On the stability of radiative shock profiles*

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Radiative hydrodynamics

- W. G. VINCENTI AND C. H. KRUGER, *Introduction to Physical Gas Dynamics*, Wiley & Sons, New York, 1965.

Euler-Poisson system ($d = 1$):

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\begin{aligned}
\rho_t + (\rho u)_x &= 0,\\
(\rho u)_t + (\rho u^2 + p)_x &= 0,\\
\left(\rho(e + \tfrac{1}{2}u^2)\right)_t + \left(\rho u(e + \tfrac{1}{2}u^2) + pu\right)_x &= -q_x,\\
-q_{xx} + aq + b(\theta^4)_x &= 0,
\end{aligned}
\qquad \text{(EP)}
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$$\begin{aligned}
\rho &= \text{mass density},\\
u &= \text{velocity},\\
p &= \text{pressure},\\
e &= \text{internal energy density},\\
\theta &= \text{temperature}.
\end{aligned}$$

$$p = p(\rho,\theta),\ e = e(\rho,\theta): \quad p_\rho > 0,\ p_\theta \neq 0,\ e_\theta > 0.$$

$q = \rho\chi_x$, radiative heat flux, $\chi =$ radiation energy density.

$a, b > 0$: Absortion coefficient α; Stefan-Boltzmann constant σ: $a = 3\alpha^2, \quad b = 4\alpha\sigma.$

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Hamer's model

- K. HAMER, *Quart. J. Mech. Appl. Math.* **24** (1971).

$$\begin{aligned} u_t + \tfrac{1}{2}(u^2)_x &= -q_x, \\ -q_{xx} + q &= -u_x, \end{aligned} \tag{H}$$

$q, u \in \mathbb{R}$, $(x,t) \in \mathbb{R} \times [0, +\infty)$. Burgers' flux function: $f(u) = \frac{1}{2}u^2$.

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Radiative shock profiles

Trveling wave solution

$$(u,q)(x,t) = (\bar{U},\bar{Q})(x-st), \quad (\bar{U},\bar{Q})(\xi) \to (U_\pm, 0),\ \xi \to \pm\infty,$$

$(U_+, U_-, s) =$ classical shock front

of the underlying hyperbolic system

- Hydrodynamics and transport determine wave propagation and wave structure.
- *Non-equilibrium diffusion regime:* Radiation and gas have different temperature ($\theta \neq q^{1/4}$); gas interacts with radiation via energy exchanges.
- Radiation is described by an stationary diffusion process.
- *Gray non-equilibrium diffusion hypothesis*: All photons have the same frequency (Lowrie, Edwards, *Shock waves* **18** (2008)).

Regularization

Hamer's model:

$$q = -(1-\partial_x)^{-1} u_x =: -\mathcal{K} u_x$$

$$\mathcal{K} f(x) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-|x-y|} f(y)\,dy = K * f, \quad K = \frac{1}{2} e^{-|x|},$$

$$q_x = u - \mathcal{K} u,$$

$$u_t + u u_x = -u + \mathcal{K} u,$$

Rosenau's regularization (ROSENAU, Phys. Rev. A **40** (1989); SCHOCHET, TADMOR, Arch. Ration. Mech. Anal. **119** (1992)):

Regularization (truncation of the Chapman-Enskog expansion).

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Symmetric and normal forms

Conserved variables:

$$U := \left(\rho,\ \rho u,\ \rho(e + \tfrac{1}{2}u^2)\right)^\top$$

Euler-Poisson (EP):

$$\begin{aligned} U_t + f(U)_x &= -Lq_x, \\ -q_{xx} + Rq + \nu(U)g(U)_x &= 0, \end{aligned}$$

$$\begin{aligned} f(U) &= \left(\rho u, \rho u^2 + p, \rho u(e + \tfrac{1}{2}u^2) + pu\right)^\top, \\ L &= (0,\ 0,\ 1)^\top, \\ R &= a \equiv 1, \\ 0 < \nu(U) &= 4b\theta^5, \\ g(U) &= -1/\theta \end{aligned}$$

Entropy (mathematical):

$$\eta = -\rho s,$$

$$W := (D_U \eta)^\top = \big(-\rho + (e - \tfrac{1}{2}u^2 + p\rho^{-1})/\theta,\, u/\theta,\, -1/\theta \big)^\top,$$

$$D_U W = D_U^2 \eta > 0$$

$$U \mapsto W$$

Symmetrization:

$$\begin{aligned} A_0(W)W_t + A(W)W_x + Lq_x &= 0, \\ -q_{xx} + q + \tilde{\nu}(W)W_x &= 0, \end{aligned}$$

$$\begin{aligned} A_0(W) &= (D_W U)(W) = (D_U^2 \eta)^{-1} \\ A(W) &= D_W(f(U(W))) = (D_U f)(D_U^2 \eta)^{-1} \end{aligned}$$

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Normal form:

$$W \mapsto V := (\rho, u, \theta)^\top$$

Mult. by $(D_V W)^\top$:

$$\begin{aligned} \bar{A}_0(V)V_t + \bar{A}(V)V_x + \bar{L}(V)q_x &= 0, \\ -q_{xx} + q + \bar{\nu}(V)\bar{L}(V)^\top V_x &= 0, \end{aligned}$$

$$\begin{aligned} \bar{A}_0(V) &= (D_V W)^\top A_0(W) D_V W = (D_V W)^\top D_V U, \\ \bar{A}(V) &= (D_V W)^\top A(W) D_V W = (D_V W)^\top D_V f(U), \\ \bar{L}(V) &= (D_V W)^\top L = (D_V W)^\top (0,0,1)^\top = (0,0,1)^\top/\theta^2, \\ \bar{\nu}(V) &= \nu(U(W(V))) = 4b\theta^5 > 0, \end{aligned}$$

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$$\bar{A}(V) = \frac{1}{\theta}\begin{pmatrix} p_\rho u/\rho & p_\rho & 0 \\ p_\rho & \rho u & p_\theta \\ 0 & p_\theta & p e_\theta u/\theta \end{pmatrix}$$

$$\bar{\nu}(V)\bar{L}(V)V_x = 4b\theta^3(0,0,\theta_x)^\top =: b\bar{g}(V)_x$$

$$\bar{g}(V) = (0,0,\theta^4),$$

General hyperbolic-elliptic system:

$$V_t + f(V)_x + Lq_x = 0,$$
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Positive “diffusion”

Eigenvector (characteristic field $p = 1$):

$$\bar{A} r_1 = (u+c)\bar{A}_0 r_1, \quad l_1 \bar{A} = (u+c) l_1 \bar{A}_0$$

$$r_1^\top = l_1 = (\rho,\, c,\, (c^2 - p_\rho)\rho/p_\theta) = (\rho, c, z_*)$$

$$B(V) := (D_V \bar{g})^\top = (0, 0, 4\theta^3)$$

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$$l_1 L B r_1 = (\rho, c, z_*)(0,0,1)^\top (0,0,4\theta^3)(\rho, c, z_*)^\top = z_*^2 4\theta^3 > 0$$

Positive diffusion coefficient in the characteristic direction $p = 1$ in the Chapman-Enskog expansion.

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Chapman-Enskog expansion:

$$\begin{aligned} \mathbf{u}_t + \nabla p &= \sum_0^\infty \varepsilon^{2n+1} \Delta^n (\mu_n \Delta \mathbf{u} + \alpha_n \nabla(\nabla \cdot \mathbf{u})) + \text{other terms}, \\ &= \varepsilon \mu_0 \Delta \mathbf{u} + \varepsilon^3 \mu_1 \Delta^2 \mathbf{u} + \ldots \end{aligned}$$

$\mu = \varepsilon \mu_0 \ll 1,$ kinematic viscosity coefficient (non-dimensional)

$\varepsilon^3 \mu_1,$ – Burnett's coefficient

Rosenau's model:

$$\sum_0^\infty \varepsilon^{2n+1}\Delta^n(\mu_n \Delta u) \sim \frac{\mu}{1-\varepsilon^2 m^2 \Delta}\Delta u, \quad m = \mu_1/\mu_0 > 0.$$

$$\mu_* = \frac{\mu}{1-\varepsilon^2 m^2 \Delta}, \qquad \hat{\mu}_*(k) = \frac{\mu}{1+\varepsilon^2 m^2 k^2}$$

Scalar model:

$$u_t + (\tfrac{1}{2}u)_x = \varepsilon\partial_x^2\Big(\frac{1}{1+m^2\varepsilon^2 k^2}\hat{u}(k)\Big)^\vee(x) = -u + \mathcal{K}u$$

