Modulational and spectral (in)stability of periodic traveling wave solutions to the nonlinear Klein-Gordon equation

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1 Introduction

2 Analysis of the monodromy map

3 Modulational instability index

4 Spectral (in)stability results
The nonlinear Klein-Gordon equation

Nonlinear Klein-Gordon with periodic potential:

\[ u_{tt} - u_{xx} + V'(u) = 0. \]  \hspace{1cm} (nKG)

for \((x, t) \in \mathbb{R} \times [0, +\infty), u\) scalar, \(V \in C^2\), periodic.

Sine-Gordon equation:

\[ u_{tt} - u_{xx} + \sin u = 0. \]  \hspace{1cm} (SG)

\[ V(u) = 1 - \cos u. \]
The nonlinear Klein-Gordon equation

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for \((x, t) \in \mathbb{R} \times [0, +\infty), u \text{ scalar}, \ V \in C^2, \text{ periodic}.\)

Sine-Gordon equation:

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\[ V(u) = 1 - \cos u. \]
Applications (sine-Gordon):

- Surfaces with negative Gaussian curvature (Eisenhart, 1909)
- Propagation of crystal dislocations (Frenkel and Kontorova, 1939)
- Elementary particles (Perring and Skyrme, 1962)
- Propagation of magnetic flux on a Josephson line (Scott, 1969)
- Dynamics of fermions in the Thirring model (Coleman, 1975)
- Oscillations of a rigid pendulum attached to a stretched rubber band (Drazin, 1983)
Assumptions on the potential:

(a) $V : \mathbb{R} \rightarrow \mathbb{R}$ is of class $C^2$ in all its domain and it is periodic with fundamental period $P$.

(b) $V$ has only non-degenerate critical points.

(c) $V'(u)^4 \left( V(u)/V'(u)^2 \right)'' \geq 0$ for all $u$ under consideration.

Assumption (c) implies monotonicity of the period map with respect to the energy.
Traveling waves

\[ u(x, t) = f(x - ct), \quad z = x - ct, \text{ solution to the nonlinear pendulum equation:} \]

\[ (c^2 - 1)f_{zz} + V'(f(z)) = 0, \]

Sine-Gordon case:

\[ (c^2 - 1)f_{zz} + \sin(f(z)) = 0, \]

\[ c \in \mathbb{R} \text{ (wave speed), } c^2 \neq 1. \]
Upon integration:

\[ \frac{1}{2} (c^2 - 1) f_z^2 = E - V(f), \]

\( E = \text{constant (energy)} \). Under assumptions:

\( 0 < E_0 = \max V(u) \)

Sine-Gordon case: \( V(u) = 1 - \cos u, \ E_0 = 2, \)

\[ \frac{1}{2} (c^2 - 1) f_z^2 = E - 1 + \cos f(z). \]
Classification

First dichotomy (wave speed):

- **Subluminal** waves: \( c^2 < 1 \)
- **Superluminal** waves: \( c^2 > 1 \)

Second dichotomy (energy \( E \)):

- **Librational** wavetrain: \( f(z + T) = f(z) \). Closed trajectory inside the separatrix in the phase portrait.
- **Rotational** wavetrain: \( f(z + T) = f(z) \pm P \). Open trajectory outside the separatrix in the phase plane. Sign \( f_z \) is fixed.
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Figure: Phase portrait: subluminal (left); superluminal (right).
**Superluminal librational:** \( c^2 > 1, \, 0 < E < E_0. \)

\( \mathcal{K}(E) = \{ u \in \mathbb{R} : (E - V(u))/(c^2 - 1) \geq 0 \} = \) disjoint union of intervals in \((0, P)\). In \((v_1, v_2)\), only one non-degenerate zero of \( V' \). Librational (closed) periodic orbit.

\[
f_z = \frac{\sqrt{2}}{\sqrt{c^2 - 1}} \sqrt{E - V(f)},
\]

where \( f \in (v_1, v_2) \subset \mathcal{K}(E) \).

\[
T = \sqrt{2} \sqrt{c^2 - 1} \int_{v_1}^{v_2} \frac{d\eta}{\sqrt{E - V(\eta)}}.
\]

Sine-Gordon: wave oscillates around \( f = 0 \), in \((v_1, v_2) = (-\arccos(-E + 1), \arccos(-E + 1))\)
Superluminal librational: \( c^2 > 1, 0 < E < E_0 \).

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\((v_1, v_2) = (-\text{Arc cos}(-E + 1), \text{Arc cos}(-E + 1))\)
**Subluminal librational:** $c^2 < 1, \ 0 < E < E_0$.

