

Lección 3.8 : Demostración del teorema de representación de Lax.

$$f \in C^2(\Omega; \mathbb{R}^n), \quad A(u) = Df(u) \in C^1(\Omega; \mathbb{R}^{n \times n})$$

$$\begin{aligned} f(u) - f(u_0) &= \int_0^1 \frac{d}{ds} (f(u_0 + s(u - u_0))) ds \\ &= \int_0^1 A(u_0 + s(u - u_0))(u - u_0) ds \\ &= \underbrace{\left[\int_0^1 A(u_0 + s(u - u_0)) ds \right]}_{\in \mathbb{R}^{n \times n}} (u - u_0) \end{aligned}$$

Para cada $u, v \in \Omega$, $s \in (0, 1)$ tal que $u + s(u - v) \in \Omega$ definimos

$$A(u, v) := \int_0^1 A(u + s(v - u)) ds$$

Rankine-Hugoniot se puede expresar como un problema espectral :

$$A(u_0, u)(u - u_0) = \sigma(u_0, u)(u - u_0)$$

$$\Rightarrow \begin{array}{ll} \sigma(u_0, u) & \text{valor propio de } A(u_0, u) \\ u - u_0 & \text{vector " " " " " "} \end{array}$$

$A(u_0) = A(u_0, u_0)$ tiene :

- $\lambda_p(u_0) \in \mathbb{R}$ valor propio simple
- base $r_j(u_0), l_j(u_0)$ con
 $l_j(u_0) r_p(u_0) = 0$ si $j \neq p$.
 vectores propios.

$A(u)$ hiperbólica - $\lambda_p(u)$ valor propio simple $\forall u \in \Omega$.

por continuidad del mapeo

$$u \mapsto A(u_0, u) \in C^1(\Omega; \mathbb{R}^{n \times n})$$

existe una vecindad $N(u_0) \subset \Omega$ tal que la matriz

$A(u_0, u)$ es hiperbólica en $u \in N(u_0)$

y con bases completas de vectores propios

$$l_j(u_0, u), \quad r_j(u_0, u), \quad 1 \leq j \leq n$$

asociados a valores propios

$$\lambda_1(u_0, u) \leq \dots \leq \lambda_{p-1}(u_0, u) < \lambda_p(u_0, u) < \lambda_{p+1}(u_0, u) \leq \dots \leq \lambda_n(u_0, u)$$

$$\forall u \in N(u_0).$$

$\Rightarrow \lambda_p(u_0, u)$ valor propio simple de $A(u_0, u)$, $\forall u \in N(u_0)$.

Además,
$$l_j(u_0, u) r_p(u_0, u) = 0$$

$$\forall j \neq p, \\ \forall u \in N(u_0).$$

De esta forma, un estado

$$u \in N(u_0) \cap H(u_0)$$

con velocidad de conexión $\sigma(u_0, u) \sim \lambda_p(u_0)$ debe satisfacer

$$(1) \dots \begin{cases} u - u_0 = \theta r_p(u_0, u) \\ \sigma(u_0, u) = \lambda_p(u_0, u) \end{cases}$$

para algún $\theta \in \mathbb{R}$.

Definimos ahora

$$(2) \dots \begin{cases} G_p(u, \theta) := u - u_0 - \theta r_p(u_0, u) \\ \forall u \in N(u_0), \forall \theta \in \mathbb{R}. \end{cases}$$

Notamos que :

- $G_p(u_0, 0) = 0$
- $D_u G_p(u_0, 0) = I$

Por el teorema de la función implícita existen una vecindad de $\theta = 0$,

$$N_\theta = (-\theta_0, \theta_0), \quad 0 < \theta_0 \ll 1,$$

una vecindad $\tilde{N}_u(u_0) \subset N(u_0) \subset \Omega$, y n funciones de clase C^2 :

$$\begin{cases} u(\theta) := (u_1(\theta), \dots, u_n(\theta))^T \\ u(\theta) : N_\theta \rightarrow \tilde{N}_u(u_0) \subset N(u_0) \end{cases}$$

tales que $\cdot u(\theta)|_{\theta=0} = u(0) = u_0$

$\cdot G_p(u(\theta), \theta) = 0$ para $\theta \in N_\theta$

$G_p(u, \theta) = 0$, con $\theta \in N_\theta$
ssi $u = u(\theta)$.

