Normal modes analysis and dynamic stability of subsonic phase boundaries in elastic materials

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Introduction

Equations (non-thermal elasticity, no external forces):

$$U_t - \nabla_x V = 0,$$
$$V_t - \operatorname{div}_x \sigma(U) = 0,$$

with

$$(x,t) \in \mathbb{R}^d \times [0,+\infty),\ d \geq 2$$

$U \in \mathbb{R}_+^{d\times d}$ – local deformation gradient,

$V \in \mathbb{R}^d$ – local velocity,

$\sigma(U)$ – (First) Piola-Kirchhoff stress tensor,

$$\sigma(U) = \frac{\partial W}{\partial U},$$

where

$$W : \mathbb{R}^{d\times d} \to \mathbb{R}$$

is an *energy density function* (hyperelastic material).

Physical constraint:

$$\text{curl}_x\, U = 0.$$

Acoustic tensor:

$$\mathcal{N}(\xi, U) = D^2 W(U)(\xi, \xi).$$

W is rank-one convex at U iff $\eta^\top \mathcal{N}(\xi, U)\eta > 0$, for all $\eta, \xi \in \mathbb{R}^d$. (Legendre-Hadamard condition).

W rank-one convex in $U \implies$ system is hyperbolic at U.

Subsonic phase-boundaries:

$$(U,V)(x,t) = \begin{cases} (U^-,V^-), & x\cdot N < st, \\ (U^+,V^+), & x\cdot N > st \end{cases}$$

$N \in S^{d-1}$, and s = shock speed, subsonic:

$$s^2 < \min\{\kappa_j(N,U^\pm) \text{ eigenvalues of } \mathcal{N}(N,U^\pm)\}.$$

Rankine-Hugoniot jump conditions:

$$-s[U] - [V]\otimes N = 0,$$
$$-s[V] - [\sigma(U)]N = 0$$

We require an additional *kinetic rule* of form

$$g((U^-,V^-),(U^+,V^+),s,N) = 0,$$

where $g : \Omega = (\mathbb{R}_+^{d\times d}\times\mathbb{R}^d)\times(\mathbb{R}_+^{d\times d}\times\mathbb{R}^d)\times\mathbb{R}\times S^{d-1}\to\mathbb{R}$. Summarize RH conditions + kinetic rule as:

$$h((U^-,V^-),(U^+,V^+),s,N) = 0$$

Static configuration:

$$(U^*, V^*)(x,t) = \begin{cases} (U^A, 0), & x \cdot N^* < 0, \\ (U^B, 0), & x \cdot N^* > 0 \end{cases}$$

$U^A \neq U^B$ local minima of W. E.g. double well potential, martensitic configuration below critical temperature.

Motion: $X : \Omega \times [0, +\infty) \to \mathbb{R}^d$ (Ω = reference configuration), so that $U = \nabla_x X, V = X_t$ for all $(x,t) \in \Omega \times [0, +\infty)$.

Continuity of tangential derivatives of $X \Rightarrow U^B = U^A + v \otimes BN^*$ for some $v \in \mathbb{R}^d$ (i. e. U^A and U^B are *rank-one connected*).

Hypotheses:

(H1) W is rank-one convex at U (local hyperbolicity).

(H2) $\forall \tilde{U} \sim U$, $\forall \xi \in \mathbb{R}^d$, $\xi \neq 0$, the eigenvalues of $\mathcal{N}(\xi, \tilde{U})$ are all semi-simple and their multiplicity is independent of $\tilde{U}$ and ξ (symmetrizability constant multiplicity –Métivier (2000)–).

(H3) $h((U^-, V^-), (U^+, V^+), s, N) = 0$, (RH jump + kinetic conditions)

(H4) The $(d^2 + d + 1) \times 2(d^2 + d)$ matrix

$$\left(d_{(U^+,V^+)}h\,,\ \ d_{(U^-,V^-)}h\right)_{|((U^-,V^-),(U^+,V^+),s,N)}$$

has full rank (non-degeneracy condition – Coulombel (2003)–).

Assumptions on the equilibrium configuration:

(E1) $\exists U^A \neq U^B$ in $\mathbb{R}_+^{d\times d}$, local minima of W, rank-one connected. W is rank-one convex both at U^A and U^B.

(E2) Symmetrizability with constant multiplicity holds at both U^A and U^B. Moreover, $h = 0$ and the non-degeneracy condition of h hold at $((U^A, 0), (U^B, 0), 0, N^*)$.

Theorem 1 *For every* $U \in \mathbb{R}_+^{d\times d}$ *satisfying hyperbolicity and any subsonic pair* $(s, N) \in \mathbb{R} \times S^{d-1}$, *there exist continuous mappings (analytic for* $\operatorname{Re}\lambda > 0$*)*

$$\widehat{R}^s_{s,N}(U) : \Gamma_N \to \mathbb{C}^{2d\times d}, \quad \widehat{R}^u_{s,N}(U) : \Gamma_N \to \mathbb{C}^{2d\times d},$$

$$\mathbb{M}_{s,N}(U) : \Gamma_N \to \mathbb{C}^{2d\times 2d}, \quad \mathcal{K}_{s,N}(U) : \Gamma_N \to \mathbb{C}^{(d^2+d)\times 2d},$$

on $\Gamma_N := \{(\lambda, \xi) \in \mathbb{C} \times \mathbb{R}^{d-1} : \operatorname{Re}\lambda \geq 0,\ \xi \cdot N = 0,\ |\lambda|^2 + |\xi|^2 = 1\}$ *so that the following holds: (i) For any subsonic phase boundary satisfying previous hypotheses for* $U = U^-$ *and* $U = U^+$, *the stability behaviour is controlled by the* **Lopatinski function**

$$\hat{\Delta}(U^-, U^+) = \det \begin{pmatrix} \widehat{R}^s_{s,N}(U^-) & \widehat{Q}(U^-, U^+) & \widehat{R}^u_{s,N}(U^+)) \\ \widehat{p}^-(U^-, U^+) & \widehat{q}(U^-, U^+) & \widehat{p}^+(U^-, U^+) \end{pmatrix} : \Gamma_N \to \mathbb{C},$$

$$\widehat{Q}(U^-, U^+)(\lambda, \xi) := \begin{pmatrix} [U]N \\ -(\lambda s[U]N + i[\sigma(U)]\xi) \end{pmatrix},$$

$$\widehat{q}(U^-, U^+)(\lambda, \xi) := -\lambda(d_s g) + i(\xi \cdot d_N)g,$$

$$\widehat{p}^-(U^-, U^+)(\lambda, \xi) := -(d_{(U^-,V^-)}g)\mathcal{K}_{s,N}(U^-)\widehat{R}^s_{s,N}(U^-),$$

$$\widehat{p}^+(U^-, U^+)(\lambda, \xi) := (d_{(U^+,V^+)}g)\mathcal{K}_{s,N}(U^+)\widehat{R}^u_{s,N}(U^+)\ .$$