$\mathcal{K}(E) = \{u \in \mathbb{R} : (V(u) - E)/(1 - c^2) \geq 0\} = \text{disjoint union of intervals in } (0, P)$. In $(\nu_3, \nu_4)$, only one non-degenerate zero of $V'$. Librational (closed) periodic orbit.

$$f_z = \frac{\sqrt{2}}{\sqrt{1 - c^2}} \sqrt{V(f) - E},$$

where $f \in (\nu_3, \nu_4) \subset \mathcal{K}(E)$.

$$T = \sqrt{2} \sqrt{1 - c^2} \int_{\nu_3}^{\nu_4} \frac{d\eta}{\sqrt{V(\eta) - E}}.$$

Sine-Gordon: wave oscillates around $f = \pi$, in $(\nu_3, \nu_4) = (-\arccos(-E + 1), 2\pi - \arccos(-E + 1))$
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**Superluminal rotational:** $c^2 > 1$, $E > E_0$, $E - V(f) > 0$ and $\mathcal{K}(E) = \mathbb{R}$. Rotation, $f_z$ has fixed sign. Orbit outside the separatrix and $f(z + T) = f(z) \pm P$ for all $z$.

$$f_z^2 = \frac{2(E - V(f))}{c^2 - 1} > 0,$$

$$T = \frac{\sqrt{c^2 - 1}}{\sqrt{2}} \int_0^P \frac{d\eta}{\sqrt{E - V(\eta)}}$$

**Subluminal rotational:** $c^2 < 1$, $E < 0$, $V(f) - E \geq 0$ and $\mathcal{K}(E) = \mathbb{R}$ with $f_z$ has fixed sign. Orbit outside the separatrix and $f(z + T) = f(z) \pm P$ for all $z$.

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Subluminal rotational: 

\[ c^2 < 1, \quad E < 0, \quad V(f) - E \geq 0 \]

and \( \mathcal{K}(E) = \mathbb{R} \) with \( f_z \) has fixed sign. Orbit outside the separatrix and \( f(z + T) = f(z) \pm P \) for all \( z \).

\[ T = \frac{\sqrt{1 - c^2}}{\sqrt{2}} \int_0^P \frac{d\eta}{\sqrt{V(\eta) - E}} \]
\( \mathcal{R}_1 = \{ c^2 < 1, 0 < E < E_0 \} \), (subluminal librational),
\( \mathcal{R}_2 = \{ c^2 < 1, E < 0 \} \), (subluminal rotational),
\( \mathcal{R}_3 = \{ c^2 > 1, 0 < E < E_0 \} \), (superluminal librational),
\( \mathcal{R}_4 = \{ c^2 > 1, E > E_0 \} \), (superluminal rotational),

\[(E, c) \in \mathcal{R} = \bigcup_{j=1}^{4} \mathcal{R}_j\]
Figure: Sketch of the open set $\mathcal{R} \subset \mathbb{R}^2$. 
Spectral problem

Solution \( f(z) + u(z, t) \), with \( u = \) perturbation:

\[
u_{tt} - 2cu_{zt} + (c^2 - 1)u_{zz} + V'(u + f) - V'(f) = 0.\]

Linearized equation:

\[
u_{tt} - 2cu_{zt} + (c^2 - 1)u_{zz} + V''(f(z))u = 0.\]

\( u = w(z)e^{\lambda t}, \lambda \in \mathbb{C}, w \in X \) Banach:

\[
(c^2 - 1)w_{zz} - 2c\lambda w_z + (\lambda^2 + V''(f(z)))w = 0. \quad (P)
\]

Quadratic “pencil” in \( \lambda \).
Spectral problem

Solution $f(z) + u(z, t)$, with $u =$ perturbation:

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Slide 16/73
Floquet spectrum

\( \lambda \in \sigma_F \) is a Floquet eigenvalue if there exists a bounded solution \( w \) to (P).

We say the wave is *spectrally stable* if \( \sigma_F \subset \{ \text{Re} \, \lambda > 0 \} \). Otherwise it is *spectrally unstable*. 
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Previous results

- Forest, MacLaughlin (1982); Murakami (1986); Ercolani, Forest, McLaughlin (1990); Parkes (1991); etc. (abridged list).
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# Summary of stability results

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<thead>
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<tbody>
<tr>
<td>Subluminal rotational</td>
<td>stable</td>
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<td>stable</td>
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<td>Subluminal librational</td>
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Whitham (1965, 1974):
Modulation theory: well established (formal) physical method based on WKB expansions. Exact wave
\[ f = f(x - ct) = \tilde{f}(kx - \omega t). \]
Allowing dependence
\[ k = k(x, t), \omega = \omega(x, t), \]
under “slow modulations”, if the PDE system on \((k, \omega)\) is well-posed then the wave is “stable”.

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Allowing dependence \( k = k(x, t), \omega = \omega(x, t) \), under “slow modulations”, if the PDE system on \((k, \omega)\) is well-posed then the wave is “stable”.

Scott (1969):

\[ y = \exp\left( \frac{-c\lambda z}{c^2 - 1} \right), \]

\[ y_{zz} + \frac{V''(f(z))}{c^2 - 1} y = \left( \frac{\lambda}{c^2 - 1} \right)^2 y =: \nu y. \quad (H) \]

Hill’s equation with period \( T \). \( \nu \in \sigma_H \) (Floquet spectrum of (H)) if there is a bounded solution \( y \).

