 0$ and $0 \geq Z''(x_0) = (Z - LM(h_+(Z'))_{x=x_0} = Z(x_0) - LM_+$, which is false. Hence, Z is monotone decreasing and $Z' < 0$. Now, assume that Z' attains a local minimum at $x = x_0$. Then the trajectory $Z' = \varphi(Z)$ in the phase plane must attain a local minimum at the same point, yielding $\varphi'(Z) = 0$ and $\varphi''(Z) \geq 0$. Thus, at $x = x_0$,

$$0 = \varphi'(Z) = Z''/Z' = (Z - LM(h_+(Z')))/Z'$$

and

$$\begin{aligned}\varphi''(Z) &= (d/dZ)((Z - LM(h_+(Z')))/Z') \\ &= 1/Z' - (dZ'/dZ)\left((Z' + LM'(h_+(Z'))h'_+(Z')Z' - LM(h_+(Z')))/(Z')^2\right).\end{aligned}$$

But $(dZ'/dZ) = \varphi'(Z) = 0$ at $x = x_0$, and thus $\varphi''(Z) = 1/Z' < 0$, which is a contradiction. This shows that Z' is strictly monotone increasing with $Z'' > 0$, and clearly $LM(h_+(Z')) \in [LM_+, LM(u_*)]$, $h_+(Z') \in [u_+, u_*]$. This shows that $Z'' = Z + \mathcal{O}(1)$ and the solution does not blow up in finite time.

By following the proof of Proposition 2.2 in [22] word by word from this point on, it is possible to show that the solution reaches the boundary of definition of the differential equation at a finite point which, by translation invariance, we can take as $x = 0$. Hence, $Z'_+(0) - Z''_+(0) = LM(u_*)$ and $Z'_+(0) = -m < 0$ hold. This concludes the proof. □

LEMMA C.2. *For the maximal solutions $Z_\pm$ of Proposition* C.1*, there holds*

$$Z_-(0) \leq LM(u_*) \leq Z_+(0).$$

Proof. This follows by mimicking the proof of Lemma 2.4 in [22]. We warn the reader to now consider the dynamical system

$$\begin{aligned} y' &= F(H(y)), \\ y(\pm\infty) &= LM_\pm. \end{aligned}$$

A comparison of the solution y of the system above with the trajectory Z_+ in the phase space yields the inequality on the right. The other inequality is analogous. See [22] for details. □

The last lemma guarantees the existence of a point of intersection for the orbits of the maximal solutions Z_+ and Z_- in the phase state field. The monotonicity of $Z_\pm$ and $Z'_\pm$ implies that the intersection is unique. Matching the two trajectories at that point provides the desired Z-profile. Hence, we have the following extension of the existence result in [22] (Theorem 2.5).

THEOREM C.3 (see [22]). *Under assumptions* (A0)–(A4)*, there exists a (unique up to translations) Z-profile of class C^1 with $Z(\pm\infty) = LM_\pm$ solution to* (C.2)*. The solution Z is of class C^2 away from a single point, where Z'' has at most a jump discontinuity. Moreover, there exists a (unique up to translations) velocity profile U with $U(\pm\infty) = u_\pm$ solution to* (C.1)*, which is continuous away from a single point, where it has at most a jump discontinuity satisfying the Rankine–Hugoniot condition and the entropy condition.*

Proof. Lemma C.2 implies the existence of a point in the (Z, Z') plane where the graphs of Z_- and Z_+ intersect. By monotonicity of the graphs the intersection is unique. Thus, after an appropriate translation, we can find a point $\bar{x} \in \mathbb{R}$ such that $(Z_-(\bar{x}), Z_-'(\bar{x})) = (Z_+(\bar{x}), Z_+'(\bar{x})) =: (\hat{Z}, \hat{Y})$, and the Z-profile is defined as

$$Z(x) := \begin{cases} Z_+(x), & x \geq \bar{x}, \\ Z_-(x), & x \leq \bar{x}. \end{cases}$$