More precisely: (i)$_1$ If $\widehat{\Delta}(U^-, U^+)$ has no zero on Γ_N, then the phase boundary is nonlinearly stable.
(i)$_2$ If $\widehat{\Delta}(U^-, U^+)$ vanishes for some $(\lambda, \xi) \in \Gamma_N$ with $\mathrm{Re}\,\lambda > 0$, then the phase boundary is strongly unstable.

(ii) $\mathbb{M}$ and $\mathcal{K}$ are given by simple explicit formulae in terms of first and second derivatives of W. $\widehat{R}^s$ and $\widehat{R}^u$ represent the left and right stable and unstable spaces of $\mathbb{M}$. In their whole domain of definition, given by

$$-\kappa_{min}(N,U) < s < \kappa_{min}(N,U),$$

$\mathbb{M}_{s,N}(U), \mathcal{K}_{s,N}(U), \widehat{R}^s_{s,N}(U), \widehat{R}^u_{s,N}(U)$, *depend continuously on* (U, s, N).

Corollary 1 *If W, U^A and U^B satisfy the hypotheses for the equilibrium configuration, then the dynamic stability of the static phase boundary is uniformly controlled by the* **static-case Lopatinski function**

$$\hat{\Delta}(U^A, U^B) : \Gamma_{N^*} \to \mathbb{C},$$

in the sense that if $\hat{\Delta}(U^A, U^B)$ has no zero on Γ_{N^*}, then any phase boundary with $h = 0$ and (U^-, U^+) sufficiently close to (U^A, U^B) is nonlinearly stable, while if $\hat{\Delta}(U^A, U^B)$ vanishes for some $(\lambda, \xi) \in \Gamma_{N^*}$ with $\mathrm{Re}\,\lambda > 0$, then any such phase boundary is strongly unstable.

Theorem 2 *Under the assumptions of Theorem 1, the left stable and the left unstable spaces of* $\mathbb{M}_{s,N}(U)$ *are represented by mappings*

$$\hat{L}^s_{s,N}(U) : \Gamma_N \to \mathbb{C}^{d\times 2d}, \qquad \hat{L}^u_{s,N}(U) : \Gamma_N \to \mathbb{C}^{d\times 2d},$$

with the same regularity properties as the $\hat{R}^s_{s,N}(U)$, $\hat{R}^u_{s,N}(U)$. *Moreover, we can define (lower-order)* $(d+1)\times(d+1)$ *determinants in terms of the "left" stable/unstable bundles* $L^{u,s}$, *namely* Δ^s, Δ^u, *such that they are equivalent to* $\hat{\Delta}(U^-, U^+)$,

$$\hat{\Delta}(U^-, U^+) \sim \hat{\Delta}^u(U^-, U^+) \sim \hat{\Delta}^s(U^-, U^+),$$

in the sense that the three differ from each other only by non-vanishing factors.

Motivation:

- Studies of moving phase boundaries for two-phase fluids by S. Benzoni-Gavage (Nonl. Anal. 1998, ARMA 1999, Phys. D. 2001), S. Benzoni-Gavage and H. Freistühler (ARMA 2004).

- Static theory of two-phase elastic media, e.g. Müller (LN Springer 1999), Ball, James.

- Nonlinear stability theory for planar shocks is available Majda (1983,1984) and Métivier (1990,1999) (classical shocks), Freistühler (1998), Coulombel (2003) (u.c. shocks).

Remark: Kinetic relation is prescribed a priori. Contribution provides a criterion for modeling dynamics of phase boundaries.

Part 1: Normal modes analysis

Hyperbolic system of conservation laws

$$u_t + \sum_{j=1}^{d} f_j(u)_{x_j} = 0,$$

$$A(\xi, u) := \sum_{j=1}^{d} \xi_j A_j(u), \quad A_j(u) := Df_j(u),$$

hyperbolic (diag. over $\mathbb{R}$), $a_1(u;\xi) \leq \cdots \leq a_n(u;\xi)$, fixed multiplicities $\alpha_1, \ldots, \alpha_n$.

Shock front:

$$u(x,t) = \begin{cases} u^+, & \text{if } x \cdot N > st, \\ u^-, & \text{if } x \cdot N < st, \end{cases}$$

Assume s non-characteristic. Counting "out-going modes"

$$a_j(N,u^-) < s < a_k(N,u^-) \quad \text{for all} \quad j \le o_-, \qquad k > o_-,$$
$$a_j(N,u^+) < s < a_k(N,u^+) \quad \text{for all} \quad j \le n - o_+, k > n - o_+,$$

for some $o_\pm$, we can define

$$l := o_+ + o_- + 1 - n = \begin{cases} 0 & \text{Lax shock,} \\ > 0 & \text{u.c. shock} \end{cases}$$

In the u.c. case we augment the RH conditions with l "kinetic conditions"

$$0 = h(u^+, u^-, s, N) := \begin{pmatrix} -s[u] + [f(u)]N \\ g(u^+, u^-, s, N) \end{pmatrix},$$

g is a $\mathbb{R}^l$ valued function of its parameters.