Scott assumed that the transformation is isospectral. This is not true. Actually:

**Lemma (JMMP1)**

If \( \lambda \in \sigma_H \cap \sigma_F \) then \( \lambda \in i\mathbb{R} \).
Scott (1969):

\[ y = \exp \left( -\frac{c\lambda z}{c^2 - 1} \right), \]

\[ y_{zz} + \frac{V''(f(z))}{c^2 - 1} y = \left( \frac{\lambda}{c^2 - 1} \right)^2 y =: vy. \quad (H) \]

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4 Spectral (in)stability results
Spectrum revisited. Evans function.

Problem (P) can be written as a first order system:

\[ W_z = \mathbb{A}(z, \lambda) W, \]

\[ W := \begin{pmatrix} w \\ w_z \end{pmatrix}, \]

\[ \mathbb{A}(z, \lambda) := \begin{pmatrix} 0 & 1 \\ -\frac{(\lambda^2 + V''(f(z)))}{c^2 - 1} & \frac{2c\lambda}{c^2 - 1} \end{pmatrix}. \]
Family of closed, densely defined operators:

$$\mathcal{T}(\lambda) : \mathcal{D} \subset X \rightarrow X$$

$$\mathcal{T}(\lambda)W := W_z - A(z, \lambda)W.$$

E.g.:

$$\mathcal{D} = H^1(\mathbb{R}; \mathbb{C}^2), \quad X = L^2(\mathbb{R}; \mathbb{C}^2),$$

Spectral stability of periodic waves with respect to localized perturbations.
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Spectral stability of periodic waves with respect to localized perturbations.
Definition (Sandstede (2002))

The *resolvent* $\zeta$, the *point spectrum* $\sigma_{pt}$ and the *essential spectrum* $\sigma_{ess}$ of problem (P) are defined as

$$\zeta := \{ \lambda \in \mathbb{C} : T(\lambda) \text{ is one-to-one and onto, and } T(\lambda)^{-1} \text{ is bounded}\},$$

$$\sigma_{pt} := \{ \lambda \in \mathbb{C} : T(\lambda) \text{ is Fredholm with zero index and has a non-trivial kernel}\},$$

$$\sigma_{ess} := \{ \lambda \in \mathbb{C} : T(\lambda) \text{ is either not Fredholm or has index different from zero}\}.$$

The *spectrum* is $\sigma = \sigma_{ess} \cup \sigma_{pt}$. ($T(\lambda)$ closed $\Rightarrow \zeta = \mathbb{C}\setminus\sigma.$)
Lemma

All spectrum of problem $\mathcal{P}$ is “continuous”, that is, $\sigma = \sigma_{\text{ess}}$ and $\sigma_{\text{pt}}$ is empty.

Monodromy matrix:

$$\mathbb{M}(\lambda) := \Phi(T, \lambda)$$

$\Phi(z, \lambda) =$ fundamental solution with $\Phi(0, \lambda) = I$.

$$\mathbb{M}(\lambda)\Phi(z, \lambda) = \Phi(z + T, \lambda)$$
Lemma

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M(\lambda)\Phi(z, \lambda) = \Phi(z + T, \lambda)
\]
Floquet multipliers:
\( \lambda \in \sigma \) if and only if there exists at least one \( \mu \in \mathbb{C} \) (Floquet multiplier) with \( |\mu| = 1 \) such that

\[
\hat{D}(\lambda, \mu) := \det(\mathbb{M}(\lambda) - \mu \mathbb{I}) = 0.
\]

\( \mu = \mu(\lambda) = e^{i\theta(\lambda)} \) are the eigenvalues of \( \mathbb{M}(\lambda) \). \( \theta = \theta(\lambda) \) are called the Floquet exponents.
Periodic Evans function (Gardner, 1997):

**Definition**

The *periodic Evans function* $D : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ is

$$D(\lambda, \kappa) := \hat{D}(\lambda, e^{i\kappa T}) = \det(M(\lambda) - e^{i\kappa T} I),$$

for each $(\lambda, \kappa) \in \mathbb{C} \times \mathbb{R}$. 
Properties: (Gardner 1997, 1998)

- \( \sigma \) is the set of all \( \lambda \in \mathbb{C} \) such that \( D(\lambda, \kappa) = 0 \) for some real \( \kappa \).
- \( D \) is analytic in \( \lambda \) and \( \kappa \).
- The order of the zero in \( \lambda \) is the multiplicity of the eigenvalue.
- \( \hat{D}(\lambda, 1) = D(\lambda, 0) \) detects spectra corresponding to perturbations which are \( T \)-periodic.
Floquet spectrum:
Boundary value problem of the form

\[(c^2 - 1)w_{zz} - 2c\lambda w_z + (\lambda^2 + V''(f(z)))w = 0,\]

\[
\begin{pmatrix}
  w(T) \\
  w_z(T)
\end{pmatrix}
= e^{i\theta}
\begin{pmatrix}
  w(0) \\
  w_z(0)
\end{pmatrix}, \quad \theta \in \mathbb{R}.
\]