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For systems:

$$\mathbb{L}A\mathbb{R} = \begin{pmatrix} A_1 & & \\ & a_p & \\ & & A_2 \end{pmatrix},$$

$$v = \mathbb{L}u$$

$$\mathbb{L}(LB(u)u_x)_x = (\mathbb{L}LB\mathbb{R})u_{xx} + \ldots,$$

Principal part:

$$(l_pLBr_p)(v_p)_{xx} + \ldots$$

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Previous work

Existence of profiles:

- SCHOCHET, TADMOR, Arch. Ration. Mech. Anal. **119** (1992). Small-amplitude profiles, Hamer's model only.
- KAWASHIMA, NISHIBATA, SIAM J. Math. Anal. **30** (1998). Hamer's model. Bounded amplitude.
- LIN, COULOMBEL, GOUDON Phys. D **218** (2006). Small-amplitude, Euler-Poisson system, ideal gas $\rho = (\gamma - 1)\rho e$.
- LATTANZIO, MASCIA, SERRE, *Indiana Univ. Math. J.* **56** (2007). General model, f general, linear coupling. Systems case problem reduces to a scalar one (!).
- LATTANZIO, MASCIA, SERRE, Proc. HYP2006, Springer (2008). Nonlinear coupling, most general result.

Miscellaneous (Cauchy problem, $t \to +\infty$, well-posedness):

- Kawashima, Nishibata, Math. Mod. Meth. Appl. Sci. **9** (1999). Cauchy problem, Hamer's model.
- Serre, Comm. Math. Sci. **1** (2003). Stability of constants in L^1, Hamer's model.
- Kawashima, Tanaka, Kyushu J. Math. **58** (2004): Stability of rerafecation waves (Hamer).
- Kawashima, Nikkuni, Nishibata, Monogr. Surv. Pure Appl. Math. **99** (1999). Cauchy problem. general hyperbolic-elliptic systems.
- Kawashima, Nikkuni, Nishibata, *Arch. Ration. Mech. Anal.* **170** (2003). Asymptotic behaviour of solutions when $t \to +\infty$.
- Lowrie, Edwards, *Shock Waves* **18** (2008). Numerical computation of profiles, Euler-Poisson (ideal gases).

Stability of radiative shock profiles:

- KAWASHIMA, NISHIBATA, SIAM J. Math. Anal. **30** (1998). L^2 stability, Hamer's (escalar) model, linear coupling.
- LIN, COULOMBEL, GOUDON, *C. R. Math. Acad. Sci. Paris* **345** (2007): Stability under zero-mass perturbations, ideal gas: $p = (\gamma - 1)\rho e$. Energy estimates of Goodman-Matsumura-Nishihara type.

New results:

- Stability of radiative shock profiles in the general scalar case (general flux function f, nonlinear coupling): C. LATTANZIO, C. MASCIA, T. NGUYEN, R. G. P, K. ZUMBRUN, *SIAM J. Math. Anal.* **41**, no. 6 (2009).
- Stability of radiative profiles for general hyperbolic-elliptic systems (small-amplitude): T. NGUYEN, R. G. P, K. ZUMBRUN, *Phys. D* **239**, no. 8 (2010).

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Viscous shock profiles

$$u_t + f(u)_x = (B(u)u_x)_x,$$

$u \in \mathbb{R}^n$, $n \geq 1$, $B(u) \equiv I$ identity viscosity; $B(u)$ degenerate (Navier-Stokes).

Viscous shock profile: traveling wave solution $u(x,t) = \bar{U}(x - st)$, $\bar{U}(x) \to u_\pm$. Here (u_+, u_-, s) is a classical shcok front of the hyperbolic system of conservation laws.

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Stability

- Viscous shock profiles *scalar case*: ILIN, OLEINIK; SATTINGER.
- Systems: GOODMAN, Arch. Ration. Mech. Anal. **95** (1986). Identity viscosity $B = I$, zero-mass perturbations. Energy estimates, diagonalization of the hyperbolic part.
- MATSUMURA, NISHIHARA, Japan J. Appl. Math. **2** (1985). Same approcah for Euler system, ideal gases.

Energy methods: LIU (1986) diffusion waves; SZEPSSY, XIN (1992) first complete result.

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Pointwise Green function bound method:

- LIU, CPAM 50 (1997): (Approximate) Green function of linearized operator.
- ZUMBRUN, HOWARD (1999) Under spectral stability assumption, resolvent kernel bounds; ponitwise bounds for the Green function.
- ZUMBRUN, MASCIA (2002-2004) degenerate viscosity.

Other cases: relaxation systems, multi-d, boundary layes, undercompressive shocks, etc.

Basic idea: Spectral stability $\Rightarrow$ nonlinear stability.

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Scalar model

$$\begin{aligned} u_t + f(u)_x + Lq_x &= 0, \\ -q_{xx} + q + M(u)_x &= 0, \end{aligned}$$

$u, q \in \mathbb{R}$, $M, f : \mathbb{R} \to \mathbb{R}$, $L \in \mathbb{R}$ (constant).

Traveling waves:

$$(u, q)(x, t) = (U, Q)(x - st), \quad (U, Q)(\pm\infty) = (u_\pm, 0),$$

$u_+ \neq u_-$. W.l.o.g. $s = 0$.

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Hypotheses:

$$f, M \in C^5, \qquad \text{(regularity)}, \tag{A0}$$

$$f''(u) > 0, \quad u \in [u_+, u_-] \qquad \text{(genuine nonlinearity)}, \tag{A1}$$

$$f(u_-) = f(u_+), \qquad \text{(Rankine-Hugoniot condition)}, \tag{A2}$$

$$u_+ < u_-, \qquad \text{(Lax's entropy condition)}, \tag{A3}$$

$$LM'(u) > 0, \quad u \in [u_+, u_-] \qquad \text{(positive diffusion)}, \tag{A4}$$

$$a(x) := f'(U), \quad b(x) := M'(U).$$

$$Lb(0) + (k + \tfrac{1}{2})a'(0) > 0, \qquad k = 1, 2, 3, 4. \qquad (\mathrm{A5}_k)$$

Structure of profiles

Traveling wave equations:

$$\begin{aligned} f(U)' + L\,Q' &= 0, \\ -Q'' + Q + M(U)' &= 0, \end{aligned}$$

$(U,Q)(\pm\infty) = (u_\pm, 0).$

Existence theory: Lattanzio, Mascia, Serre, *Indiana Univ. Math. J.* **56** (2007); Proc. HYP2006, Springer (2008).

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Proposition [LMS]
Under (A0) - (A4) there exists a unique (up to translations) traveling wave $(U,Q)(x)$*. Moreover, the velocity profile* U *is* C^2 *except, at most, in one single point where it has an entropic jumo satisfying Rankine-Hugoniot and Lax conditions.* U *is monotone decreasing* $U_x < 0$*, function* $a(x) = f'(U(x))$ *is* C^1 *a.e., it is zero only at one point which we take w.l.o.g. as* $x = 0$*:*

$$a(0) = 0.$$

If the amplitude is sufficiently small, then the profile is of class C^2*.*

Consequences:

$$a'(x) < 0 \qquad \forall\, x \in \mathbb{R}, \qquad\qquad x\, a(x) < 0 \qquad \forall\, x \neq 0.$$

Integrating:

$$LQ = f(u_\pm) - f(U) > 0,$$
$$\big(a'(x) + L\, b(x)\big)\, U' = -LQ - a(x)U'',$$

In $x = 0$, U monotone:

$$a'(0) + L\, b(0) > 0. \qquad \text{(P)}$$

($A5_1$) implies (P).

Consequences:

$$a'(x) < 0 \qquad \forall\, x \in \mathbb{R}, \qquad\qquad x\, a(x) < 0 \qquad \forall\, x \neq 0.$$

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Exponential decay

$$\left|(d/dx)^k (U - u_\pm, Q)\right| \leq C e^{-\eta|x|}, \quad k = 0, \ldots, 4,$$

as $|x| \to +\infty$, for some $\eta > 0$.

Eigenvalue equations

$$\begin{aligned} \lambda u + (a(x)u)' + Lq' &= 0, \\ -q'' + q + (b(x)u)' &= 0. \end{aligned} \tag{SP}$$

$' = d/dx$, $u, q \in L^2(\mathbb{R})$ perturbations.