Majda-Métivier theory: the nonlinear stability of shock fronts is controlled by the Lopatinski conditions (Kreiss (1971), Sakamoto (1971)):

$$\text{Uniformly stable: } \Delta(\lambda,\xi) \neq 0, \text{ for all } (\lambda,\xi) \in \Gamma_N,$$
$$\text{Weakly stable: } \Delta(\lambda,\xi) \neq 0, \text{ for all } (\lambda,\xi) \in \Gamma_N \cap \{\mathrm{Re}\,\lambda > 0\},$$
$$\text{Strongly unstable: } \Delta(\lambda,\xi) = 0 \text{ for some } (\lambda,\xi) \in \Gamma_N \cap \{\mathrm{Re}\,\lambda > 0\}$$

where

$$\Gamma_N := \{(\lambda,\xi) \in \mathbb{C} \times \mathbb{R}^d \,:\, \mathrm{Re}\,\lambda \geq 0, \xi \cdot N = 0, |\lambda|^2 + |\xi|^2 = 1\},$$

$$\Delta = \det \begin{pmatrix} R_-^s & Q & R_+^u \\ -(d_u - g)(A_N^- - sI)^{-1} R_-^s & q & (d_u + g)(A_N^+ - sI)^{-1} R_+^u \end{pmatrix}$$

(Lopatinski determinant)

$$q = q(\lambda, \xi) = -\lambda(d_s g) + i(d_N g)\xi$$

$$Q = Q(\lambda, \xi) = \lambda[u] + i[f(u)]\xi$$

Columns of $R_{\pm}^{s,u}(\lambda, \xi)$ span the stable/unstable spaces of

$$(\lambda I + iA(\xi, u^{\pm}))(A(N, u^{\pm}) - sI)^{-1}.$$

The analysis is performed by a Fourier decomposition of the constant coefficients linearized problem, and the theory of hyperbolic initial boundary value problems (ala Kreiss).

Part 2: Hyperlasticity case and the space $\mathbb{G}$

Notation:

$$U_j = j\text{-th column of } U.$$

Stress tensor:

$$\sigma(U)_j = W_{U_j}$$

Second derivatives:

$$B_i^j(U) := \frac{\partial \sigma_j}{\partial U_i} = \begin{pmatrix} W_{U_{1j}U_{1i}} & \cdots & W_{U_{1j}U_{di}} \\ \vdots & & \vdots \\ W_{U_{dj}U_{1i}} & \cdots & W_{U_{dj}U_{di}} \end{pmatrix} \in \mathbb{R}^{d\times d}.$$

B_i^i is symmetric, $(B_j^i)^\top = B_i^j$.

Then,

$$f_j(U,V) := -\begin{pmatrix} 0 \\ \vdots \\ V \\ \vdots \\ 0 \\ \sigma(U)_j \end{pmatrix} \in \mathbb{R}^{d^2+d}, \quad j = 1,\ldots,d,$$

$$A_j(U) = df_j(U) = -\begin{pmatrix} & & & 0 \\ & & & \vdots \\ & 0 & & I \\ & & & \vdots \\ & & & 0 \\ B_1^j(U) & \cdots & B_d^j(U) & 0 \end{pmatrix} \in \mathbb{R}^{(d^2+d)\times(d^2+d)}$$

Notice the 0 mode.

Under hyperbolicty and symmetrizability with constant multiplicity, for any $N \in \mathbb{R}^d \setminus \{0\}$, the characteristic speeds of $A(N,U)$ are

1. $a_0(N,U) = 0$ with constant algebraic multiplicity $\alpha_0 = d^2 - d$, and

2. $a_j^{\pm}(N,U) = \pm\sqrt{\kappa_j(N,U)}$, $j = 1,\ldots,m$, where κ_j are the m distinct semi-simple eigenvalues of $\mathcal{N}$, $m \leq d$, with constant multiplicities α_j, and with $\sum \alpha_j = d$.

3. Assuming subsonicity, and denoting o_-, o_+, l as before, a phase boundary of speed $s > 0$ (resp. $s < 0$) has

$$o_- = d, o_+ = d^2, l = 1 \quad \text{(resp. } o_- = d^2, o_+ = d, l = 1\text{)}.$$

W.l.o.g assume $N = e_1$. Suppose W is hyperbolic at U, s subsonic with respect to (e_1, U).

Matrix field:

$$\mathcal{A}(U, s, \lambda, \tilde{\xi}) = C(s)^{-1}(\lambda I + i \sum_{j \neq 1} \xi_j A_j(U))(A_1(U) - sI)^{-1} C(s)$$

where

$$C(s) := \begin{pmatrix} I_d & 0 & 0 \\ 0 & s\, I_{d^2-d} & 0 \\ 0 & 0 & I_d \end{pmatrix},$$

Time-space frequencies:

$$\Gamma = \{(\lambda, \tilde{\xi}) \in \mathbb{C} \times \mathbb{R}^{d-1} : \mathbf{Re}\, \lambda \geq 0, |\lambda|^2 + |\tilde{\xi}|^2 = 1\}$$

Here $\xi = (0, \tilde{\xi}) \perp e_1$

Note

$$C(s)^{-1}(A_1 - sI) = \begin{pmatrix} -sI & 0 & \cdots & 0 & -I \\ 0 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & 0 \\ -B_1^1 & -B_2^1 & \cdots & -B_d^1 & -sI \end{pmatrix}$$

$$(A_1 - sI)^{-1}C(s) = \begin{pmatrix} -s\hat{B} & -\hat{B}B_2^1 & \cdots & -\hat{B}B_d^1 & \hat{B} \\ 0 & -I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -I & 0 \\ \hat{B}B_1^1 & s\hat{B}B_2^1 & \cdots & s\hat{B}B_d^1 & -s\hat{B} \end{pmatrix},$$

where

$$\hat{B}(s) := (s^2 - B_1^1)^{-1}$$

are analytic matrix-fields for all subsonic s, including $s = 0$.

Define on Γ the $2d$ dimensional bundle

$$\mathbb{G}(\lambda,\tilde{\xi}) := \left\{ \begin{pmatrix} \lambda Y \\ i\xi_2 Y \\ \vdots \\ i\xi_d Y \\ Z \end{pmatrix} : \; Y, Z \in \mathbb{C}^d \right\}$$

Whence:

- $\mathbb{G}$ is invariant for $\mathcal{A}$.