For a given \(\theta \in \mathbb{R}\) we define \(\sigma_\theta \subset \mathbb{C}\) to be the set of complex \(\lambda\) for which there exists a nontrivial solution. The Floquet spectrum \(\sigma_F\) is defined then as the union over \(\theta\) of these sets:

\[
\sigma_F := \bigcup_{-P < \theta \leq P} \sigma_\theta.
\]

Clearly: \(\sigma = \sigma_F\).
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Clearly: \(\sigma = \sigma_F\).
Solutions at $\lambda = 0$

\[ f = f(z; E, c), \ (E, c) \in \mathcal{R}. \] Initial conditions:

\[ u_0(E, c) = f(0; E, c) \]

\[ = \begin{cases} 
  f(T; E, c), & E \in (0, E_0), \quad \text{(lib)}, \\
  f(T; E, c) - P, & E \in (-\infty, 0) \cup (E_0, +\infty), \quad \text{(rot)}, 
\end{cases} \]

\[ v_0(E, c) = f_z(0; E, c) = f_z(T; E, c) \]
System at $\lambda = 0$:

$$Y_z = A(z, 0) Y,$$

$$A(z, 0) = \begin{pmatrix} 0 & 1 \\ -V''(f(z))/(c^2 - 1) & 0 \end{pmatrix}.$$

**Lemma**

The two-dimensional vector space of solutions is spanned by

$$Y_0(z) = \begin{pmatrix} f_z \\ f_{zz} \end{pmatrix}, \quad \text{and} \quad Y_1(z) = \begin{pmatrix} f_E \\ f_{Ez} \end{pmatrix}.$$
System at $\lambda = 0$:

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\[ \det(Y_0(z), Y_1(z)) = f_z f_{Ez} - f_E f_{zz} = \frac{1}{c^2 - 1} \neq 0 \]

Solution matrix:

\[ Q(z, 0) := (Y_0(z), Y_1(z)) \]

\[ \Phi(z, 0) = Q(z, 0) Q(0, 0)^{-1}. \]

\[ M(0) = \Phi(T, 0) = Q(T, 0) Q(0, 0)^{-1} \]

\[ Q(z, 0)^{-1} = (c^2 - 1) \begin{pmatrix} f_{Ez} & -f_E \\ -f_{zz} & f_z \end{pmatrix}. \]
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Lemma

If $T_E \neq 0$, there exists a basis in $\mathbb{R}^2$ such that the monodromy map $\mathbb{M}(\lambda)$ at $\lambda = 0$ has the Jordan form:

$$\mathbb{M}(0) \sim \begin{pmatrix} 1 & -T_E \\ 0 & 1 \end{pmatrix}.$$ 

$Q(T,0) - Q(0,0)$ is a rank-one matrix provided that $T_E \neq 0$:

$$Q(T,0) = Q(0,0) + \begin{pmatrix} 0 & -T_E v_0(E,c) \\ 0 & -T_E \frac{V'(u_0(E,c))}{c^2-1} \end{pmatrix}.$$
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\( \mathbb{Q}(T, 0) - \mathbb{Q}(0, 0) \) is a rank-one matrix provided that \( T_E \neq 0 \):

\[
\mathbb{Q}(T, 0) = \mathbb{Q}(0, 0) + \begin{pmatrix} 0 & -T_E v_0(E, c) \\ 0 & -T_E \frac{v'(u_0(E, c))}{c^2 - 1} \end{pmatrix}
\]
Under Assumption (c), we have monotonicity of the period map (Chicone, 1987: criterion for planar Hamiltonian systems):

**Lemma**

Under assumptions there holds $T_E \neq 0$. More precisely we have:

(i) $T_E > 0$ in the rotational subluminal and librational superluminal cases.

(ii) $T_E < 0$ in the rotational superluminal and librational subluminal cases.
Lemma

If we define

\[ \bar{\Delta} := -\frac{T_E}{c^2 - 1} \]

then

(a) \( \bar{\Delta} > 0 \) for rotational waves.
(b) \( \bar{\Delta} < 0 \) for librational waves.
Solutions series expansions

\[ Q = Q(z, \lambda) \] solution to

\[ \frac{dQ}{dz} = A(z, \lambda)Q. \]

\[ Q(0, \lambda) = Q(0, 0) = (Y_0(0), Y_1(0)) \]

By analyticity, seek series expansion

\[ Q(z, \lambda) = \sum_{n=0}^{+\infty} \lambda^n Q_n(z) \]
Solutions series expansions

\[ \mathcal{Q} = \mathcal{Q}(z, \lambda) \] solution to

\[ \frac{d\mathcal{Q}}{dz} = \mathcal{A}(z, \lambda) \mathcal{Q}. \]

\[ \mathcal{Q}(0, \lambda) = \mathcal{Q}(0, 0) = (Y_0(0), Y_1(0)) \]