Spectral stability (scalar case)

Zero-mass conditions:

$$\int u = 0, \qquad \int q = 0,$$

Integrating (SP):

$$\begin{aligned} \lambda u + a(x)\, u' + Lq' &= 0, \\ -q'' + q + b(x)\, u' &= 0. \end{aligned}$$

Lemma (Spectral stability)

Let (u, q) be an eigenfuction, with $\lambda \in \mathbb{C}$ eigenvalue. Then Re $\lambda < 0$ if one of the following conditions hold:

(i) *b is constant (linear coupling), or,*

(ii) *$|u_+ - u_-|$ is sufficiently small.*

Proof (via energy estimates):

$$\text{Re } \lambda |b^{1/2} u|^2_{L^2} \leq \langle a' b u, u \rangle - \frac{L}{2} \|q\|^2_{H^1} + C \langle (|a| + |b'|) |b'| u, u \rangle.$$

$Lb > 0$, w.l.o.g. $b \geq \theta > 0$ $(q \to -q)$. $a' > 0$, $a', b' = \mathcal{O}(|u_+ - u_-|)$.

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$a', b' = \mathcal{O}(|u_+ - u_-|)$.

First order formulations

Specral problem: $p := b(x)u - q'$.

$$\begin{aligned} a(x)u' &= -\left(\lambda + a'(x) + Lb(x)\right) u + Lp, \\ q' &= b(x)u - p, \\ p' &= -q. \end{aligned} \tag{SP2}$$

Regularity near $x = 0$.

Lemma

Given $\lambda \in \mathbb{C}$, $\nu := (Re\ \lambda + a'(0) + Lb(0))/|a'(0)|$. Under (A0) - (A4), $Re\ \lambda > -Lb(0)$, every solution to (SP2) satisfies

1. $|u(x)| \leq C\,|x|^{\nu}$ *for* $x \sim 0$, *some* $C > 0$;
2. q *is A.C.,* p *is* C^1 *(for* $x \sim 0$*),*

In particular, $u \in L^1_{loc}$ *(for* $x \sim 0$*), and* $a(x)u(x) \to 0$ *if* $x \to 0$.

Construction of the resolvent kernel

Fisrt order system

$$(\Theta(x)W)' = \mathbb{A}(x,\lambda)W,$$

$$\Theta(x) := \begin{pmatrix} a(x) & 0 \\ 0 & I_2 \end{pmatrix}, \qquad \mathbb{A}(x,\lambda) := \begin{pmatrix} -(\lambda + L\,b(x)) & 0 & L \\ b(x) & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

Θ is singular at $x = 0$.

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Solutions to:

$$\partial_x \left(\Theta(x) \, \mathcal{G}_\lambda \right) - \mathbb{A}(x, \lambda) \, \mathcal{G}_\lambda = \delta_y(x) \, I,$$

$$\partial_x \left(\Theta(x) \, \mathcal{G}_\lambda \right) - \mathbb{A}(x, \lambda) \, \mathcal{G}_\lambda = 0, \qquad \text{if} \;\; x \neq y,$$

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Asymptotic systems

$$\Theta_\pm W' = \mathbb{A}_\pm(\lambda) W,$$

$$\Theta_\pm := \begin{pmatrix} a_\pm & 0 \\ 0 & I_2 \end{pmatrix}, \qquad \mathbb{A}_\pm(\lambda) := \begin{pmatrix} -(\lambda + L\, b_\pm) & 0 & L \\ b_\pm & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$a_\pm := \lim_{x \to \pm\infty} a(x) = f'(u_\pm), \qquad b_\pm := \lim_{x \to \pm\infty} b(x) = M'(u_\pm).$$

$$a_+ < 0 < a_-.$$

Dispersion relation

$$|\mu I - \Theta_{\pm}^{-1}\mathbb{A}_{\pm}(\lambda)| = \mu^3 + a_{\pm}^{-1}(\lambda + Lb_{\pm})\mu^2 - \mu - a_{\pm}^{-1}\lambda,$$

For $\lambda \in \mathbb{R}_+, \lambda \to +\infty$: 2 positive, 1 negative root for π_+. 2 negative, 1 positive for π_-.

For each Re $\lambda > 0$:

$$\dim U^+(\lambda) = 2, \qquad \dim S^+(\lambda) = 1,$$
$$\dim U^-(\lambda) = 1, \qquad \dim S^-(\lambda) = 2.$$

Dimensions are not equal.

$$\{\mathrm{Re}\,\lambda > 0\} \subset \Lambda = \text{region to the left of the dispersion curves}$$

Region of (not so) consistent splitting

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Region of (not so) consistent splitting

Small frequencies: $\lambda \sim 0$

Eigenvalues of $\Theta_\pm^{-1}\mathbb{A}_\pm(\lambda)$:

$$\begin{aligned}
\mu_2^\pm(\lambda) &= -\frac{\lambda}{a_\pm} + \mathcal{O}(|\lambda|^2),\\
\mu_1^\pm(\lambda) &= \pm\theta_1^\pm + \mathcal{O}(|\lambda|),\\
\mu_3^\pm(\lambda) &= \mp\theta_3^\pm + \mathcal{O}(|\lambda|),
\end{aligned}$$

$$\begin{aligned}
\mu_2^\pm(0) &= 0,\\
\mu_1^-(0) &= -\theta_1^- < 0 < \theta_1^+ = \mu_1^+(0),\\
\mu_3^+(0) &= -\theta_3^+ < 0 < \theta_3^- = \mu_3^-(0)\,.
\end{aligned}$$

Eigenvectors:

$$V_j^{\pm} = \begin{pmatrix} b_{\pm}^{-1}(1-\mu_j^{\pm}(\lambda)^2) \\ -\mu_j^{\pm}(\lambda) \\ 1 \end{pmatrix}.$$

$$V_2^{\pm}(\lambda) = \begin{pmatrix} \mathcal{O}(1) \\ \mathcal{O}(\lambda) \\ \mathcal{O}(1) \end{pmatrix}, \qquad V_j^{\pm}(\lambda) = \mathcal{O}(1), \quad j = 1, 3.$$

Lemma

For each $\lambda \in \Lambda$, the asymptotic systems has solutions

$$e^{\mu_j^{\pm}(\lambda)x} V_j^{\pm}(\lambda), \quad x \gtrless 0, \; j = 1, 2, 3.$$

For $|\lambda| \sim 0$, it is possible to find analytic representations for $\mu_j^{\pm}$ and $V_j^{\pm}$, namely, two "slow" modes

$$\mu_2^{\pm}(\lambda) = -a_{\pm}^{-1}\lambda + \mathcal{O}(\lambda^2),$$

and four "fast" modes:

$$\mu_1^{\pm}(\lambda) = \pm\theta_1^{\pm} + \mathcal{O}(\lambda), \qquad \mu_3^{\pm}(\lambda) = \mp\theta_3^{\pm} + \mathcal{O}(\lambda),$$

Solutions to the system $\Theta W' = \mathbb{A}(x,\lambda)W$

Thanks to Conjugation Lemma (Mascia, Zumbrun, *Indiana Univ. Math. J.* **51** (2002): exponential decay of the waves in the hyperbolic region implies the existence of projections $P_\pm(x,\lambda) = I + \Phi_\pm$, uniformly bounded, which relate the solutions Z to the asymptotic system, with the solutions W to the variable coefficient system, $W = P_\pm Z$. Moreover, $|\partial_\lambda^j \partial_x^k \Phi_\pm| \lesssim e^{-\eta|x|}$.

Lemma

For $|\lambda| \sim 0$ there exist $\psi_j^{\pm}(x,\lambda)$, $j = 1,2$, explosive modes, and $\phi_3^{\pm}(x,\lambda)$ decaying modes, in $x \gtrless 0$, of class C^1 in x, analytic in λ, such that

$$\psi_j^{\pm}(x,\lambda) = e^{\mu_j^{\pm}(\lambda)\,x}\, V_j^{\pm}(\lambda)(I + \mathcal{O}(e^{-\eta|x|})), \quad j = 1,2,$$
$$\phi_3^{\pm}(x,\lambda) = e^{\mu_3^{\pm}(\lambda)\,x}\, V_3^{\pm}(\lambda)(I + \mathcal{O}(e^{-\eta|x|})),$$

where $\eta > 0$ is the rate of exponential decay of the profiles.