- $\dim \mathbb{G} = 2d$

- The action of $\mathcal{A}$ on $\mathbb{G}$,

$$\mathbb{M}(U,s,\lambda,\tilde{\xi}) \begin{pmatrix} Y \\ Z \end{pmatrix} := \begin{pmatrix} M_1^1 & M_1^2 \\ M_2^1 & M_2^2 \end{pmatrix} \begin{pmatrix} Y \\ Z \end{pmatrix} = \begin{pmatrix} \tilde{Y} \\ \tilde{Z} \end{pmatrix},$$

has the $d \times d$-block components:

$$M_1^1 := -\widehat{B}(\lambda s I + i \sum_{j\neq 1} \xi_j B_j^1),$$

$$M_1^2 := \widehat{B},$$

$$M_2^1 := (\lambda s I + i \sum_{j\neq 1} \xi_j B_1^j)\widehat{B}(\lambda s I + i \sum_{j\neq 1} \xi_j B_j^1) - \lambda^2 I - \sum_{i,j\neq 1} \xi_i \xi_j B_j^i,$$

$$M_2^2 := -(\lambda s I + i \sum_{j\neq 1} \xi_j B_1^j)\widehat{B}$$

$\mathbb{M}$ *is well-defined and smooth for all subsonic* s *including* 0.

The restriction $\mathcal{A}_{|\mathbb{G}}$ has a unique analytic extension to $s = 0$, even though $\mathcal{A}$ is not defined there.

We investigate only those modes of $\mathcal{A}(U, s, \cdot, \cdot)$ the amplitudes of which lie in $\mathbb{G}$.

Lemma 1 *For $(\lambda, \tilde{\xi}) \in \Gamma$ and s subsonic, the eigenvalues $-i\mu$ of $\mathbb{M}(U, s, \lambda, \tilde{\xi})$ satisfy*

$$\det(\tilde{\mathcal{N}}(\mu, \tilde{\xi}, U) + (i\mu s - \lambda)^2 I) = 0,$$

and $(Y, Z)^\top \in \mathbb{C}^{2d}$ is an eigenvector of $\mathbb{M}$ if and only if

$$Y \in \ker(\tilde{\mathcal{N}}(\mu, \tilde{\xi}) + (i\mu s - \lambda)^2 I),\ Y \neq 0,\ \textit{and}$$

$$Z = \left(s(\lambda - i\mu s)I + i\mu B_1^1 + i \sum_{j \neq 1} \xi_j B_j^1 \right) Y.$$

Moreover, for $\operatorname{Re} \lambda > 0$, d of these eigenvalues (counting multiplicities) have $\operatorname{Im} \mu > 0$, while the remaining d of them have $\operatorname{Im} \mu < 0$.

Lemma 2 *The matrix $\mathbb{M}$ satisfies the block structure assumption of Majda.*

(This follows form the results of Métivier (2000)).

Consequence: the stable/unstable bundles can be extended continuously to $\operatorname{Re}\lambda = 0$.

Lemma 3 *There exist continuous mappings (analytic for $\operatorname{Re}\lambda > 0$)*

$$\widehat{R}^u_s(U) : \Gamma \to \mathbb{C}^{2d\times d}, \quad \widehat{L}^u_s(U) : \Gamma \to \mathbb{C}^{d\times 2d},$$
$$\widehat{R}^s_s(U) : \Gamma \to \mathbb{C}^{2d\times d}, \quad \widehat{L}^s_s(U) : \Gamma \to \mathbb{C}^{d\times 2d},$$

with $\widehat{L}^u_s(U)\widehat{R}^u_s(U) = I_d$, $\widehat{L}^s_s(U)\widehat{R}^s_s(U) = I_d$, spanning right and left invariant spaces of $\mathbb{M}(U, s, \lambda, \tilde{\xi})$, spaces that are unstable, respectively stable (at least) for $\operatorname{Re}\lambda > 0$. The matrix fields

$$\widehat{R}^u_s(U), \widehat{L}^u_s(U), \widehat{R}^s_s(U), \widehat{L}^s_s(U)$$

depend continuously on U and $s \in (-\sqrt{\kappa_{min}(e_1, U)}, \sqrt{\kappa_{min}(e_1, U)})$.

The whole characteristic polynomial of $\mathcal{A}$ is

$$\pi(\mu) = (i\mu s - \lambda)^{d^2-d} \det(\tilde{\mathcal{N}}(\mu, \tilde{\xi}) + (i\mu s - \lambda)^2 I).$$

Thus, there is also a Lopatinski frequency

$$\beta_* = -i\mu_* = -\frac{\lambda}{s},$$

that creates a bad singularity around $s = 0$.

However, thanks to the curl-free constraint, the Fourier analysis can be performed on a $2d$-dimensional space excluding the blowing-up Lopatinski frequency μ_*. *And this bundle is precisely $\mathbb{G}$!*

Lemma 4 *If*

$$(U,V)(x,t) = (\hat{U}(x_1 - st), \hat{V}(x_1 - st)) \,\mathrm{exp}(i\tilde{\xi}\cdot\tilde{x} + \lambda t),$$

are solutions to the equations and curl $U = 0$*, where* $x = (x_1, \tilde{x})$*,* $\tilde{x} = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$ *and* $(\lambda, \tilde{\xi}) \in \Gamma$*, then, necessarily,*

$$C(s)^{-1}(A_1 - sI)(\hat{U}(\cdot), \hat{V}(\cdot))^\top \in \mathbb{G}(\lambda, \tilde{\xi}).$$

We “cut down” the modes associated to the singular mode μ_* in the original expression of Δ, resulting into an equivalent uniform Lopatinski condition in terms of a lower-dimensional determinant containing the representative modes compatible with the constraint.

For all $(\lambda, \xi) \in \Gamma$ there is an isomorphism $\mathcal{J}(\lambda, \tilde{\xi}) : \mathbb{C}^{2d} \to \mathbb{G}$

$$\mathcal{J} = \begin{pmatrix} \lambda I & 0 \\ i\xi_2 I & 0 \\ \vdots & \vdots \\ i\xi_d I & 0 \\ 0 & I \end{pmatrix},$$

which translates between $\mathbb{G}$ to its natural coordinates.