By analyticity, seek series expansion

\[ \mathcal{Q}(z, \lambda) = \sum_{n=0}^{+\infty} \lambda^n \mathcal{Q}_n(z) \]
Collecting like powers of $\lambda$ we obtain a hierarchy:

\[
(c^2 - 1) \frac{dQ_1}{dz} = A_0(z)Q_1 + A_1 Q_0
\]

\[
(c^2 - 1) \frac{dQ_n}{dz} = A_0(z)Q_n + A_1 Q_{n-1} + A_2 Q_{n-2}, \quad n = 2, 3, \ldots
\]

Solution by variation of parameters:

\[
Q_1(z) = \frac{Q_0(z)}{c^2 - 1} \int_0^z Q_0(y)^{-1} A_1 Q_0(y) \, dy
\]

\[
Q_n(z) = \frac{Q_0(z)}{c^2 - 1} \int_0^z Q_0(y)^{-1} (A_1 Q_{n-1}(y) + A_2 Q_{n-2}) \, dy, \quad n \geq 2
\]
Collecting like powers of $\lambda$ we obtain a hierarchy:

$$(c^2 - 1) \frac{dQ_1}{dz} = A_0(z)Q_1 + A_1 Q_0$$

$$(c^2 - 1) \frac{dQ_n}{dz} = A_0(z)Q_n + A_1 Q_{n-1} + A_2 Q_{n-2}, \quad n = 2, 3, \ldots$$

Solution by variation of parameters:

$$Q_1(z) = \frac{Q_0(z)}{c^2 - 1} \int_0^z Q_0(y)^{-1} A_1 Q_0(y) dy$$

$$Q_n(z) = \frac{Q_0(z)}{c^2 - 1} \int_0^z Q_0(y)^{-1} (A_1 Q_{n-1}(y) + A_2 Q_{n-2}) dy, \quad n \geq 2$$
By Abel’s identity:

**Lemma**

For all $z \in \mathbb{R}$, $\lambda \in \mathbb{C}$, there holds

$$\det Q(z, \lambda) = \frac{\exp\left(\frac{2c\lambda z}{c^2 - 1}\right)}{c^2 - 1}.$$
After (tedious) computations:

**Lemma**

\[ \text{tr } Q_0 (T) Q_0 (0)^{-1} = 2. \]

\[ \text{tr } Q_1 (T) Q_0 (0)^{-1} = \frac{2cT}{c^2 - 1}. \]

\[ \text{tr } Q_2 (T) Q_0 (0)^{-1} = \frac{c^2 T^2}{(c^2 - 1)^2} - \frac{T_E}{c^2 - 1} \int_0^T f_z (y)^2 \, dy. \]
Perturbation of the Jordan block

By analyticity of the monodromy map:

\[ M(\lambda) = \sum_{n=0}^{+\infty} \frac{\lambda^n}{n!} \frac{d^n M}{d\lambda^n}(0). \]

(Standard perturbation theory, Kato.) In general, the Floquet multipliers bifurcate from \( \lambda = 0 \) in Pusieux series.

Fundamental matrix:

\[ \Phi(z, \lambda) = Q(z, \lambda)Q_0(0)^{-1} = \sum_{n=0}^{+\infty} \lambda^n Q_n(z)Q_0^{-1} =: \sum_{n=0}^{+\infty} \lambda^n \Phi_n(z) \]
Perturbation of the Jordan block

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Lemma

We have convergent series expansions

\[ M(\lambda) = \sum_{n=0}^{+\infty} \lambda^n Q_n(T) Q_0(0)^{-1}, \]

\[ \text{tr} M(\lambda) = \sum_{n=0}^{+\infty} \lambda^n \text{tr} Q_n(T) Q_0(0)^{-1}, \]

and

\[ \det M(\lambda) = \sum_{n=0}^{+\infty} \left( \frac{2cT}{c^2 - 1} \right)^n \frac{\lambda^n}{n!}. \]
Introduction

Analysis of the monodromy map

Modulational instability index

Spectral (in)stability results
Expansion of the Floquet multipliers

$\mu$, solutions to:

$$\hat{D}(\lambda, \mu) = \det (M(\lambda) - \mu I) = \mu^2 - (\text{tr} M(\lambda))\mu + \det M(\lambda) = 0$$

$$\mu_{\pm}(\lambda) = \frac{1}{2} \left( \text{tr} M(\lambda) \pm \left( (\text{tr} M(\lambda))^2 - 4 \det M(\lambda) \right)^{1/2} \right)$$
Expanding:

\[
\begin{align*}
\text{tr} \mathbb{M}(\lambda)^2 - 4 \det \mathbb{M}(\lambda) &= \\
&= \left( \text{tr} Q_0(T)Q_0(0)^{-1} + \lambda \text{tr} Q_1(T)Q_0(0)^{-1} + \lambda^2 \text{tr} Q_2(T)Q_0(0)^{-1} \right)^2 + \\
&\quad - 4 \left( 1 + \frac{2cT}{c^2 - 1} \right) \lambda + \frac{2c^2 T^2}{(c^2 - 1)^2} \lambda^2 + O(\lambda^3) \\
&= 4 \Delta \lambda^2 + O(\lambda^3),
\end{align*}
\]

\[
\Delta := - \frac{T_E}{c^2 - 1} \int_0^T f_z(y)^2 \, dy
\]
The two Floquet multipliers are analytic functions of $\lambda$ at $\lambda = 0$. Asymptotic form:

$$
\mu_{\pm}(\lambda) = 1 + \left( \frac{cT}{c^2 - 1} \pm \Delta^{1/2} \right) \lambda + O(\lambda^2)
$$

Definition

We define the modulational instability index to be the quantity

$$
\rho := \text{sgn} \Delta.
$$

Clearly $\text{sgn} \Delta = \text{sgn} \bar{\Delta}$. 