Denotamos

$$\begin{aligned} S_p(u_0) &= \{ u(\theta) \in \Omega : \theta \in (-\theta_0, \theta_0) \} \\ &\subset \tilde{N}_u(u_0) \subset N(u_0). \quad \dots (3) \end{aligned}$$

Dado que $G_p(u(\theta), \theta) = 0 \quad \forall \theta \sim 0$ tenemos

$$u(\theta) - u_0 = \theta r_p(u_0, u(\theta))$$

$\Rightarrow u(\theta) - u_0 \perp q_j(u_0, u(\theta))$ con $j \neq p$

Esto implica que

$$f(u(\theta)) - f(u_0) = A(u_0, u(\theta)) (u(\theta) - u_0)$$

$$= \lambda_p(u_0, u(\theta)) (u(\theta) - u_0)$$

$$= \sigma(u_0, u(\theta)) (u(\theta) - u_0)$$

$$=: \sigma(\theta)$$

$$\Rightarrow S_p(u_0) \subset T(u_0).$$

Reparametrización :

$$\cdot = \frac{d}{d\theta} \quad (\text{notación})$$

Derivando obtenemos : $f(u(\theta)) - f(u_0) = \sigma(\theta)(u(\theta) - u_0)$

$$A(u(\theta))\dot{u}(\theta) = \dot{\sigma}(\theta)(u(\theta) - u_0) + \sigma(\theta)\dot{u}(\theta) \quad \dots \quad (4)$$

$$\Rightarrow A(u_0)\dot{u}(0) = \sigma(0)\dot{u}(0)$$

Asimismo, $u(\theta) - u_0$ es ortogonal a $l_j(u_0, u(\theta))$, $\forall \theta \sim 0$

$$\Rightarrow l_j(u_0, u(\theta)) \frac{(u(\theta) - u_0)}{\theta} = 0$$

$$\forall j \neq p, \\ \forall \theta \neq 0, \theta \sim 0.$$

Tomando el límite cuando $\theta \rightarrow 0$

$$l_j(u_0)\dot{u}(0) = 0 \quad \forall j \neq p.$$

$$\Rightarrow \begin{cases} \dot{u}(0) = \alpha r_p(u_0), \\ \sigma(0) = \lambda_p(u_0) \end{cases} \quad \text{para cierto } \alpha \in \mathbb{R}.$$

Podemos tomar
w.l.o.g. $\alpha \equiv 1$

Sea ahora :

$$\dot{A}(u(\theta)) = D^2 f(u(\theta)) \dot{u}(\theta) \in \mathbb{R}^{n \times n}$$

Derivando (4) :

$$\begin{aligned} \dot{A}(u(\theta)) \dot{u}(\theta) + A(u(\theta)) \ddot{u}(\theta) \\ = \ddot{\sigma}(\theta)(u(\theta) - u_0) + 2\dot{\sigma}(\theta) \dot{u}(\theta) + \\ + \sigma(\theta) \ddot{u}(\theta) \quad \dots (6) \end{aligned}$$

Evaluable en $\theta = 0$:

$$\begin{aligned} \dot{A}(u_0) r_p(u_0) + A(u_0) \ddot{u}(0) = \\ = 2\dot{\sigma}(0) r_p(u_0) + \lambda_p(u_0) \ddot{u}(0) \quad \dots (7) \end{aligned}$$

Dado que también,

$$A(u(\theta)) r_p(u(\theta)) = \lambda_p(u(\theta)) r_p(u(\theta)), \quad \forall \theta \sim 0$$

derivamos y evaluamos en $\theta = 0$:

$$\begin{aligned} \dot{A}(u_0) r_p(u_0) + A(u_0) \dot{r}_p(u_0) = \\ \dot{\lambda}_p(u_0) r_p(u_0) + \lambda_p(u_0) \dot{r}_p(u_0) \quad \dots (8) \end{aligned}$$

donde

$$\begin{aligned} \dot{\lambda}_p(u(\theta)) &:= \frac{d}{d\theta} [\lambda_p(u(\theta))] \\ \dot{r}_p(u(\theta)) &:= \frac{d}{d\theta} [r_p(u(\theta))] \end{aligned}$$