Z is C^1 and satisfies $Z \to Lb_\pm$ as $x \to \pm\infty$. Moreover, Z is C^2 except at $x = \bar{x}$. The velocity profile is now defined via

$$U := H(Z - Z''),$$

with the described regularity properties due to regularity of Z and the fact that $H = (LM)^{-1}$ is of class at least C^2. Likewise, at the only possible discontinuity $x = \bar{x}$ of U it is possible to prove that U satisfies the Rankine–Hugoniot condition, $U(\bar{x} - 0) = U(\bar{x} + 0)$, and the entropy condition $U(\bar{x} - 0) = h_-(\hat{Y}) > h_+(\hat{Y}) = U(\bar{x} + 0)$. ∎

LEMMA C.4 (monotonicity). *The constructed profile U is strictly monotone decreasing.*

Proof. Let $x_2 > x_1$, with $x_i \neq \bar{x}$, and suppose that $U(x_2) \geq U(x_1)$, that is, $H(Z - Z'')_{|x=x_2} \geq H(Z - Z'')_{|x=x_1}$. Since H is strictly monotone increasing we readily have that

$$LM(h_\pm(Z_\pm'(x_2))) = (Z - Z'')_{|x=x_2} \geq (Z - Z'')_{x=x_1} = LM(h_\pm(Z_\pm'(x_1))),$$

where the $\pm$ sign depends on which side of $x = \bar{x}$ we are evaluating the Z-profile. Suppose x_1, x_2 are on the same side, say, $\bar{x} < x_1 < x_2$ (the symmetric case, $x_1 < x_2 < \bar{x}$, is analogous). Since LM is monotonically increasing, the last condition implies that $h_+(Z_+'(x_2)) \geq h_+(Z_+'(x_1))$. But this contradicts the fact that Z_+' is monotone increasing and h_+ is strictly decreasing, yielding $h_+(Z_+'(x_2)) < h_+(Z_+'(x_1))$. The case $x_1 < \bar{x} < x_2$ leads to the condition $h_+(Z_+'(x_2)) \geq h_-(Z_-'(x_1))$, which is obviously false in view of the fact that $h_+ : [-m, 0] \to [u_+, u_*]$ and $h_- : [-m, 0] \to [u_*, u_-]$, yielding again a contradiction. Finally, we remark that at the only point of discontinuity of U, namely, at $x = \bar{x}$, the jump is entropic, satisfying $U(\bar{x} - 0) > U(\bar{x} + 0)$. Therefore U is strictly monotone decreasing in all $x \in \mathbb{R}$. ∎

Remark C.5. Observe that the constructed velocity profile is continuous, except, at most, at one point where it observes an entropic jump. The regularity of U increases as long as the strength of the profile decreases below an explicit threshold [16, 22], becoming continuous and, moreover, of class C^2. We remark, however, that away from the possible discontinuity $x = \bar{x}$, the profile has the same regularity of Z'', independently of the shock strength, because of the smoothness of H. From regularity assumption (A0) and by differentiating equation (C.3), Z'' is of class C^2 away from $x = \bar{x}$, and so is U. Finally, thanks to translation invariance, we have chosen $x = 0$ to be the point where the equations for the profiles $Z_\pm$ reach $LM(u_*)$, where u_* is the only zero of $\frac{df}{du}(u)$; this implies that $U(0) = H(Z - Z'')_{x=0} = H(LM(h_\pm(-m))) = u_*$, so that $a(x) = \frac{df}{du}(U)$ vanishes only at $x = 0$.

In view of our last remark we have the following.

COROLLARY C.6. *Except for a possible single point $x = \bar{x}$, the profile U is of class C^2 and satisfies $U' < 0$ a.e. Moreover, the function $a(x) := \frac{df}{du}(U)$ is of class C^1, except at a point $x = \bar{x}$, and vanishes only at $x = 0$ (by translation invariance).*

Finally, we state the regularity properties for the convex flux, based on the analysis in [22, section 3]. The proof is, once again, an adaptation of the general G case of the proof of Proposition 3.3 in [22], which we omit.

COROLLARY C.7. *Under convexity of the velocity flux,* $\frac{d^2 f}{du^2} > 0$*, if the shock amplitude* $|u_+ - u_-|$ *is sufficiently small, then the profile is of class* C^2 *and* $U'(x) < 0$ *for all* $x \in \mathbb{R}$.

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