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**Definition**

We define the *modulational instability index* to be the quantity

$$\rho := \text{sgn} \Delta.$$ 

Clearly $\text{sgn} \Delta = \text{sgn} \bar{\Delta}$. 
Expansion of $D$ near the origin

**Lemma**

The periodic Evans function $D(\lambda, \kappa)$, for $(\lambda, \kappa) \in \mathbb{C} \times \mathbb{R}$, has the following expansion in a neighborhood of $(\lambda, \kappa) = (0, 0)$,

$$D(\lambda, \kappa) = -\Delta \lambda^2 + \left( i\kappa - \frac{cT}{c^2 - 1} \lambda \right)^2 + O(3),$$

where $O(3)$ denotes terms of order three or higher in $(\lambda, k)$.
Lemma

If $\rho = 1$ then the solutions to $D(\lambda, \kappa) = 0$ near $(\lambda, \kappa) = (0, 0)$ emerge from the origin tangentially to the imaginary axis in the complex $\lambda$-plane:

$$\lambda(\kappa) = -iv\kappa + O(\kappa^2),$$

with $v \in \mathbb{R}$, for $|\kappa| \ll 1$.

If $\rho = -1$ then the solutions emerge from the origin tangentially to two lines passing through the origin and forming non-zero angles with the imaginary axis:

$$\lambda(\kappa) = -(\alpha + i\beta)\kappa + O(\kappa^2),$$

with $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, for $|\kappa| \ll 1$. 
Figure: Qualitative sketch of $\sigma$ near the origin. $\rho = 1$ (left); $\rho = -1$ (right).
Theorem

Under assumptions (a), (b) and (c):

- $\rho = -1$ for librational waves. Spectrally unstable.
- $\rho = 1$ for rotational waves. The spectrum is tangent to the imaginary axis at $\lambda = 0$.

Theorem

Under the non-degeneracy condition $T_E \neq 0$ if the modulational instability index is $\rho = -1$ then the underlying periodic traveling wave is spectrally unstable.
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Theorem

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Relation to Whitham’s modulation theory


WKB approximations of the form:

\[ u(x,t) = f\left(\frac{z(x,t)}{\xi}\right) + O(\varepsilon), \]

\(k, \omega\) are no longer constant (and hence, \(E\) and \(c\)). We have \(c = \omega/k\) and \(k = \theta_x, \omega = -\theta_t, \theta = kx - \omega t\). Conservation of fluxons:

\[ k_t + \omega_x = 0 \]
Averaged Lagrangian

\[ I[u] = \int \int L(u, u_x, u_t) \, dx \, dt, \]

\[ L(u, u_x, u_t) = \frac{1}{2} u_t^2 - \frac{1}{2} u_x^2 - V(u). \]

In the wave \( u = f(x - ct) = \Phi(kx - \omega t) \):

\[ L(u, u_x, u_t) = \frac{1}{2} (\omega^2 - k^2) \Phi_\theta(\theta)^2 - V(\Phi(\theta)) \]

Averaged Lagrangian:

\[ \langle L \rangle = \frac{1}{kT} \int_0^{kT} \frac{1}{2} (\omega^2 - k^2) \Phi_\theta(\theta)^2 - V(\Phi(\theta)) \, d\theta = \tilde{L}(\omega, k, E). \]
Averaged Lagrangian variational principle

\[ \delta \int \int \tilde{L}(\omega, k, E) \, dx \, dt = 0, \]

\[ \tilde{L}_E = 0, \text{ dispersion relation} \]

\[ k_t + \omega_x = 0 \]

\[ (\tilde{L}_\omega)_t - (\tilde{L}_k)_x = 0. \] (*

If the last system (*) is hyperbolic (Cauchy problem well-posed) then the wave is \textit{stable under slow modulations} (Whitham, 1974).
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Equivalently (Whitha, 1965) we may express (*) in terms of $E$ and $c$. Averaged Lagrangian:

$$\langle L \rangle = \frac{1}{T} \int_0^T \frac{1}{2} (c^2 - 1) f_z(z)^2 - V(f(z)) \, dz$$

$$= \frac{\sqrt{2}}{T} \int ((c^2 - 1)(E - V(\eta)))^{1/2} \, d\eta - E =: \mathcal{L}(E, c).$$
\[ \mathcal{L}(E, c) = \frac{2\sqrt{2}}{T} \sqrt{c^2 - 1} \int_{v_1}^{v_2} \sqrt{E - V(\eta)} \, d\eta - E, \quad \text{(sup, lib)}, \]