Restando (7) de (8) :

$$\begin{aligned} A(u_0) \left(\ddot{u}(0) - \dot{r}_p(u_0) \right) &= \\ &= \lambda_p(u_0) \left[\ddot{u}(0) - \dot{r}_p(u_0) \right] + \\ &+ \left[2\dot{\sigma}(0) - \dot{\lambda}_p(u_0) \right] r_p(u_0) \quad \dots (9) \end{aligned}$$

Multiplicando (por la izq.) por $l_p(u_0)$:

$$\left[2\dot{\sigma}(0) - \dot{\lambda}_p(u_0) \right] \underbrace{\left(l_p(u_0) r_p(u_0) \right)}_{\neq 0} = 0$$

$$\Rightarrow \dot{\sigma}(0) = \frac{1}{2} \dot{\lambda}_p(u_0)$$

es decir,

$$\begin{aligned} \dot{\sigma}(0) &= \left. \frac{d}{d\theta} (\sigma(u(\theta))) \right|_{\theta=0} \\ &= \frac{1}{2} \left. \frac{d}{d\theta} (\lambda_p(u(\theta))) \right|_{\theta=0} \end{aligned}$$

$$= \frac{1}{2} D\lambda_p(u_0)^T \dot{u}(0)$$

$$= \frac{1}{2} D\lambda_p(u_0)^T r_p(u_0)$$

$$\left\{ \begin{array}{l} \neq 0 \text{ si g.n.l.} \\ \text{p-campo} \\ = 0 \text{ si p-campo} \\ \text{es lin. degen.} \end{array} \right.$$

Sustituyendo en (9) :

$$(A(u_0) - \lambda_p(u_0)I)(\ddot{u}(0) - \dot{r}_p(u_0)) = 0$$

$\Rightarrow \exists \beta \in \mathbb{R}$ tal que

$$\begin{aligned}\ddot{u}(0) - \dot{r}_p(u_0) &= \ddot{u}(0) - D r_p(u_0) \dot{u}(0) \\ &= \ddot{u}(0) - D r_p(u_0) r_p(u_0) \\ &= \beta r_p(u_0)\end{aligned}$$

Reparametrizamos mediante:

$$\theta(\varepsilon) := \varepsilon - \frac{1}{2} \beta \varepsilon^2$$

$$\cdot = \frac{d}{d\theta}, \quad ' = \frac{d}{d\varepsilon}$$

$$\Rightarrow \begin{cases} \theta'(\varepsilon) = 1 - \beta\varepsilon, & \theta(0) = 0 \\ \theta''(\varepsilon) = -\beta, & \theta'(0) = 1 \\ & \theta''(0) = -\beta \end{cases}$$

Definimos,

$$u(\varepsilon) := u(\theta(\varepsilon))$$

$$u'(\varepsilon) = (1 - \beta\varepsilon) \dot{u}(\theta(\varepsilon))$$

$$u''(\varepsilon) = -\beta \dot{u}(\theta(\varepsilon)) + (1 - \beta\varepsilon)^2 \ddot{u}(\theta(\varepsilon))$$

com $u'(0) = \dot{u}(0) = r_p(u_0)$

$$u''(0) = -\beta \dot{u}(0) + \ddot{u}(0)$$

$$\Rightarrow u''(0) + \cancel{\beta u'(0)} = D r_p(u_0) r_p(u_0) \\ = \cancel{\beta u'(0)}$$

$$\Rightarrow u''(0) = D r_p(u_0) r_p(u_0)$$

Assí, temos

$$u(\varepsilon) = u_0 + \varepsilon r_p(u_0) + \frac{1}{2} \varepsilon^2 D r_p(u_0) r_p(u_0) \\ + O(\varepsilon^3) \quad \dots (11)$$

Iguamente,

$$\sigma(\varepsilon) := \sigma(\theta(\varepsilon))$$

$$\gamma \quad \sigma(\varepsilon) = \sigma(u_0, u(\varepsilon))$$

$$= \lambda_p(u_0) + \frac{1}{2} \varepsilon D \lambda_p(u_0)^T r_p(u_0) \\ + O(\varepsilon^2) \quad \dots (12)$$

□