The stable/unstable bundles of $\mathbb{M}$ "lift" to stable/unstable bundles of $\mathcal{A}$

$$\check{R}^s(\lambda, \tilde{\xi}) := \mathcal{J}(\lambda, \tilde{\xi})\hat{R}^s(\lambda, \tilde{\xi}),$$
$$\check{R}^u(\lambda, \tilde{\xi}) := \mathcal{J}(\lambda, \tilde{\xi})\hat{R}^u(\lambda, \tilde{\xi}).$$

which are compatible with the constraint. *It suffices to work with the $\hat{R}$'s directly.*

All the ingredients of the Lop. determinant have equivalent representations.

$$Q = \begin{pmatrix} \lambda[U_1] \\ is\xi_2[U_1] \\ \vdots \\ is\xi_d[U_1] \\ -(\lambda s[U_1] + i\sum_{j\neq 1}\xi_j[\sigma(U)_j]) \end{pmatrix}.$$

$$Q = C(s)\mathcal{J}\hat{Q} \quad \text{with} \quad \hat{Q} = \begin{pmatrix} [U_1] \\ -(\lambda s[U_1] + i\sum_{j\neq 1}\xi_j[\sigma(U)_j]) \end{pmatrix},$$

and we work directly with $\hat{Q}$. The matrix field $\mathcal{K}(U^{\pm})$ of the theorem is defined, consequently, as

$$\mathcal{K}(U^{\pm}) := (A_1(U^{\pm}) - sI)^{-1}C(s)\mathcal{J}.$$

By existent nonlinear theory, this shows Theorem 1.

Part 3: An example

Dimension $d = 2$ (two-dimensional crystal lattice)

$$W(U) = \tfrac{1}{8}(\beta_1 - (1+\delta^2))^2 + (\beta_2 - 1)^2 + \gamma(\beta_3^2 - \delta^2)^2,$$

with

$$\beta_1 := |U_1|^2,\ \beta_2 := |U_2|^2,\ \beta_3 := U_1^\top U_2,$$

$U_j = j$-th column of U, $\gamma > 0$, $\delta \neq 0$.

W is rank-one convex at the two wells

$$U^A = \begin{pmatrix} 1 & 0 \\ -\delta & 1 \end{pmatrix}, \quad U^B = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix},$$

rank-one connected.

This W satisfies all previous hypotheses, plus “frame-indifference”.

Kinetic relation: *generalized Hugoniot rule* (conservation of energy)

$$g = [W(U)] - N^\top [U]^\top \langle \sigma(U) \rangle N.$$

Here $\langle f \rangle$ denotes $\frac{1}{2}(f^+ + f^-)$ for any f.

Perturbations of it:

$$g = [W(U)] - N^\top [U]^\top \langle \sigma(U) \rangle N + \tilde{g},$$

where $\tilde{g} \in C^1$ satisfies,

$$\tilde{g} = 0 \quad \text{for} \quad s = 0,$$

$$\tilde{g} > 0 \quad \text{for} \quad s < 0, \qquad \tilde{g} < 0 \quad \text{for} \quad s > 0,$$

$$\text{and,} \qquad d_s \tilde{g} < 0.$$

E.g. (artificial example): $\tilde{g} = -\epsilon s$, with $\epsilon > 0$.

We can compute numerically the mapping

$$\lambda \mapsto \hat{\Delta}(\lambda, \pm 1)$$

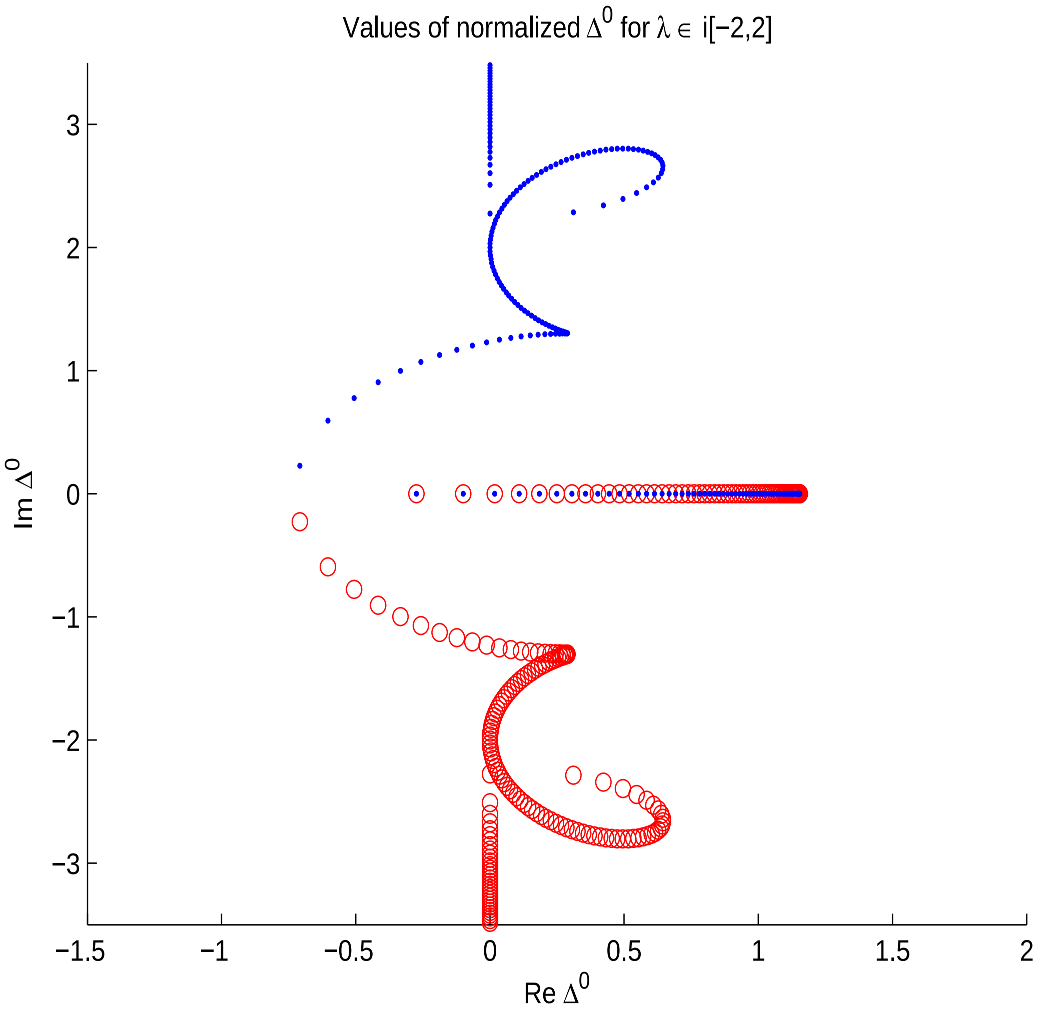
for a appropriately normalized version of $\hat{\Delta}$, and λ along a suff. large contour