\[ \mathcal{L}(E, c) = -\frac{2\sqrt{2}}{T} \sqrt{1 - c^2} \int_{v_3}^{v_4} \sqrt{V(\eta) - E} \, d\eta - E, \quad \text{(sub, lib)}, \]

\[ \mathcal{L}(E, c) = \frac{\sqrt{2}}{T} \sqrt{c^2 - 1} \int_{0}^{P} \sqrt{E - V(\eta)} \, d\eta - E, \quad \text{(sup, rot)}, \]

\[ \mathcal{L}(E, c) = -\frac{\sqrt{2}}{T} \sqrt{1 - c^2} \int_{0}^{P} \sqrt{V(\eta) - E} \, d\eta - E, \quad \text{(sub, rot)}. \]
Define:

\[ W(E, c) = \sqrt{2} \int ((c^2 - 1)(E - V(\eta)))^{1/2} d\eta, \]

\[ W(E, c) := \text{sgn}(c^2 - 1)\sqrt{|c^2 - 1|J(E)}, \]

\[ J(E) := \begin{cases} J_L(E), & \text{librations,} \\ J_R(E), & \text{rotations,} \end{cases} \]

\[ J_R(E) := \sqrt{2} \int_0^P \sqrt{\text{sgn}(c^2 - 1)(E - V(\eta))} d\eta \]

\[ J_L(E) := 2\sqrt{2} \int_{v_i}^{v_f} \sqrt{\text{sgn}(c^2 - 1)(E - V(\eta))} d\eta \]
Lemma

For each of the four cases under consideration (sub- or superluminal, libration or rotation) there hold

\[ W_E = T, \]  
\[ W_c = \frac{cW}{c^2 - 1}. \]
Taking average of conservation of energy and momentum equations we can express the Whitham modulation system (*) as:

\[
\left( \frac{W_c}{T} \right)_t + \left( \frac{cW_c}{T} - E \right)_x = 0, \tag{**}
\]

\[
\left( \frac{1}{T} \right)_t + \left( \frac{c}{T} \right)_x = 0.
\]
Lemma

*Whitham's system of equations (**) is equivalent to the system:*

\[
\begin{pmatrix} E \\ c \end{pmatrix}_t + A(E, c) \begin{pmatrix} E \\ c \end{pmatrix}_x = 0, \quad \text{(Wh)}
\]

\[
A(E, c) = \frac{1}{N(E, c)} \begin{pmatrix} c(J(E)J''(E) + J'(E)^2) & -J(E)J'(E) \\ (c^2-1)^2J'(E)J''(E) & c(J(E)J''(E) + J'(E)^2) \end{pmatrix},
\]

\[
N(E, c) = J(E)J''(E) + c^2J'(E)^2.
\]
Lemma

*Whitham system* (Wh) is hyperbolic if and only if

\[ J''(E) < 0. \]

Characteristic velocities:

\[
c(J(E)J''(E) + J'(E)^2) - s_{\pm} = \pm |c^2 - 1| \left( -J(E)J''(E)J'(E)^2 \right)^{1/2}.
\]
Proof of Whitham’s modulational instability

Lemma

\[ \text{sgn} J''(E) = -\rho. \]

Proof:

\[ T_E = W_{EE} = \text{sgn} (c^2 - 1) \sqrt{|c^2 - 1| J''(E)}. \]
**Corollary**

*The quasilinear Whitham system (Wh) is hyperbolic if and only if* $\rho = 1$. *In this case we say that the underlying periodic traveling wave is modulationally stable (otherwise we say it is modulationally unstable).*

**Theorem (Proof of Whitham’s instability)**

*Under the non-degenerate condition* $T_E \neq 0$, *if the periodic traveling wave is modulationally unstable in the sense defined by Whitham then it is spectrally unstable.*
Corollary

The quasilinear Whitham system (Wh) is hyperbolic if and only if $\rho = 1$. In this case we say that the underlying periodic traveling wave is modulationally stable (otherwise we say it is modulationally unstable).

Theorem (Proof of Whitham’s instability)

Under the non-degenerate condition $T_E \neq 0$, if the periodic traveling wave is modulationally unstable in the sense defined by Whitham then it is spectrally unstable.
Corollary

Under the non-degenerate condition $T_E \neq 0$, a necessary condition for the spectral stability of a periodic wave is that the modulational instability index is $\rho = 1$, or equivalently, that the Whitham modulation system is hyperbolic.

Finally we recover:

Theorem (Whitham, 1974)

- Both super- and subluminal rotational waves are modulationally stable,
- Both super- and subluminal librational waves are modulationally unstable (and whence, spectrally unstable).
Corollary

Under the non-degenerate condition $T_E \neq 0$, a necessary condition for the spectral stability of a periodic wave is that the modulational instability index is $\rho = 1$, or equivalently, that the Whitham modulation system is hyperbolic.

