Lección 1.2: Ejemplos (continuación): ecuaciones de Euler.

Ejemplos

(A) Ecuaciones de Euler

$$\rho_t + \sum_{j=1}^{d} (\rho u_j)_{x_j} = 0 \qquad (1a)$$

$$(\rho u_i)_t + \sum_{j=1}^{d} (\rho u_i u_j)_{x_j} + p_{x_j} = 0, \quad i=1,\dots,d \qquad (1b)$$

$$\Big(\rho\big(e+\tfrac{1}{2}|u|^2\big)\Big)_t + \sum_{j=1}^{d} \Big(\rho u_j \big(e+\tfrac{1}{2}|u|^2\big) + p u_j\Big)_{x_j} = 0 \qquad (1c)$$

$\rho > 0$ – densidad de masa

$u = \begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} \in \mathbb{R}^d$ – campo de velocidades

Ecuación de estado: $\quad p = \hat{p}(\rho, e)$

e – densidad de energía interna

E – " " " total

$E = e + \frac{1}{2}|u|^2$

El sistema (1) tiene la forma de un (SLC):

$n = d+2$ cantidades conservadas

$$U = \begin{pmatrix} \rho \\ \rho u_1 \\ \vdots \\ \rho u_d \\ \rho E \end{pmatrix} =: \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{d+1} \\ U_d \end{pmatrix} \in \mathbb{R}^n = \mathbb{R}^{d+2}$$

$$\longrightarrow U_t + \sum_{j=1}^{d} f^j(U)_{x_j} = 0$$

$$f^j \in C^2(\mathbb{R}^n;\mathbb{R}^n)$$

$$F := \begin{bmatrix} f^1 & f^2 & \cdots & f^d \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \in \mathbb{R}^{n\times d} = \mathbb{R}^{(d+2)\times d}$$

Para cada $1 \leq j \leq d$:

$$f^j(U) = \begin{bmatrix} \rho u_j \\ \rho u_1 u_j + \delta_1^j P \\ \vdots \\ \rho u_d u_j + \delta_d^j P \\ (\rho(e + \frac{1}{2}|u|^2) + P) u_j \end{bmatrix}$$

Ejemplo: $d = 3$

$$F = \begin{bmatrix} \rho u_1 & \rho u_2 & \rho u_3 \\ \rho u_1^2 + P & \rho u_1 u_2 & \rho u_1 u_3 \\ \rho u_1 u_2 & \rho u_2^2 + P & \rho u_2 u_3 \\ \rho u_1 u_3 & \rho u_2 u_3 & \rho u_3^2 + P \\ \underbrace{(\rho(e + \frac{1}{2}|u|^2) + P)}_{=:\Phi} u_1 & \Phi u_2 & \Phi u_3 \end{bmatrix}$$

$$= \begin{bmatrix} U_2 & U_3 & U_4 \\ (U_2)^2/U_1 + P(U) & \vdots & \vdots \\ U_2 U_3 / U_1 & & \\ U_2 U_4 / U_1 & & \\ U_2 U_5 / U_1 + (U_2/U_1) P & & \end{bmatrix}$$

Conjunto de variables de estado :

$$\mathcal{U} = \Big\{ U \in \mathbb{R}^{d+2} : \quad U_1 = \rho > 0,$$

$$\underbrace{\frac{U_5}{U_1} - \frac{1}{2}\left(U_2^2 + U_3^2 + U_4^2\right)/U_1^2}_{} = e > 0 \Big\}$$

$$P = \hat{P}(\rho, e) = \hat{P}(V_1, \qquad) = \bar{P}(U)$$

Euler en 1-D

caso $d=1$: $\qquad n = d+2 = 3$

$$\rho_t + (\rho u)_x = 0 \qquad (2a)$$

$$(\rho u)_t + (\rho u^2 + P)_x = 0 \qquad (2b)$$

$$\left(\rho\left(e + \tfrac{1}{2}u^2\right)\right)_t + \left(\rho\left(e + \tfrac{1}{2}u^2\right)u + Pu\right)_x = 0 \qquad (2c)$$

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho\left(e + \frac{1}{2}u^2\right) \end{pmatrix} =: \begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} \quad \text{variables conservadas}$$

$j=1$, $\qquad f(U) = \begin{pmatrix} \rho u \\ \rho u^2 + \hat{P}(\rho, e) \\ \rho\left(e + \frac{1}{2}u^2\right)u + Pu \end{pmatrix} = F$

$$= \begin{pmatrix} U_2 \\ U_2^2/U_1 + \bar{P}(U) \\ U_2\left(U_3 + \bar{P}(U)\right)/U_1 \end{pmatrix} \in C^2(\mathcal{U}; \mathbb{R}^3)$$

$$\mathcal{U} = \{ \rho > 0,\ e > 0 \}$$
$$= \{ U_1 > 0,\ U_3/U_1 - \tfrac{1}{2} U_2^2/U_1^2 > 0 \}$$

Variables lagrangianas

Caso unidimensional : $d = 1$.