Summary: solutions for $\lambda \sim 0$, $x \gtrless 0$

In $x \geq x_0 > 0$:

$$\psi_1^+(x,\lambda) = e^{(\theta_1^+ + \mathcal{O}(|\lambda|))x}\, V_1^+(\lambda)(I + \mathcal{O}(e^{-\eta|x|})), \qquad \text{(fast growing)},$$

$$\psi_2^+(x,\lambda) = e^{(-\lambda/a_+ + \mathcal{O}(|\lambda|^2))x}\, V_2^+(\lambda)(I + \mathcal{O}(e^{-\eta|x|})), \quad \text{(slowly growing}$$

$$\phi_3^+(x,\lambda) = e^{(-\theta_3^+ + \mathcal{O}(|\lambda|))x}\, V_3^+(\lambda)(I + \mathcal{O}(e^{-\eta|x|})), \qquad \text{(fast decaying)}$$

In $x \leq x_0 < 0$:

$$\psi_1^-(x,\lambda) = e^{(-\theta_1^- + \mathcal{O}(|\lambda|))x}\, V_1^-(\lambda)(I + \mathcal{O}(e^{-\eta|x|})), \qquad \text{(fast growing)},$$

$$\psi_2^-(x,\lambda) = e^{(-\lambda/a_- + \mathcal{O}(|\lambda|^2))x}\, V_2^-(\lambda)(I + \mathcal{O}(e^{-\eta|x|})), \quad \text{(slowly growin}$$

$$\phi_3^-(x,\lambda) = e^{(\theta_3^- + \mathcal{O}(|\lambda|))x} V_3^-(\lambda)\,(I + \mathcal{O}(e^{-\eta|x|})), \qquad \text{(fast decaying}$$

Solutions for $x \sim 0$

$$\xi = \int_{\epsilon_0}^{x} \frac{dz}{a(z)},$$

$\xi(\epsilon_0) = 0$, $\xi \to +\infty$ if $x \to 0^+$.

$$u' = \frac{du}{dx} = \frac{1}{a(x)}\frac{du}{d\xi} = \frac{1}{a(x)}\dot{u},$$

$\dot{} = d/d\xi$.

$$\dot{W} = \tilde{\mathbb{A}}(\xi,\lambda)W \quad \text{where} \quad \tilde{\mathbb{A}}(\xi,\lambda) := \begin{pmatrix} -\omega & 0 & L \\ \tilde{a}\,\tilde{b} & 0 & -\tilde{a} \\ 0 & -\tilde{a} & 0 \end{pmatrix},$$

$$\omega(\xi) := \lambda + a'(x(\xi)) + L\, b(x(\xi)), \quad \tilde{a}(\xi) := a(x(\xi)), \quad \tilde{b}(\xi) := b(x(\xi)).$$

For $\lambda \sim 0$, $0 < \epsilon_0 \ll 1$

$$\operatorname{Re}\,\omega(\xi) \sim \operatorname{Re}\,\omega(0) = \eta := \operatorname{Re}\lambda + a'(0) + L\, b(0) > 0,$$

for $\xi \in [0, +\infty)$.

Block diagonalization:

$$\dot{Z} = \begin{pmatrix} -\omega & 0 \\ 0 & 0 \end{pmatrix} Z + \tilde{a}\,\hat{\Theta}(\xi) Z,$$

$$\hat{\Theta} = \begin{pmatrix} 0 & L/\omega & L(a'' + L\,b')/\omega^2 \\ \tilde{b} & 0 & -1 + L\,\tilde{b}/\omega \\ 0 & -1 & 0 \end{pmatrix}$$

"Fast" and "slow" coordinates

$$\dot{Z}_1 = -\omega\, Z_1 + \mathcal{O}(\tilde{a}) Z_1,$$
$$\dot{Z}_2 = \mathcal{O}(\tilde{a}) Z_2.$$

$Z_1 \to 0$, $\xi \to +\infty$:

$$e^{-\int_0^\xi \omega(z)\,dz} \lesssim e^{-(\mathrm{Re}\,\lambda + \frac{1}{2}\eta)\xi} \to 0,$$

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Lemma (Solutions near $x = 0$).
Under (A0) - (A4) y $(\mathrm{A}5_2)$*, there exist* $0 < \epsilon_0 \ll 1$ *small such that, for* $\lambda \sim 0$*, the solutions in* $(-\epsilon_0, 0) \cup (0, \epsilon_0)$ *are generated by "fast" modes,*

$$w_2^{\pm}(x,\lambda) = \begin{pmatrix} u_2^{\pm} \\ q_2^{\pm} \\ p_2^{\pm} \end{pmatrix} = \begin{pmatrix} Z_1(x) \\ 0 \\ 0 \end{pmatrix} (1+\mathcal{O}(a(x))), \qquad \pm\epsilon_0 \gtrsim x \gtrsim 0,$$

and "slow" modes,

$$z_j^{\pm}(x,\lambda) = \begin{pmatrix} u_j^{\pm} \\ q_j^{\pm} \\ p_j^{\pm} \end{pmatrix}, \qquad \pm\epsilon_0 \gtrsim x \gtrsim 0, \quad j = 1, 3,$$

bounded as $x \to 0^{\pm}$*. moreover, the fast modes decay as*

$$u_2^{\pm} \sim |x|^{\nu} \to 0, \qquad \begin{pmatrix} q_2^{\pm} \\ p_2^{\pm} \end{pmatrix} \sim \mathcal{O}(|x|^{\nu} a(x)) \to 0,$$

when $x \to 0^{\pm}$*, where* $\nu := (Re\ \lambda + a'(0) + Lb(0))/|a'(0)|$.

Decaying modes

Resolvent kernel construction: complete set of decaying modes

Let $\epsilon_0 > 0$, small. Objective: two decaying modes in $+\infty$, W_j^+, $j = 1, 2$; one decaying in $-\infty$, W_3^-.

$$W_3^-(x,\lambda) := \begin{cases} \phi_3^-(x,\lambda), & x < -\epsilon_0, \\ (\gamma_1 z_1^- + \gamma_3 z_3^- + \gamma_2 w_2^-)(x,\lambda), & -\epsilon_0 < x < 0. \end{cases}$$

$$W_1^+(x,\lambda) := \begin{cases} \phi_3^+(x,\lambda), & x > \epsilon_0, \\ (\alpha_1 z_1^+ + \alpha_3 z_3^+ + \alpha_2 w_2^+)(x,\lambda), & 0 < x < \epsilon_0, \\ (\beta_1 z_1^- + \beta_3 z_3^- + \beta_2 w_2^-)(x,\lambda), & -\epsilon_0 < x < 0 \\ (\delta_1 \psi_1^- + \delta_2 \psi_2^- + \delta_3 \phi_3^-)(x,\lambda), & x < -\epsilon_0. \end{cases}$$

$$W_2^+(x,\lambda) := \begin{cases} 0, & x > 0, \\ w_2^-(x,\lambda), & -\epsilon_0 < x < 0, \\ (\kappa_1 \psi_1^- + \kappa_2 \psi_2^- + \kappa_3 \phi_3^-)(x,\lambda), & x < -\epsilon_0. \end{cases}$$

w_2^- is the fast decaying mode at $x \to 0^-$.

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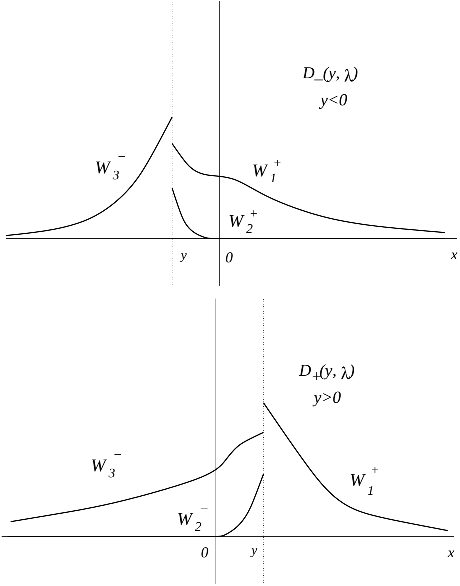

Figura: Two Evans function: D_+ for $y > 0$, and D_- for $y < 0$.

Analogously we select two modes W_2^-, W_3^- decaying at $-\infty$, and one W_1^+, decaying at $+\infty$.

We define *two* Evans functions:

$$D_\pm(y,\lambda) := \det\big(W_1^+ \; W_2^\mp \; W_3^-\big)(y,\lambda), \qquad \text{for } y \gtrless 0,$$

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Properties

(i) For $\lambda \sim 0$

$$D_\pm(y,\lambda) = -a(y)^{-1}\lambda[u]\det \begin{pmatrix} q_1^+ & q_2^\mp \\ p_1^+ & p_2^\mp \end{pmatrix}_{|\lambda=0} + \mathcal{O}(|\lambda|^2),$$

where $[u] = u_+ - u_-$.

