Lección 2.16 : Propiedades de la solución viscosa.

Problema de Cauchy:

$$\left.\begin{aligned} u_t + f(u)_x &= 0, \\ u(x,0) &= u_0(x), \end{aligned}\right\} \quad (1)$$

$x \in \mathbb{R}$, $t>0$, $f \in C^2(\mathbb{R})$, $u_0 \in L^\infty(\mathbb{R})$.

Problema viscoso:

$$\left.\begin{aligned} u_t^\varepsilon + f(u^\varepsilon)_x &= \varepsilon u_{xx}^\varepsilon, \\ u^\varepsilon(x,0) &= u_0(x), \end{aligned}\right\} \quad (2)$$

$\forall\ \varepsilon > 0$.

Propiedades de la solución viscosa:

Lema 1 Sean $u_0, \bar{u}_0 \in L^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$, que toman valores en un conjunto compacto $[a,b]$. Para cada $\varepsilon>0$ sean u^ε y $\bar{u}^\varepsilon$ las soluciones al problema viscoso (2) con condiciones iniciales $u^\varepsilon(x,0) = u_0(x)$, y $\bar{u}^\varepsilon(x,0) = \bar{u}_0(x)$, respectivamente. Entonces:

(i) $\| u^\varepsilon(\cdot,t) - \bar{u}^\varepsilon(\cdot,t) \|_{L^1} \leq \| u_0 - \bar{u}_0 \|_{L^1}$

$\forall\ t>0$ fijo.

(ii) Si $u_0(x) \leq \bar{u}_0(x)$ c.d.s. en $x \in \mathbb{R}$ entonces $u^\varepsilon(x,t) \leq \bar{u}^\varepsilon(x,t)$ c.d.s. en $(x,t) \in \mathbb{R} \times (0,\infty)$. Mas aún, los rangos de $u^\varepsilon, \bar{u}^\varepsilon \subset [a,b]$.

Demostración: Como $u_0, \bar{u}_0 \in L^\infty(\mathbb{R})$ por el Teorema 3 $\exists$ soluciones suaves de (2) que satisfacen

$$\|u^\varepsilon(\cdot,t)\|_{L^\infty} \leq \|u_0\|_{L^\infty}$$

$$\|\bar{u}^\varepsilon(\cdot,t)\|_{L^\infty} \leq \|\bar{u}_0\|_{L^\infty}$$

$\forall\, \varepsilon > 0$, $\forall\, t > 0$ fijo.

Además, $u^\varepsilon(\cdot,t), \bar{u}^\varepsilon(\cdot,t) \in L^1(\mathbb{R})$ $\forall\, t>0$ fijo, $\forall\, \varepsilon > 0$. De hecho:

$$\left.\begin{array}{l} \|u^\varepsilon(\cdot,t)\|_{L^1} \leq \|u_0\|_{L^1} \\ \|\bar{u}^\varepsilon(\cdot,t)\|_{L^1} \leq \|\bar{u}_0\|_{L^1} \end{array}\right\} \quad \cdots \quad (3)$$

Probaremos (3) para u^ε. Tenemos

$$u^\varepsilon_t = \varepsilon u^\varepsilon_{xx} - f(u^\varepsilon)_x$$

Multiplicando por $\operatorname{sgn} u^\varepsilon$ obtenemos:

$$(\operatorname{sgn} u^\varepsilon)\, u^\varepsilon_t = \varepsilon (\operatorname{sgn} u^\varepsilon)\, u^\varepsilon_{xx} - (\operatorname{sgn} u^\varepsilon) f(u^\varepsilon)_x$$

Integrando en $x \in \mathbb{R}$:

$$\int_{\mathbb{R}} (\operatorname{sgn} u^\varepsilon) u^\varepsilon_t \, dx = \frac{d}{dt} \int_{\mathbb{R}} |u^\varepsilon(x,t)| \, dx$$

$$= \frac{d}{dt} \|u^\varepsilon(\cdot,t)\|_{L^1}$$

ya que $\partial_t |u| = \partial_t \left((\operatorname{sgn} u) u \right) = (\operatorname{sgn} u) u_t$ c.d.s. como $(\operatorname{sgn} u^\varepsilon)_x = 0$ c.d.s. integrando por partes