$$C_\rho = C_\rho^+ \cup C_\rho^0,$$

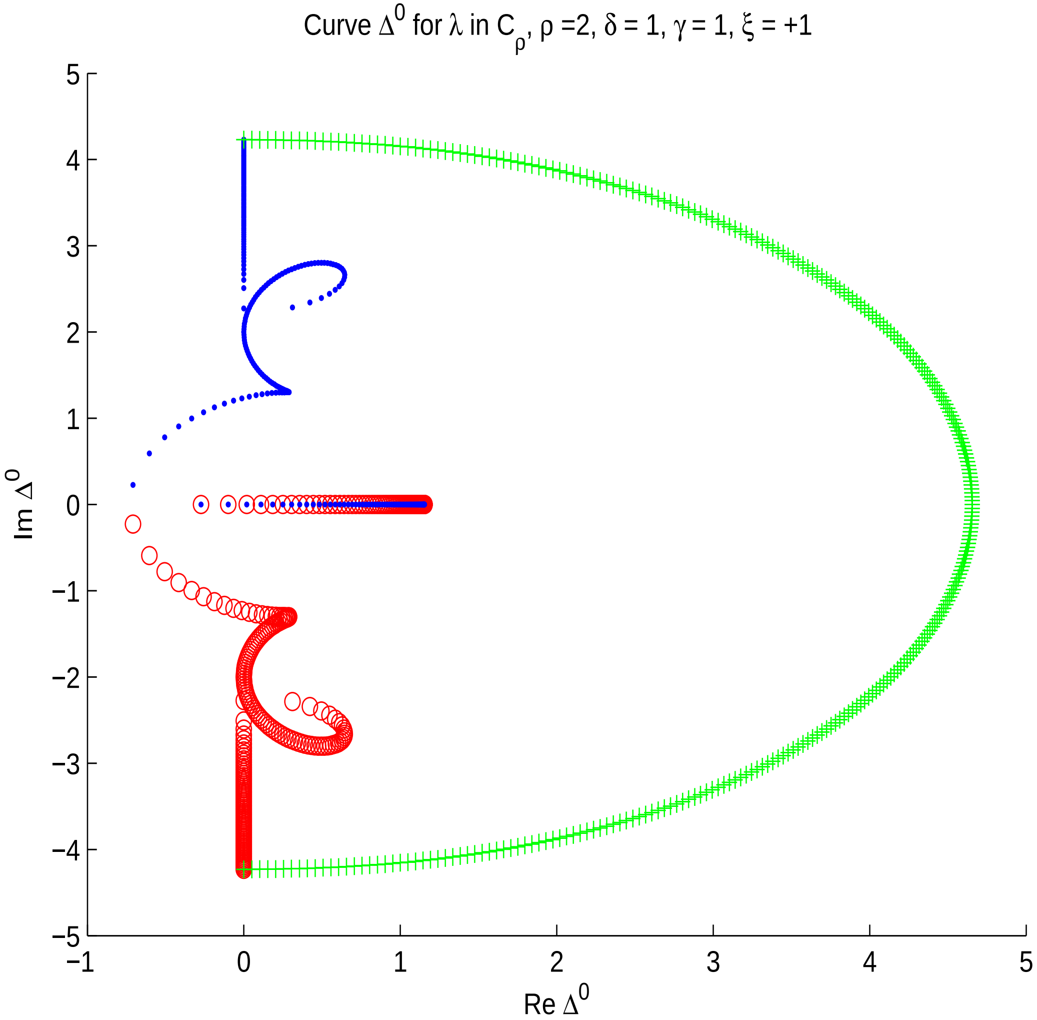
with

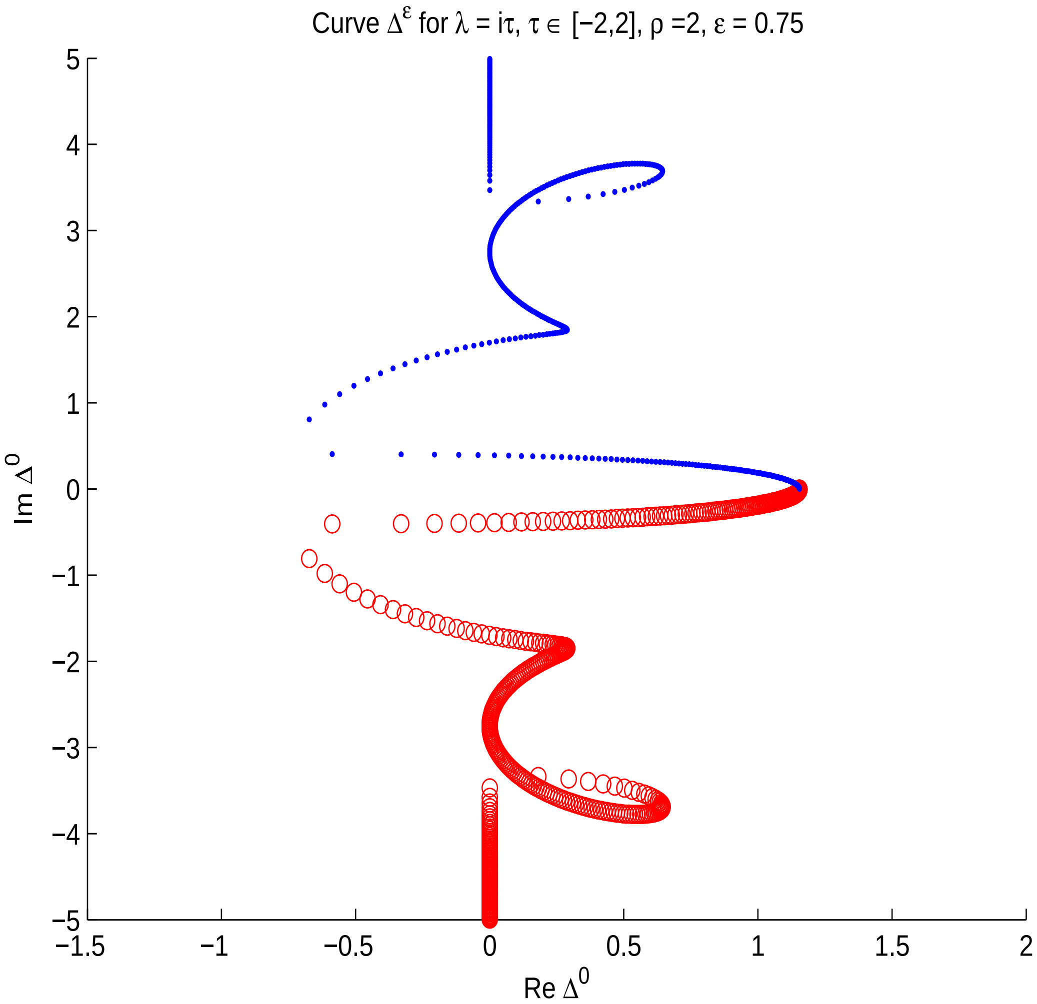
$$C_\rho^+ := \{\lambda \in \mathbb{C}\,;\ |\lambda| = \rho,\ \mathrm{Re}\,\lambda > 0\} \qquad \text{(half circle)},$$
$$C_\rho^0 := \{\lambda \in \mathbb{C}\,;\ \lambda = i\tau,\ \tau \in [-\rho, \rho]\} \qquad \text{(imaginary axis)},$$

for some $\rho > 0$.

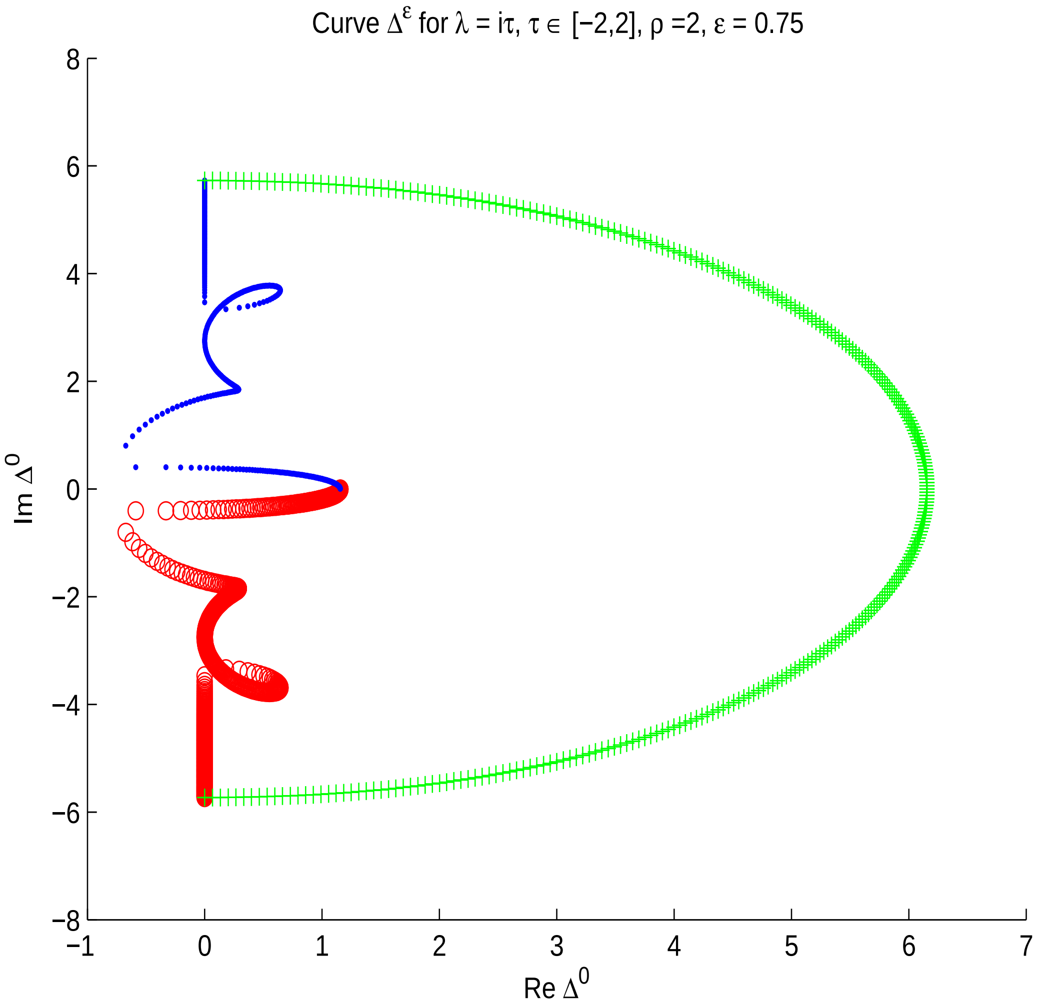


Values of normalized Δ^0 for $\lambda \in i[-2,2]$
Im Δ^0
3
2
1
0
−1
−2
−3
−1.5
−1
−0.5
0
0.5
1
1.5
2
Re Δ^0


Curve Δ^0 for λ in C_ρ, ρ =2, δ = 1, γ = 1, ξ = +1
Im Δ^0
Re Δ^0
5
4
3
2
1
0
−1
−2
−3
−4
−5
−1
0
1
2
3
4
5



Curve Δ^{ε} for $\lambda = i\tau$, $\tau \in [-2,2]$, $\rho = 2$, $\varepsilon = 0.75$
5
4
3
2
1
0
−1
−2
−3
−4
−5
Im Δ^0
−1.5
−1
−0.5
0
0.5
1
1.5
2
Re Δ^0



Curve Δ^{ε} for $\lambda = i\tau$, $\tau \in [-2,2]$, $\rho = 2$, $\varepsilon = 0.75$
8
6
4
2
0
−2
−4
−6
−8
Im Δ^0
−1
0
1
2
3
4
5
6
7
Re Δ^0

The end.