(ii) We define

$$D_\pm(\lambda) := D_\pm(\pm 1, \lambda).$$

Then, $D_+(\lambda) = mD_-(\lambda) + \mathcal{O}(|\lambda|^2)$, where $m \neq 0$.

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Resolvent kernel bounds for $\lambda \sim 0$

W.l.o.g. $y < 0$. Jump conditions in $x = y$:

$$[\mathcal{G}_\lambda(\cdot, y)] = \begin{pmatrix} a(y)^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\mathcal{G}_\lambda(x, y)$ is constructed in terms of decaying solutions:

$$\mathcal{G}_\lambda(x, y) = \begin{cases} W_1^+(x,\lambda)C_1^+(y,\lambda) + W_2^+(x,\lambda)C_2^+(y,\lambda), & x > y, \\ -W_3^-(x,\lambda)C_3^-(y,\lambda), & x < y \end{cases}$$

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By Cramer's rule:

$$C_{11}^{+}(y,\lambda) = a(y)^{-1} D_{-}(y,\lambda)^{-1} \begin{vmatrix} q_2^{+} & q_3^{-} \\ p_2^{+} & p_3^{-} \end{vmatrix} (y,\lambda),$$

$$C_{21}^{+}(y,\lambda) = a(y)^{-1} D_{-}(y,\lambda)^{-1} \begin{vmatrix} q_3^{-} & q_1^{+} \\ p_3^{-} & p_1^{+} \end{vmatrix} (y,\lambda),$$

$$C_{31}^{-}(y,\lambda) = a(y)^{-1} D_{-}(y,\lambda)^{-1} \begin{vmatrix} q_1^{+} & q_2^{+} \\ p_1^{+} & p_2^{+} \end{vmatrix} (y,\lambda).$$

The only coefficients with possible jumps are in the first column.

Bounds for $y \sim 0$.

Lemma

(i) For $y \sim 0$

$$C_1^+(y,\lambda) = \frac{1}{\lambda}[u]^{-1}(1,\ -L,\ 0) + \mathcal{O}(1),$$
$$C_3^-(y,\lambda) = -\frac{1}{\lambda}[u]^{-1}(1,\ -L,\ 0) + \mathcal{O}(1),$$
$$C_2^+(y,\lambda) = a(y)^{-1}|y|^{-\nu}\mathcal{O}(1).$$

(ii) Under (A0) - (A5$_k$), $y < 0$, near zero,

$$\mathcal{G}_\lambda(x,y) = \lambda^{-1}[u]^{-1}\bar{W}'(1,\ -L,\ 0) + \mathcal{O}(e^{-\eta|x|}), \quad y < 0 < x,$$

$$\mathcal{G}_\lambda(x,y) = \lambda^{-1}[u]^{-1}\bar{W}'(1,\ -L,\ 0) + \mathcal{O}(1)\left(1 + \frac{|x|^\nu}{a(y)|y|^\nu}\right), \quad y < x < 0$$

$$\mathcal{G}_\lambda(x,y) = \lambda^{-1}[u]^{-1}\bar{W}'(1,\ -L,\ 0) + \mathcal{O}(e^{-\eta|x|}), \quad x < y < 0,$$

for some $\eta > 0$. The $y > 0$ case is analogous. $\bar{W}' =$ derivative of the profile.

Bounds for $y \to -\infty$.

Lemma

Under (A0) - (A5$_k$), $y < 0$, for $|y|$ large,

$$\begin{aligned}\mathcal{G}_\lambda(x,y) &= \lambda^{-1}[u]^{-1}e^{-\mu_2^- y}\bar{W}'(1,\ -L,\ 0)\\ &\quad + \mathcal{O}((e^{-\mu_2^- y} + e^{-\mu_1^- y})e^{\mu_3^+ x}), \quad y < 0 < x,\end{aligned}$$

$$\begin{aligned}\mathcal{G}_\lambda(x,y) &= \lambda^{-1}[u]^{-1}e^{-\mu_2^- y}\bar{W}'(1,\ -L,\ 0)\\ &\quad + \mathcal{O}(e^{\mu_1^- (x-y)}) + \mathcal{O}(e^{\mu_2^- (x-y)}) + \mathcal{O}(e^{-\mu_2^- y}e^{\mu_3^- x}), \quad y < x < 0,\end{aligned}$$

$$\begin{aligned}\mathcal{G}_\lambda(x,y) &= -\lambda^{-1}[u]^{-1}e^{-\mu_2^- y}\bar{W}'(1,\ -L,\ 0)\\ &\quad + \mathcal{O}(e^{-\mu_2^- y}e^{\mu_3^- x}) + \mathcal{O}(e^{\mu_3^- (x-y)}), \quad x < y < 0.\end{aligned}$$

The $y > 0$ case is analogous.

Low frequency estimates

"Low frequency" Green function

$$G^I(x,t;y) := \frac{1}{2\pi i} \int_{\Gamma \bigcap \{|\lambda| \leq r\}} e^{\lambda t} \mathcal{G}_\lambda(x,y) d\lambda$$

Γ = contour near $\lambda = 0$, away from essential spectrum, $0 < r \ll 1$ small such that the bounds for G_λ hold.

Lemma

Under (A0) - (A5$_k$), we have the decomposition for $y < 0$,

$$G^I(x,t;y) = E + \widetilde{G}^I + R,$$

$$E(x,t;y) := \bar{U}_x(x)[u]^{-1} e(y,t),$$

$$e(y,t) := \left(\operatorname{errfn}\left(\frac{y + a_- t}{\sqrt{4 L\, b_-\, t}}\right) - \operatorname{errfn}\left(\frac{y - a_- t}{\sqrt{4 L\, b_-\, t}}\right)\right);$$

$$|\partial_x^\kappa \partial_y^\beta \widetilde{G}^I(x,t;y)| \leq C_1\, t^{-(|\beta|+|\kappa|)/2-1/2} e^{-(x-y-a_- t)^2/C_2\, t},$$

$$R(x,t;y) = \mathcal{O}(e^{-\eta(|x-y|+t)}) + \mathcal{O}(e^{-\eta t})\chi(x,y)\left[1 + \frac{1}{a(y)}(x/y)^\nu\right],$$

for some $\eta, C_1, C_2 > 0$, where $\beta, \kappa = 0, 1$ and $\nu = \frac{Lb(0)+a'(0)}{|a'(0)|}$ and

$$\chi(x,y) = \begin{cases} 1 & -1 < y < x < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Symmetric bounds for $y \geq 0$.

With this decomposition and bounds we prove:

Lemma

Under (A0) - (A5$_k$), for $1 \leq q \leq p \leq +\infty$,

$$\Big| \int_{-\infty}^{+\infty} \partial_y^\beta \widetilde{G}^I(\cdot, t; y) f(y)\, dy \Big|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1/q - 1/p) - |\beta|/2} |f|_{L^q},$$

$$\left. \begin{array}{r} |e_y(\cdot,t)|_{L^p}, |e_t(\cdot,t)|_{L^p}, \leq Ct^{-\frac{1}{2}(1-1/p)}, \\ |e_{yt}(\cdot,t)|_{L^p} \leq Ct^{-\frac{1}{2}(1-1/p)-1/2}. \end{array} \right\}, \quad t > 0, C > 0, p \geq 1.$$

$$\Big| \int_{-\infty}^{+\infty} R(\cdot, t; y) f(y)\, dy \Big|_{L^p} \leq Ce^{-\eta t}(|f|_{L^p} + |f|_{L^\infty}),$$

High frequencies

$$\mathcal{S}_2(t) = \frac{1}{2\pi i} \int_{-\gamma_1 - i\infty}^{-\gamma_1 + i\infty} \chi_{\{|\mathrm{Im}\,\lambda| \geq \gamma_2\}} e^{\lambda t} (\lambda - \mathcal{L})^{-1} d\lambda,$$

Small constants $\gamma_1, \gamma_2 > 0$, χ_I = characteristic function.