$$\frac{d}{dt} \|u^\varepsilon(\cdot,t)\|_{L^1} = \varepsilon \int_{\mathbb{R}} (\operatorname{sgn} u^\varepsilon) u^\varepsilon_{xx} \, dx - \int_{\mathbb{R}} (\operatorname{sgn} u^\varepsilon) f(u^\varepsilon)_x \, dx$$

$$\leq 0 \qquad \cdots (4)$$

int. x partes

$$\Rightarrow \quad \|u^\varepsilon(\cdot,t)\|_{L^1} \leq \|u_0\|_{L^1}$$

Para probar (4) multiplicamos por el alisamiento de la función signo :

$$S_\delta(u) = [\eta_\delta * \operatorname{sgn}](u)$$

$$= \int_{-\delta}^{\delta} \eta_\delta(y) \operatorname{sgn}(u-y) \, dy$$

donde $\delta > 0$, $\eta_\delta(\cdot)$ - alisador de Friedrichs.

Así,

$$S_\delta(u^\varepsilon)u^\varepsilon_t = \varepsilon\, S_\delta(u^\varepsilon)u^\varepsilon_{xx} - S_\delta(u^\varepsilon)f(u^\varepsilon)_x \quad \cdots (5)$$

$$\Rightarrow \quad \partial_t\, \sigma_\delta(u^\varepsilon) = \varepsilon\Big(S_\delta(u^\varepsilon)u^\varepsilon_x\Big]_x - \varepsilon\, S_\delta'(u^\varepsilon)(u^\varepsilon_x)^2 +$$
$$- F_\delta(u^\varepsilon)_x \quad \cdots (6)$$

donde :

$$\sigma_\delta(u) := \int_0^u S_\delta(\xi)\, d\xi$$

$$F_\delta(u) := S_\delta(u)(f(u) - f(0)) \quad +$$
$$- \int_0^u S_\delta'(\xi)(f(\xi) - f(0))\, d\xi$$

En efecto :

- $\sigma_\delta'(u) = S_\delta(u) \Rightarrow \partial_t\, \sigma_\delta(u^\varepsilon) = S_\delta(u^\varepsilon)u^\varepsilon_t$
- $F_\delta(u^\varepsilon)_x = S_\delta(u^\varepsilon)f(u^\varepsilon)_x \quad +$
$$+ \cancel{S_\delta'(u^\varepsilon)(f(u^\varepsilon) - f(0))u^\varepsilon_x} \quad +$$
$$- \cancel{S_\delta'(u^\varepsilon)(f(u^\varepsilon) - f(0))u^\varepsilon_x}$$

Por definición de $S_\delta(\cdot)$ tenemos :

$$
S_\delta(u) = \begin{cases} \displaystyle\int_{-\delta}^{\delta} \eta_\delta(y)\, dy = 1, & \text{si } u > \delta \\ \displaystyle -\int_{-\delta}^{\delta} \eta_\delta(y)\, dy = -1, & \text{si } u < -\delta \\ \displaystyle\int_{-\delta}^{u} \eta_\delta(y)\, dy - \int_{u}^{\delta} \eta_\delta(y)\, dy, & \text{si } u \in (-\delta, \delta) \end{cases}
$$

Si $u \in (-\delta, \delta)$ entonces

$$
S_\delta'(u) = 2\eta_\delta(u) \geq 0
$$

En otro caso, si $u \notin (-\delta, \delta)$ entonces $S_\delta'(u) = 0$. Por lo tanto,

$$
S_\delta'(u) \geq 0 \quad \forall u.
$$

Integrando (6) en $\mathbb{R}$:

$$
\int_{\mathbb{R}} \partial_t \sigma_\delta(u^\varepsilon)\, dx = \underbrace{\varepsilon \left(S_\delta(u^\varepsilon) u_x^\varepsilon \right) \Big|_{x=-\infty}^{x=\infty}}_{=0} \underbrace{- \left(F_\delta(u^\varepsilon) \right) \Big|_{x=-\infty}^{x=\infty}}_{=0} \underbrace{- \varepsilon \int_{\mathbb{R}} S_\delta'(u^\varepsilon) (u_x^\varepsilon)^2\, dx}_{\leq 0}
$$

ya que $u^\varepsilon(\cdot, t) \in L^p(\mathbb{R}) \quad \forall t > 0$

$$
\Rightarrow \quad u^\varepsilon(\cdot, t) \to 0 \quad \text{si } |x| \to \infty
$$

y $\quad F_\delta(0) = 0 = \lim\limits_{|x|\to\infty} F_\delta(u^\varepsilon(x,t)) \quad \forall\, t>0$ fijo.