Thanks.

1. How to show that $\mathbb{M}$ satisfies the block structure of Majda. Have to check (Métivier (2000)) the conditions:

(i) When $\underline{\eta} > 0$, then $\det(i\mu I + \mathbb{M}(\underline{z})) \neq 0$, for all $\mu \in \mathbb{R}$.

(ii) When $\underline{z} \in \mathbb{R}_+^{d\times d} \times \mathbb{R} \times \Sigma_0$, then for all $\underline{\mu} \in \mathbb{R}$ such that $\det(i\underline{\mu} I + \mathbb{M}(\underline{z})) = 0$, there are a positive integer $\alpha \in \mathbb{Z}^+$ and C^∞ functions $\nu(\mu, \tilde{\xi}, U, s)$ and $\theta(z, \mu)$ defined on neighborhoods of $(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}, \underline{s})$ in $\mathbb{C} \times \mathbb{R}^{d-1} \times \mathbb{R}_+^{d\times d} \times \mathbb{R}$, and $(\underline{z}, \underline{\mu}) \in \mathcal{O} \times \mathbb{C}$, respectively, holomorphic in μ and such that

$$\det(i\mu I + \mathbb{M}(z)) = \theta(z, \mu)(\eta + i\tau + i\nu(\mu, \tilde{\xi}, U, s))^\alpha.$$

Moreover, ν is real when μ is real, and $\theta(\underline{z}, \underline{\mu}) \neq 0$. In addition, there is a C^∞ matrix-valued function $\mathbb{P}(\mu, \tilde{\xi}, U, s)$ on a neighborhhod of $(\underline{\mu}, \underline{\tilde{\xi}}, \underline{U}, \underline{s})$, holomorphic in μ, such that $\mathbb{P}$ is a projection of rank α and

$$\ker(i\mu I + \bar{\mathbb{M}}(z)) = \mathbb{P}(\mu, \tilde{\xi}, U, s)\mathbb{C}^{2d},$$

when $\eta + i\tau + i\nu(\mu, \tilde{\xi}, U, s) = 0$.

The appropriate projections are given by $\Pi_j : \mathbb{C}^d \to \mathbb{C}^d$, defined as

$$\Pi_j(\mu,\tilde{\xi},U) := -\frac{1}{2\pi i}\int_{|\zeta-\kappa_j(\mu,\tilde{\xi},U)|\le\varepsilon} (\mathcal{N}(\mu,\tilde{\xi},U)-\zeta)^{-1}\, d\zeta,$$

with $\varepsilon > 0$ sufficiently small, is a projector of constant rank α_j, C^∞ function of $(\mu,\tilde{\xi},U)$, for $(\mu,\tilde{\xi}) \neq (0,0)$. Thus,

$$\ker(\mathcal{N}(\underline{\mu},\underline{\tilde{\xi}},\underline{U}) - (\underline{\tau}-\underline{\mu}\underline{s})^2 I) = \Pi_j(\underline{\mu},\underline{\tilde{\xi}},\underline{U})\mathbb{C}^d,$$

and define $\mathbb{P}_j(\underline{\mu},\underline{\tilde{\xi}},\underline{U},\underline{s}) : \mathbb{C}^{2d} \to \mathbb{C}^{2d}$ as

$$\mathbb{P}_j(\underline{\mu},\underline{\tilde{\xi}},\underline{U},\underline{s}) := \begin{pmatrix} \Pi_j(\underline{\mu},\underline{\tilde{\xi}},\underline{U}) & 0 \\ i(\underline{s}(\underline{\tau}-\underline{\mu}\underline{s})I + \underline{\mu}B_1^1 + \sum_{k\neq 1}\underline{\xi}_k B_k^1)\Pi_j(\underline{\mu},\underline{\xi},\underline{U}) & 0 \end{pmatrix}$$

and,

$$\ker(i\underline{\mu}I + \mathbb{M}(\underline{z})) = \mathbb{P}_j(\underline{\mu},\underline{\tilde{\xi}},\underline{U},\underline{s})\mathbb{C}^{2d}.$$

2. How to decrease the order of Lopatinski determinants

$$\Delta = \det \begin{pmatrix} R^s_- & Q & R^u_+ \\ -(d_{u-}g)(A^-_N - sI)^{-1} R^s_- & q & (d_{u+}g)(A^+_N - sI)^{-1} R^u_+ \end{pmatrix}$$

First, if $l > 0$, multiplying the upper block from the left by $(d_{u-}g)(A^-_N - sI)^{-1}$ and subtracting from the lower $l \times (n+l)$ block, we get a matrix of form

$$\begin{pmatrix} R^s_- & Q & R^u_+ \\ 0 & q^u & p^u \end{pmatrix}.$$

Observing,

$$\begin{pmatrix} (R^s_-)^\top & 0 \\ L^u_- & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} R^s_- & Q & R^u_+ \\ 0 & q^u & p^u \end{pmatrix} = \begin{pmatrix} (R^s_-)^\top R^s_- & * & * \\ 0 & L^u_- Q & L^u_- R^u_+ \\ 0 & q^u & p^u \end{pmatrix},$$

we get

$$\Delta^u := \det \begin{pmatrix} L_-^u Q & L_-^u R_+^u \\ q^u & p^u \end{pmatrix},$$

where

$$p^u := \left((d_{u+}g)(A_N^+ - sI)^{-1} + (d_{u-}g)(A_N^- - sI)^{-1}\right) R_+^u,$$
$$q^u := q + d_{u-}g\,(A_N^- - sI)^{-1}Q.$$

Same procedure for a reduction on the right column.