Linear problem

$$u_t + (a(x)u)_x + Lq_x = \varphi,$$
$$-q_{xx} + q + (b(x)u)_x = \psi,$$

is recast as

$$u_t + (a(x)u)_x + \mathcal{J}u = \varphi - L\,\partial_x(\mathcal{K}\psi),$$
$$u(x,0) = u_0(x)$$

$$\mathcal{J}u := -L\,\partial_x \mathcal{K}\,\partial_x(b(x)u), \quad \mathcal{L} := -(a(x)u)_x - \mathcal{J}u.$$

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High frequency bounds

$$|(\lambda - \mathcal{L})^{-1}(\varphi - L\partial_x(\mathcal{K}\psi))|_{H^1} \leq C\Big(|\varphi|^2_{H^1} + |\psi|^2_{L^2}\Big),$$

$$|(\lambda - \mathcal{L})^{-1}(\varphi - L\partial_x(\mathcal{K}\psi))|_{L^2} \leq \frac{C}{|\lambda|^{1/2}}\Big(|\varphi|^2_{H^1} + |\psi|^2_{L^2}\Big),$$

under (A0) - (A5$_k$), $R, C > 0$ large, $\gamma > 0$ small, *and for all* $|\lambda| > R$, *Re* $\lambda \geq -\gamma$.

Mid-frequency bounds

$$|(\lambda - \mathcal{L})^{-1}\varphi|_{L^2} \leq C\,|\varphi|_{H^1} \quad \text{for } R^{-1} \leq |\lambda| \leq R \text{ and Re } \lambda \geq -\gamma,$$

R and $C = C(R)$ large, and $\gamma = \gamma(R)$ small.

High frequency bounds

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R and $C = C(R)$ large, and $\gamma = \gamma(R)$ small.

Lemma

Under (A0) - (A5$_k$), we have the bounds

$$|\partial_x^\kappa \mathcal{S}_2(t)(\varphi - L\,\partial_x(\mathcal{K}\psi))|_{L^2} \leq Ce^{-\eta_1 t}\Big(|\psi|_{H^{\kappa+2}} + |\varphi|_{H^{\kappa+2}}\Big), \;\; \kappa = 0, 1,$$

for some $\eta > 0$.

Nonlinear perturbation

$$\begin{pmatrix} u \\ q \end{pmatrix}(x,t) := \begin{pmatrix} \tilde{u} \\ \tilde{q} \end{pmatrix}(x+\alpha(t),t) - \begin{pmatrix} U \\ Q \end{pmatrix}(x),$$

$$u_t + (a(x)\,u)_x + L\,q_x = N_1(u)_x + \dot{\alpha}(t)\,(u_x + U_x),$$
$$-q_{xx} + q + (b(x)\,u)_x = N_2(u)_x,$$

$$N_j(u) = O(|u|^2)$$

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$$N_j(u) = O(|u|^2)$$

Auxiliar estimate

Lemma

Under (A0) - (A5$_k$), if $|u|_{W^{2,\infty}}$ *y* $|\dot{\alpha}|$ *remain small,*

$$|u|^2_{H^k}(t) \leq Ce^{-\eta t}|u|^2_{H^k}(0)+C\int_0^t e^{-\eta(t-s)}(|u|^2_{L^2}+|\dot{\alpha}|^2)(s)\,ds, \quad \eta > 0,$$

for $k = 1, \ldots, 4$.

Crucial: $Lb > 0$, uniformly. For systems it is not trivial!

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Under (A0) - (A5$_k$), if $|u|_{W^{2,\infty}}$ y $|\dot{\alpha}|$ remain small,

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Crucial: $Lb > 0$, uniformly. For systems it is not trivial!

Green function decomposition

$$G(x,t;y) = G^I(x,t;y) + G^{II}(x,t;y)$$

$$\widetilde{G}^I(x,t;y) = G^I(x,t;y) - E(x,t;y) - R(x,t;y)$$

$$\widetilde{G}^{II}(x,t;y) = G^{II}(x,t;y) + R(x,t;y).$$

From previous estimates:

$$\Big| \int_{-\infty}^{+\infty} \partial_y^\beta \widetilde{G}^I(\cdot,t;y) f(y)\,dy \Big|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1/q-1/p)-|\beta|/2} |f|_{L^q},$$

for $1 \leq q \leq p, \beta = 0,1,$

$$\Big| \int_{-\infty}^{+\infty} \widetilde{G}^{II}(x,t;y) f(y)\,dy \Big|_{L^p} \leq C e^{-\eta t} |f|_{H^3},$$

for $2 \leq p \leq \infty$.

Duhamel principle

Integral representation for the perturbation:

$$
\begin{aligned}
u(x,t) &= \int_{-\infty}^{+\infty} (\widetilde{G}^I + \widetilde{G}^{II})(x,t;y)u_0(y)dy \\
&- \int_0^t \int_{-\infty}^{+\infty} \widetilde{G}^I_y(x,t-s;y)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y N_2(u) + \dot{\alpha}\,u\Big)(y,s)\,dy\,ds \\
&+ \int_0^t \int_{-\infty}^{+\infty} \widetilde{G}^{II}(x,t-s;y)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y N_2(u) + \dot{\alpha}\,u\Big)_y(y,s)\,dy\,ds \\
q(x,t) &= (\mathcal{K}\partial_x)(N_2(u) - b\,u)(x,t),
\end{aligned}
$$

$$
\begin{aligned}
\alpha(t) = &-\int_{-\infty}^{+\infty} e(y,t)u_0(y)dy \\
&+ \int_0^t \int_{-\infty}^{+\infty} e_y(y,t-s)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y N_2(u) + \dot{\alpha}\,u\Big)(y,s)\,dy
\end{aligned}
$$

$$
\begin{aligned}
\dot{\alpha}(t) = &-\int_{-\infty}^{+\infty} e_t(y,t)u_0(y)dy \\
&+ \int_0^t \int_{-\infty}^{+\infty} e_{yt}(y,t-s)\Big(N_1(u) - L\,\mathcal{K}\,\partial_y N_2(u) + \dot{\alpha}\,u\Big)(y,s)\,d
\end{aligned}
$$

Theorem

Under (A0) - (A5$_k$), and spectral stability assumption, the profile (U, Q) is asymptotically orbitally stable. The solution to the nonlinear problem with initial data $\tilde{u}_0$ satisfies

$$|\tilde{u}(x,t) - U(x - \alpha(t))|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1-1/p)}|u_0|_{L^1 \cap H^4}$$
$$|\tilde{u}(x,t) - U(x - \alpha(t))|_{H^4} \leq C(1+t)^{-1/4}|u_0|_{L^1 \cap H^4}$$

$u_0 := \tilde{u}_0 - U$ sufficiently small in $L^1 \cap H^4$, $p \geq 2$, with $\alpha(t)$ such that $\alpha(0) = 0$

$$|\alpha(t)| \leq C|u_0|_{L^1 \cap H^4}, \qquad |\dot{\alpha}(t)| \leq C(1+t)^{-1/2}|u_0|_{L^1 \cap H^4}.$$

More details in:

C. Lattanzio, C. Mascia, T. Nguyen, R. G. P, K. Zumbrun, *SIAM J. Math. Anal.* **41**, no. 6 (2009).

Hyperbolic-elliptic systems

$$\begin{aligned} u_t + f(u)_x + Lq_x &= 0, \\ -q_{xx} + q + g(u)_x &= 0, \qquad (x,t) \in \mathbb{R} \times [0,+\infty), \end{aligned} \tag{HE}$$

$$\begin{aligned} \mathbb{R}^n \supseteq \mathcal{U} \ni u &- \text{state variables, } n \geq 1, \\ \mathbb{R} \ni q &- \text{general heat flux function,} \\ \mathbb{R}^{n\times 1} \ni L &- \text{constant vector (column),} \\ f \in C^2(\mathcal{U};\mathbb{R}^n) &- \text{flux function,} \\ g \in C^2(\mathcal{U};\mathbb{R}) &- \text{non-linear coupling.} \end{aligned}$$

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$$
\begin{aligned}
A(u) &:= Df(u) \in \mathbb{R}^{n\times n}, \\
B(u) &:= Dg(u) \in \mathbb{R}^{1\times n}, \qquad u \in \mathcal{U}.
\end{aligned}
$$

Hyperbolicity: eigenvalues of A, real, semi-simple,

$$
a_1 \leq \cdots \leq a_n.
$$

Eigenvectors associated to a_j,

$$
Ar_j = a_j r_j, \qquad l_j A_j = a_j l_j.
$$

Structural hypotheses

$$f, g \in C^2 \quad \text{(regularity)}, \tag{S0}$$

For each $u \in \mathcal{U}$ there exists A_0 symmetric, positive definite, such that $A_0 A$ symmetric, $A_0 LB$ symmetric, positive semi-definite of rank one. Moreover, principal eigenvalue a_p of A, $1 \leq p \leq n$, is simple. (S1)

No eigenvector of A lies in $\ker LB$ (genuine coupling). (S2)

Theorem (Kawashima-Shizuta)

Under (S0) y (S1), condition (S2) is equivalent to the existence of a skew-symmetrix $K : \mathcal{U} \to \mathbb{R}^{n\times n}$ *such that*

$$\mathrm{Re}\left(KA + A_0LB\right) > 0, \tag{K}$$

for all $u \in \mathcal{U}$.