$\Rightarrow$
$$\int_{\mathbb{R}} \partial_t \sigma_\delta(u^\varepsilon)\, dx \leq 0$$

Tomando el límite cuando $\delta\to 0^+$ y por convergencia uniforme de $S_\delta \to \text{sgn}$

$$\frac{d}{dt}\|u^\varepsilon(\cdot,t)\|_{L^1} = \lim_{\delta\to 0^+}\int_{\mathbb{R}} \partial_t \sigma_\delta(u^\varepsilon)\, dx$$
$$\leq 0$$

$\Rightarrow$ (3).

Prueba de (i):

Sea $\delta>0$. Definimos:

$$\theta_\delta(w) := \begin{cases} 0, & \text{si } w\in(-\infty,0] \\ \dfrac{w^2}{4\delta}, & \text{si } w\in(0,2\delta] \\ w-\delta, & \text{si } w\in(2\delta,\infty) \end{cases}$$

$$\theta_\delta \in C(\mathbb{R}), \qquad \theta_\delta(w) \to \begin{cases} 0, & w\leq 0 \\ w, & w>0 \end{cases}$$
cuando $\delta\to 0^+$.

Además,

$$\theta_\delta'(w) = \begin{cases} 0, & w \in (-\infty, 0) \\ \dfrac{w}{2\delta}, & w \in (0, 2\delta) \\ 1, & w \in (2\delta, \infty) \end{cases}$$

$$\theta_\delta''(w) = \begin{cases} 0, & w \in (-\infty,0) \cup (2\delta, \infty) \\ \dfrac{1}{2\delta}, & w \in (0, 2\delta). \end{cases}$$

- $\theta_\delta', \theta_\delta''$ $\exists$ c.d.s. en $w \in \mathbb{R}$
- $\theta_\delta \in C^1$ c.d.s. en $w \in \mathbb{R}$
- $\theta_\delta'' \geq 0$ " " .

$u^\varepsilon, \bar{u}^\varepsilon$ son soluciones suaves de (2).

$$\partial_t \left[\theta_\delta (u^\varepsilon - \bar{u}^\varepsilon) \right] = \theta_\delta'(u^\varepsilon - \bar{u}^\varepsilon)(u_t^\varepsilon - \bar{u}_t^\varepsilon)$$

$$= \theta_\delta'(u^\varepsilon - \bar{u}^\varepsilon) \Big[\varepsilon u_{xx}^\varepsilon - f(u^\varepsilon)_x + $$

$$- \varepsilon \bar{u}_{xx}^\varepsilon + f(\bar{u}^\varepsilon)_x \Big]$$

Calculando
- $\left[\theta_\delta'(u^\varepsilon - \bar{u}^\varepsilon)(f(u^\varepsilon) - f(\bar{u}^\varepsilon)) \right]_x = \ldots$
- $\left[\theta_\delta (u^\varepsilon - \bar{u}^\varepsilon) \right]_{xx} = \ldots$

(regla de la cadena)

obtenemos

$$\partial_t\left[\theta_\delta(u^\varepsilon-\bar{u}^\varepsilon)\right] + \left[\theta_\delta'(u^\varepsilon-\bar{u}^\varepsilon)\left(f(u^\varepsilon)-f(\bar{u}^\varepsilon)\right)\right]_x +$$

$$- \theta_\delta''(u^\varepsilon-\bar{u}^\varepsilon)(u^\varepsilon_x-\bar{u}^\varepsilon_x)\left(f(u^\varepsilon)-f(\bar{u}^\varepsilon)\right)$$

$$= \varepsilon\left[\theta_\delta(u^\varepsilon-\bar{u}^\varepsilon)\right]_{xx} \underbrace{- \varepsilon\,\theta_\delta''(u^\varepsilon-\bar{u}^\varepsilon)(u^\varepsilon_x-\bar{u}^\varepsilon_x)^2}_{\leq 0}$$

Fijamos $0<s<t<\infty$ e integramos:

$$\int_{\mathbb{R}} \theta_\delta\left(u^\varepsilon(x,t)-\bar{u}^\varepsilon(x,t)\right)dx - \int_{\mathbb{R}} \theta_\delta\left(u^\varepsilon(x,s)-\bar{u}^\varepsilon(x,s)\right)dx$$

$$\leq \int_s^t\int_{\mathbb{R}} \theta_\delta''(u^\varepsilon-\bar{u}^\varepsilon)(u^\varepsilon_x-\bar{u}^\varepsilon_x)\left(f(u^\varepsilon)-f(\bar{u}^\varepsilon)\right)dx\,dt$$

ya que:

- $$\int_{\mathbb{R}}\left[\theta_\delta(u^\varepsilon-\bar{u}^\varepsilon)\right]_{xx}dx = \left[\theta_\delta'(u^\varepsilon-\bar{u}^\varepsilon)(u^\varepsilon_x-\bar{u}^\varepsilon_x)\right]\Big|_{-\infty}^{\infty} = 0$$

Por otro lado $\theta_\delta''(u^\varepsilon-\bar{u}^\varepsilon)(f(u^\varepsilon)-f(\bar{u}^\varepsilon))$ es acotado uniformemente en $\delta>0$. Esto es porque

$$u^\varepsilon \in [-M_0, M_0], \quad M_0 = \|u_0\|_{L^\infty}$$

$$\bar{u}^\varepsilon \in [-\bar{M}_0, \bar{M}_0], \quad \bar{M}_0 = \|\bar{u}_0\|_{L^\infty}$$

cuando $\delta \to 0^+$:

$$\theta_\delta(u^\varepsilon - \bar{u}^\varepsilon) \longrightarrow \begin{cases} 0, & \text{si } u^\varepsilon \leq \bar{u}^\varepsilon \\ u^\varepsilon - \bar{u}^\varepsilon & \text{si } u^\varepsilon > \bar{u}^\varepsilon \end{cases}$$

$$=: [u^\varepsilon - \bar{u}^\varepsilon]^+$$

Además,

$$\theta_\delta''(u^\varepsilon(x,t) - \bar{u}^\varepsilon(x,t))(f(u^\varepsilon(x,t)) - f(\bar{u}^\varepsilon(x,t)))$$

$$\longrightarrow 0 \quad \text{si } \delta \to 0^+$$

c.d.s. en (x,t)

Invirtiendo los roles de u^ε y $\bar{u}^\varepsilon$:

$$\int_{\mathbb{R}} |u^\varepsilon(x,t) - \bar{u}^\varepsilon(x,t)|\, dx - \int_{\mathbb{R}} |u^\varepsilon(x,s) - \bar{u}^\varepsilon(x,s)|$$

$$\leq 0$$

Tomando $s \to 0^+$:

$$\| u^\varepsilon(\cdot,t) - \bar{u}^\varepsilon(\cdot,t) \|_{L^1} \leq \| u_0 - \bar{u}_0 \|_{L^1}$$

$$\Rightarrow (i)$$

Para probar (ii) observamos que si $u_0 \leq \bar{u}_0$ c.d.s. entonces

$$\underbrace{\int_{\mathbb{R}} \theta_\delta \left(u^\varepsilon(x,t) - \bar{u}^\varepsilon(x,t)\right) dx}_{\substack{\downarrow\, \delta\to 0^+ \\ \int_{\mathbb{R}} [u^\varepsilon - \bar{u}^\varepsilon]^+ \, dx}} \leq \underbrace{\int_{\mathbb{R}} \theta_\delta\left(u_0(x) - \bar{u}_0(x)\right) dx}_{\substack{\downarrow\, \delta\to 0^+ \\ \int_{\mathbb{R}} u_0(x) - \bar{u}_0(x) dx \\ \leq 0}}$$

$$\Rightarrow \quad 0 \leq \int_{\mathbb{R}} [u^\varepsilon - \bar{u}^\varepsilon]^+ \, dx \leq 0$$

$$\Rightarrow \quad [u^\varepsilon - \bar{u}^\varepsilon]^+ = 0 \quad \text{c.d.s.}$$

$$\Rightarrow \quad u^\varepsilon \leq \bar{u}^\varepsilon \quad \text{c.d.s. en } x \in \mathbb{R}, \ \forall t > 0.$$

Por el principio del máximo, con

$$-\infty < a < b < \infty$$

tal que $\begin{matrix} [-M_0, M_0] \\ [-\bar{M}_0, \bar{M}_0] \end{matrix} \subset [a,b]$

entonces $u^\varepsilon(x,t), \ \bar{u}^\varepsilon(x,t) \in [a,b]$

$\forall x \in \mathbb{R}$ c.d.s., $\forall t > 0$

$\Rightarrow$ (ii) $\square$