Finally we recover:

Theorem (Whitham, 1974)

- Both super- and subluminal rotational waves are modulationally stable,
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1 Introduction

2 Analysis of the monodromy map

3 Modulational instability index

4 Spectral (in)stability results
(In)stability in the rotational case

**Theorem**

*Under assumptions we have:*

(A) *Superluminal rotational waves are spectrally unstable.*

(B) *Subluminal rotational waves are spectrally stable.*

*That is: if \( \lambda \in \sigma \) then \( \lambda \) is purely imaginary.*
Part (A):  
Define $G : \mathbb{C} \to \mathbb{R}$ by 

$$G(\lambda) = \log |\mu_+(\lambda)| \log |\mu_-(\lambda)|.$$ 

$G$ continuous in $\mathbb{R}^2$ and $\lambda \in \sigma$ if and only if $G(\lambda) = 0$. Fact: if $\mu(\lambda) \in \sigma \mathbb{M}(\lambda)$ (Floquet mult. for (P) then 
$\eta(\lambda) = \exp(-\lambda cT/(c^2 - 1)) \in \sigma \mathbb{M}_H(\lambda)$ (Floquet mult. for (H)). By Abel’s identity: 

$$G(\lambda) = \left( \text{Re} \frac{c\lambda T}{c^2 - 1} \right)^2 - (\log |\eta_+(\lambda)|)^2$$ 

$$= \left( \text{Re} \frac{c\lambda T}{c^2 - 1} \right)^2 - (\log |\eta_-(\lambda)|)^2.$$


Slide 67/73
Thus, for $\lambda \in i\mathbb{R}$, $G \leq 0$. Moreover, $G(i\beta) = 0$ iff $i\beta \in \sigma \cap i\mathbb{R} = \sigma^H \cap i\mathbb{R}$. Thus,

**Corollary**

Suppose $\beta \in \mathbb{R}$ is such that \[
\left(\frac{i\beta}{c^2 - 1}\right)^2 \notin \sigma^H.
\] Then $G(i\beta) < 0$. 
Moreover, we can show:

**Lemma**

*For a superluminal rotational wave, \( G(\lambda) > 0 \) for \( \lambda \in \mathbb{R} \), \( \lambda \gg 1 \), and there is a \( i\beta_* \) in the spectral gap of \( \sigma_H \), that is, \( G(i\beta) < 0 \).*

By continuity, there must be an eigenvalue

\[
\lambda = \alpha_* t + i\beta_* (1 - t)
\]

for some \( t \in (0, 1) \), where \( G(\alpha_*) > 0 \), \( \alpha_* \) large and real, such that \( G(\lambda) = 0 \). Clearly, \( \text{Re} \lambda > 0 \).

This shows (A).
Moreover, we can show:

**Lemma**

*For a superluminal rotational wave, \( G(\lambda) > 0 \) for \( \lambda \in \mathbb{R} \), \( \lambda \gg 1 \), and there is a \( i\beta_* \) in the spectral gap of \( \sigma_H \), that is, \( G(i\beta) < 0 \).*

By continuity, there must be an eigenvalue \( \lambda = \alpha_* t + i\beta_*(1 - t) \) for some \( t \in (0, 1) \), where \( G(\alpha_*) > 0 \), \( \alpha_* \) large and real, such that \( G(\lambda) = 0 \). Clearly, \( \text{Re} \, \lambda > 0 \). This shows (A).
Figure: Numerical plots of the Floquet spectrum $G(\lambda) = 0$ for sine-Gordon.
Part (B): Spectral stability of subluminal rotations.

By energy estimates: define the Hamiltonian operator $H = d^2/dz^2 + V''(f)/(c^2 - 1)$ so that the spectral equation (P) is:

$$(c^2 - 1)Hw(z) - 2c\lambda w_z(z) + \lambda^2 w(z) = 0$$

**Lemma**

*The operator $H$ is negative semidefinite in the case of a rotational wave. For librations, $H$ is indefinite.*
If $\lambda \in \sigma$, multiply eq. by $w^*$ and integrate by parts on a fundamental period $[0, T]$:

$$(c^2 - 1)\langle w, Hw \rangle - 2im\lambda + \|w\|^2\lambda^2 = 0,$$

$$m := -ic \int_0^T w(z)^*w_z(z)\,dz \in \mathbb{R}$$

$m \in \mathbb{R}$ using the periodicity of $w$ and integrating by parts. The roots of the quadratic are:

$$\lambda = \frac{1}{\|w\|^2} \left[ im \pm \sqrt{-m^2 - (c^2 - 1)\|w\|^2 \langle w, Hw \rangle} \right].$$

$\lambda \in i\mathbb{R}$ whenever $c^2 < 1$. This shows (B).
If $\lambda \in \sigma$, multiply eq. by $w^*$ and integrate by parts on a fundamental period $[0, T]$:

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$\lambda \in i\mathbb{R}$ whenever $c^2 < 1$. This shows (B).
Thank you!