Shock profiles

Traveling wave solutions

$$u(x,t) = U(x-st), \qquad q(x,t) = Q(x-st),$$

$$U(x) \to u_\pm, \qquad Q(x) \to 0, \qquad \text{if } \; x \to \pm\infty,$$

$u_\pm \in \mathcal{U} \subseteq \mathbb{R}^n$ constant states $u_- \neq u_+$, $s \in \mathbb{R}$ shock speed. The triple (u_+, u_-, s) is a front (weak solution) of the underlying system of conservation laws: $u_t + f(u)_x = 0$. It satisfies Rankine-Hugoniot:

$$f(u_+) - f(u_-) - s(u_+ - u_-) = 0,$$

plus Lax entropy conditions.

Traveling wave equations:

$$\begin{aligned} f(U)_x + LQ_x &= 0, \\ -Q_{xx} + Q + g(U)_x &= 0. \end{aligned}$$

W.l.o.g. $s = 0$ (stationary wave).

Hypotheses on the shock:

$$f(u_+) = f(u_-), \qquad \text{(Rankine-Hugoniot)}, \qquad \text{(H0)}$$

$$\begin{aligned} a_p(u_+) < 0 < a_{p+1}(u_+), \\ a_{p-1}(u_-) < 0 < a_p(u_-), \end{aligned} \qquad \text{(Lax entropy condition)}, \qquad \text{(H1)}$$

$$(\nabla a_p)^\top r_p \neq 0, \quad \text{for all } u \in \mathcal{U}, \qquad \text{(genuine nonlinearity)}, \qquad \text{(H2)}$$

$$l_p(u_\pm) L B(u_\pm) r_p(u_\pm) > 0, \qquad \text{(positive diffusion)}. \qquad \text{(H3)}$$

Eliminating the q variable:

$$u_t + f(u)_x = (LB(u)u_x)_x + (u_t + f(u)_x)_{xx},$$

Positive diffusion hypothesis (H3):

$$l_p \cdot (B \otimes L^\top r_p) > 0,$$

It provides the positive along the p-characteristic field in the Chapman-Enskog expansion.

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Stability results

- T. NGUYEN, R. G. P, K. ZUMBRUN, *Phys. D* **239**, no. 8 (2010).

Theorem 1 (Spectral stability)
Under (S0) - (S2), (H0) - (H3), radiative shock profiles are spectraly stable for $\epsilon = |u_+ - u_-|$ *sufficiently small.*

Theorem 2 (Nonlinear orbital stability)
Under (S0) - (S2), (H0) - (H3) and $\epsilon = |u_+ - u_-|$ sufficiently small, radiative shock profiles are nonlinear orbitally stable, that is, the solution (u, q) to system (HE) with initial data u_0 satisfies

$$|\tilde{u}(x,t) - U(x - \alpha(t))|_{L^p} \leq C(1+t)^{-\frac{1}{2}(1-1/p)}|u_0|_{L^1\cap H^4},$$
$$|\tilde{q}(x,t) - Q(x - \alpha(t))|_{W^{1,p}} \leq C(1+t)^{-\frac{1}{2}(1-1/p)}|u_0|_{L^1\cap H^4},$$

provided that $u_0 - U$ is sufficiently small in $L^1 \cap H^4$, $p \geq 2$, and for some $\alpha(t)$ satisfying $\alpha(0) = 0$, and

$$|\alpha(t)| \leq C|u_0|_{L^1\cap H^4}$$
$$|\dot{\alpha}(t)| \leq C(1+t)^{-1/2}|u_0|_{L^1\cap H^4}.$$

Spectral problem

$$\begin{aligned} \lambda u + (Au)_x + Lq_x &= 0, \\ -q_{xx} + q + (Bu)_x &= 0. \end{aligned}$$

$$A := A(U(x)), \quad B := B(U(x))$$

$$u, q \in L^2$$

Zero-mass conditions

$$\int u\,dx = 0, \qquad \int q\,dx = 0,$$

Equivalent spectral problem:

$$\lambda u + Au_x + Lq_x = 0,$$
$$-q_{xx} + q + Bu_x = 0.$$

$u_\pm \in \mathcal{N}(u_*)$, open neighborhood.

$$0 < \max_{u \in \mathcal{N}} |u - u_*| \leq \epsilon \ll 1, \quad |u_* - u_\pm|, |u_- - u_+| = \mathcal{O}(\epsilon).$$

"Scalar" structure of the profile:

$$U_x = \mathcal{O}(\epsilon^2) e^{-\eta\epsilon|x|}(r_p(u_*) + \mathcal{O}(\epsilon)),$$
$$U_{xx} = \mathcal{O}(\epsilon^3) e^{-\theta\epsilon|x|},$$

$\theta, \eta > 0$. Principal characteristic speed: $a_p := a_p(U(x))$,

$$(a_p)_x = \mathcal{O}(U_x) < 0, \quad \text{(monotonicity)},$$
$$(a_p)_{xx} = \mathcal{O}(U_{xx}).$$

Lemma

Under (S0) - (S2) there exists $\beta = \beta(u) > 0$, such that

$$(A_0 L)^\top = \beta B, \quad \forall u \in \mathcal{U}.$$

Basic Friedrichs-type estimate

If u, q solutions with Re $\lambda \geq 0$, then for $\epsilon \ll 1$ sufficiently small,

$$(\text{Re }\lambda)|u|_{L^2}^2 + |q|_{L^2}^2 + |q_x|_{L^2}^2 \leq C \int |U_x||u|^2\,dx,$$

$$|\text{Im }\lambda| \int |U_x||u|^2\,dx \leq C \int |U_x|\big(\delta|u|^2 + \delta^{-1}|q|^2\big)\,dx,$$

for some $C > 0$, any $\delta > 0$.

Corollary:

$$0 \leq \text{Re }\lambda \leq C\epsilon^2,$$

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Kawashima-type estimate

For $0 < \epsilon \ll 1$ small and $\mathrm{Re}\,\lambda \geq 0$, there is $C > 0$ such that

$$(\mathrm{Re}\,\lambda)|u|_{L^2}^2 + |u_x|_{L^2}^2 \leq C \int |U_x||u|^2 \, dx \qquad \text{(KE)}$$

Basic ideas:

- Control of the $|u_x|_{L^2}^2$ term.
- L^2 weighted product with the *skew*-symmetric form K.

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Goodman-type estimate

For $0 < \epsilon \ll 1$ small and $\text{Re}\,\lambda \geq 0$, there exists $\bar{C} > 0$ such that

$$(\text{Re}\,\lambda)\big(|u|_{L^2}^2 + |u_x|_{L^2}^2\big) + \bar{C}\int |U_x||u|^2\,dx \leq \bar{C}\epsilon|u_x|_{L^2}^2 \qquad \text{(GE)}$$

Basic ideas:

- Control of the $\int |U_x||u|^2$ term.
- Weighted norms in the characteristic direction.
- Diagonalization of the hyperbolic part along the whole trajectory of the profile.

Goodman-type estimate

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Basic ideas:

- Control of the $\int |U_x||u|^2$ term.
- Weighted norms in the characteristic direction.
- Diagonalization of the hyperbolic part along the whole trajectory of the profile.

Adding $\bar{C}\epsilon$ times (GE) to (KE):

$$(\mathrm{Re}\ \lambda)(1+\bar{C}\epsilon)|u|_{L^2}^2 + (\bar{C} + C\bar{C}\epsilon)\int |U_x||u|^2\,dx \leq 0.$$

$$\Longrightarrow \quad \mathrm{Re}\ \lambda < 0, \quad \text{i.e., } \textit{spectral stability}.$$

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$$\Longrightarrow \; \mathrm{Re}\,\lambda < 0, \text{ i.e., } \textit{spectral stability}.$$

Sketch of the Kawashima-type estimate

Notation:

$$\bar{A} := A_0 A((U(x)), \quad \bar{L} := A_0(U(x))L,$$

$$K := K(u(x)), \quad \beta = \beta(U(x)),$$

$$\beta_x, \bar{L}_x, \bar{A}_x, K_x = \mathcal{O}(|U_x|) = \mathcal{O}(\epsilon^2).$$

Suffices to control the $|u_x|_{L^2}$ term for Re $\lambda \geq 0$, $\lambda \neq 0$,

$$|u_x|_{L^2}^2 \leq \bar{C}\Big((\text{Re } \lambda)\eta |u|_{L^2}^2 + \int |U_x||u|^2 \, dx\Big), \qquad (*)$$

for some $C > 0$, $\eta > 0$, such that $\epsilon^2/\eta \ll 1$.