Ley de conservación de masa :

$$\rho_t + (\rho u)_x = 0 \qquad (2a)$$

Cambio de coordenadas $(x,t) \longmapsto (y,t)$

Variable lagrangiana :

$$y(x,t) = \int_0^x \rho(\xi,t)\, d\xi$$

(2a) $\Rightarrow$ $dy = \rho\, dx - \rho u\, dt$ es una diferencial exacta.

$\Leftrightarrow$

$$\rho_t = -(\rho u)_x$$

El volumen específico del gas es :

$$v := \frac{1}{\rho}, \qquad \rho > 0$$

Haciendo el cambio de variables :

$$y_t = \int_0^x \rho_t(\xi, t)\, d\xi \overset{(2a)}{=} -\int_0^x (\rho u)_\xi \, d\xi$$

$$= -\rho u$$

$$\rho_t = -\frac{1}{v^2}\left(\rho_y y_t + v_t\right), \qquad y_x = \rho$$

sustituyendo en (2a)

$$0 = \rho_t + (\rho u)_x = \rho_t + \rho_x u + \rho u_x$$

$$= \frac{u \cancel{v_y}}{\cancel{v^3}} - \frac{v_t}{v^2} + \frac{\rho u_y}{v} - \frac{u \cancel{v_y}}{\cancel{v^3}}$$

$$= \frac{1}{v^2}\left(-v_t + u_y\right). \qquad v \neq 0$$

$$(2a) \iff v_t = u_y \quad \cdots \quad (3a)$$

Supongamos que tenemos una ley de conservación:

$$\partial_t W_j + \partial_x \Phi_j = 0 \qquad (4)$$

<u>Proposición</u> (4) se transforma en

$$\partial_t (v W_j) + \partial_y \left(\Phi_j - u W_j \right) = 0 \quad \cdots \quad (5)$$

<u>Dem.</u> $(4) \Rightarrow d\left(\Phi_j\, dt - W_j\, dx \right) = 0$

Así, calculamos

$$d\left[(\Phi_j - uW_j)\,dt - v W_j\, dy \right)$$

$$= d\left((\Phi_j - uW_j)\,dt - vW_j(\rho\, dx - \rho u\, dt)\right)$$

$$= d\left(\Phi_j\, dt - \cancel{uW_j}\, dt - W_j\, dx + \cancel{uW_j}\, dt \right)$$

$$= d\left(\Phi_j\, dt - W_j\, dx \right) = 0$$

$\square$

Si $\quad W_1 \equiv \rho, \quad \Phi_1 = \rho u$:

$$(5)_{j=1} \Rightarrow \quad \partial_t (v\rho) + \partial_y(\rho u - u\rho)$$

$$= \partial_t(1) + \partial_y(0) = 0$$

Si $\quad W_4 \equiv 1, \quad \Phi_4 \equiv 0$:

$$(5)_{j=4} \Rightarrow \quad \partial_t(v) + \partial_y(-u) = 0$$

$$\rightarrow \quad v_t = u_y \qquad \cdots \quad (3a)$$

Tomando $\begin{cases} W_2 = \rho u, & \Phi_2 = \rho u^2 + p \\ W_3 = \rho e + \frac{1}{2}\rho u^2, & \Phi_3 = \left[\rho e + \frac{1}{2}\rho u^2 + p\right] u \end{cases}$

encontramos :

$$v_t - u_y = 0 \qquad (3a)$$

$$u_t + p_y = 0 \qquad (3b)$$

$$\left(e + \tfrac{1}{2}u^2\right)_t + (up)_y = 0 \qquad (3c)$$

Euler en variables lagrangianas

$$p = \hat{p}(\rho, e) = \hat{p}(1/v, e) =: \check{p}(v, e)$$

$$U = \begin{pmatrix} v \\ u \\ e + \frac{1}{2}u^2 \end{pmatrix} \in \mathcal{U} \subset \mathbb{R}^3$$

$$f(U) = \begin{pmatrix} -u \\ \check{p}(v, e) \\ u\check{p}(v, e) \end{pmatrix}$$

$$\mathcal{U} = \{ v > 0, \ e > 0 \}.$$