Taking $\eta = \mathcal{O}(\epsilon)$ small and with the Friedrichs-type estimate we get (KE).

Ingredients:

- Take L^2 product of u equation with Ku_x, use K *skew*-symmetric, $\text{Im}\,\langle Ku_x, u\rangle = -\frac{1}{2}\langle K_x u, u\rangle$:

$$\text{Re}\,\langle u_x, KAu_x\rangle = \text{Re}\,(\lambda\langle Ku_x, u\rangle) + \text{Re}\,\langle Ku_x, Lq_x\rangle,$$

$$\text{Re}\,(\lambda\langle Ku_x, u\rangle) \leq C(\text{Re}\,\lambda)\big(\eta^{-1}|u_x|^2_{L^2} + \eta|u|^2_{L^2}\big) + C|\text{Im}\,\lambda| \int |U_x||u|^2\,dx$$

- $\bar{L}B$ symmetric, positive semi-definite; $\text{Re}\,(KA + \bar{L}B) > 0$:

$$\text{Re}\,\langle u_x, KAu_x\rangle + \langle u_x, \bar{L}Bu_x\rangle \geq \frac{1}{C}|u_x|^2_{L^2}$$

- L^2 product of u_x with equation for q:

$$\langle u_x, \bar{L}Bu_x\rangle = -\langle u_{xx}, \bar{L}q_x\rangle - \langle u_x, \bar{L}_x q_x\rangle - \langle u_x, \bar{L}q\rangle.$$

- Use $\bar{A}$ symmetric, $\bar{L}B$ symmetric, positive semi-definide estimate term by term:

$$\langle u_x, \bar{L}Bu_x\rangle \leq C\epsilon |u_x|_{L^2}^2 + C\int |U_x||u|^2\, dx.$$

- Substitution into the $|u_x|_{L^2}^2$ estimate. Re $\lambda = \mathcal{O}(\epsilon^2)$, $\epsilon^2/\eta \ll 1$ small. The result is (*).

Sketch of the Goodman-type estimate

Lemma (Goodman)
There exist smooth matrix field $\mathbb{R}(u), \mathbb{L}(u)$ such that

$$\mathbb{L}A\mathbb{R} = \begin{pmatrix} A_- & & \\ & a_p & \\ & & A_+ \end{pmatrix}$$

where $A_\pm$ are symmetric, $A_- \leq \delta < 0$, $A_+ \geq \delta > 0$. If $\mathbb{L} = \mathbb{L}(U), \mathbb{R} = \mathbb{R}(U)$,

$$(\mathbb{L}\mathbb{R}_x)_{pp} = (\mathbb{L}_x\mathbb{R})_{pp} = 0,$$

$$\mathbb{L}LB\mathbb{R} \geq -C\epsilon$$

$$\mathbb{R} = \Gamma\check{\mathbb{R}}, \quad \mathbb{L} = \Gamma^{-1}\check{\mathbb{L}},$$

$$\check{\mathbb{R}} = (A_0)^{1/2}O^\top, \quad \check{L} = O(A_0)^{1/2},$$

O orthogonal, real, block-diagonalizes $(A_0)^{1/2}A(A_0)^{-1/2}$,

$$\Gamma = \begin{pmatrix} I_{p-1} & & \\ & \alpha & \\ & & I_{n-p} \end{pmatrix}$$

α solves the ODE

$$\alpha_x = -\check{l}_p(\check{r}_p)_x\alpha, \qquad \alpha(0) = 1.$$

$$\alpha = e^{\int_0^x -\check{l}_p(\check{r}_p)_x} = e^{\mathcal{O}(\int |U_x|)} = 1 + \mathcal{O}(\epsilon).$$

From (H3): $l_p^\pm L B^\pm r_p^\pm > 0$, by continuity, $U \sim u_\pm$,

$$(l_p L B r_p)_{|u=U} > 0.$$

$$\mathrm{Re}\, \mathbb{L} L B \mathbb{R} \geq -C\epsilon,$$

$$(\mathbb{L} L B \mathbb{R})_{pp} \geq \theta > 0$$

$$\tilde{A}(x) := (\mathbb{L}A\mathbb{R})(U(x)) = \begin{pmatrix} A_- & & \\ & a_p & \\ & & A_+ \end{pmatrix}$$

$$\tilde{L}(x) := \mathbb{L}(U(x))L, \quad \tilde{B}(x) := B(U(x))\mathbb{R}(U(x)),$$

$$v := \mathbb{L}u,$$

$$\begin{aligned} \lambda v + \tilde{A}v_x + \tilde{L}q_x &= \tilde{A}\mathbb{L}_x\mathbb{R}v, \\ -q_{xx} + q + \tilde{B}v_x &= -B\mathbb{R}_x v \end{aligned}$$

$$\tilde{A}(x) := (\mathbb{L}A\mathbb{R})(U(x)) = \begin{pmatrix} A_- & & \\ & a_p & \\ & & A_+ \end{pmatrix}$$

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Weighted norms:

$$W := \begin{pmatrix} w_- I_{p-1} & & \\ & w_p & \\ & & w_+ I_{n-p} \end{pmatrix}$$

$w_p, w_\pm$ scalar functions:

$$w_p \equiv 1,$$

$$(w_\pm)_x = -c_* |U_x| w_\pm / a_\pm, \quad w_\pm(0) = 1.$$

$$\Rightarrow \; w_\pm = \exp\Big(\int_0^x c_* |U_x| / \bar{a}_\pm \Big) = 1 + \mathcal{O}(\epsilon),$$

$$(w_\pm)_x = \mathcal{O}(|U_x|), \; (w_p)_x = 0.$$

$c_* > 0$ sufficiently large, such that

$$W\tilde{A}_x + W_x\tilde{A} \leq C \begin{pmatrix} -c_* I_p & & \\ & -\theta & \\ & & -c_* I_{n-p} \end{pmatrix}, \quad C > 0.$$

Ingredients:

- L^2 product of Wv versus the equation for v; integrating by parts, $\tilde{A}$ symmetric, $v := (v_-, v_p, v_+)^\top$, previous inequality:

$$(\text{Re } \lambda)|v|^2_{L^2} + \tfrac{c_*}{2}\langle v_\pm, |U_x| v_\pm\rangle + \tfrac{1}{2}\theta\langle v_p, |U_x| v_p\rangle + \\ + \text{Re } \langle Wv, \tilde{L} q_x\rangle = \text{Re } \langle Wv, \tilde{A}\mathbb{L}_x\mathbb{R}v\rangle.$$

Bound $\text{Re } \langle Wv, \tilde{A}\mathbb{L}_x\mathbb{R}v\rangle \leq C\langle v_\pm, |U_x| v_\pm\rangle + C\epsilon\langle v_p, |U_x| v_p\rangle$; and taking c_* large, $\epsilon \ll 1$,

$$(\text{Re } \lambda)|v|^2_{L^2} + C\int |U_x||v|^2 \leq -\text{Re } \langle Wv, \tilde{L} q_x\rangle$$

- Control of the product of v with q_x; L^2 product of Wv_x with $\tilde{L}$ times the equation for la q, integrate by parts,

$$-\langle Wv, \tilde{L}q_x \rangle = -\langle Wv_x, \mathbb{L}LB\mathbb{R}_x v\rangle - \langle Wv_x, \tilde{L}\tilde{B}v_x\rangle + \\ + \langle Wv_x, \tilde{L}q_{xx}\rangle + \langle Wv, \tilde{L}_x q\rangle + \langle W_x v, \tilde{L}q\rangle.$$

- Bound all the terms, in particular

$$\mathrm{Re}\ \langle Wv_{xx}, \tilde{L}q_x\rangle \leq C\epsilon \int |U_x||v|^2 + C\epsilon^2 |v_x|^2_{L^2},$$

one gets

$$-\mathrm{Re}\ \langle Wv, \tilde{L}q_x\rangle \leq C\epsilon \int |U_x||v|^2 + C\epsilon |v_x|^2_{L^2}$$

Combining with last estimate, back into the u variables, we obtain (GE).

- Control of the product of v with q_x; L^2 product of Wv_x with $\tilde{L}$ times the equation for la q, integrate by parts,

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Thank you!