Lección 5.6 : Estabilidad para sistemas de 2x2.

Sistema en $\mathbb{R}^2$:

$$y' = Ay \qquad \dots (1)$$

$A \in \mathcal{L}(\mathbb{R}^2)$, $y = 0$ punto de equilibrio.
único si $\det A \neq 0$

$\{\lambda_1, \lambda_2\} = \sigma(A)$. Tenemos:

$$A = S^{-1} B S$$

con, (a) $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, $\lambda_1 \neq \lambda_2$, $\lambda_j \in \mathbb{R}$

(b) $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda \in \mathbb{R}$

(c) $B = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$, $\lambda = \alpha + i\beta$, $\beta \neq 0$.

Haciendo $x = Sy$, $S \in \mathbb{R}^{2\times 2}$, $\exists S^{-1}$

Hipótesis: $\det A \neq 0$

$$\Rightarrow \quad x' = Sy' = SAy = SAS^{-1}x$$
$$= Bx$$

soluciones de

(2) $\begin{cases} x' = Bx \\ x(0) = x_0 := Sy(0) \end{cases}$ $\Rightarrow$ soluciones del sist. (1)

Lema 1

(a) $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \underset{\lambda_i \in \mathbb{R}}{\Rightarrow} e^B = \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$

(b) $\underset{\lambda \in \mathbb{R}}{B} = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix} \Rightarrow e^B = e^{\lambda} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$

(c) $B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \Rightarrow e^B = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$

Dem. (a) $B^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}$

$$\Rightarrow \quad e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}$$

$$= \begin{pmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{pmatrix}$$

(b) $B = \lambda I + \underbrace{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}}_{Q} =: \lambda I + Q$

λI conmuta con Q:

$$e^B = e^{\lambda I + Q} = e^{\lambda I} e^{Q}$$

$Q^2 = 0 \quad \Rightarrow \quad e^Q = I + Q$

(a) $\Rightarrow e^{\lambda I} = e^{\lambda} I \quad \therefore \quad e^B = e^{\lambda} I (I + Q)$

$$= e^{\lambda} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

(c) Por inducción sobre $k \in \mathbb{N}$

si $z = a + ib$

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^k = \begin{pmatrix} \operatorname{Re} z^k & -\operatorname{Im} z^k \\ \operatorname{Im} z^k & \operatorname{Re} z^k \end{pmatrix}$$

$$\Rightarrow \quad e^B = \begin{pmatrix} \operatorname{Re} e^z & -\operatorname{Im} e^z \\ \operatorname{Im} e^z & \operatorname{Re} e^z \end{pmatrix}$$

$$= e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}$$

$\square$

Corolario (a) $$e^{tB} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

(b) $$e^{tB} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

(c) $$e^{tB} = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}$$

Soluciones de (2):

(a) $$X(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} x_0$$

(b) $x(t) = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x_0$

(c) $x(t) = e^{\alpha t} \begin{pmatrix} \cos \beta t & -\sin \beta t \\ \sin \beta t & \cos \beta t \end{pmatrix}$

$\lambda = \alpha + i\beta$

Las soluciones se clasifican topológicamente de la sig. forma:

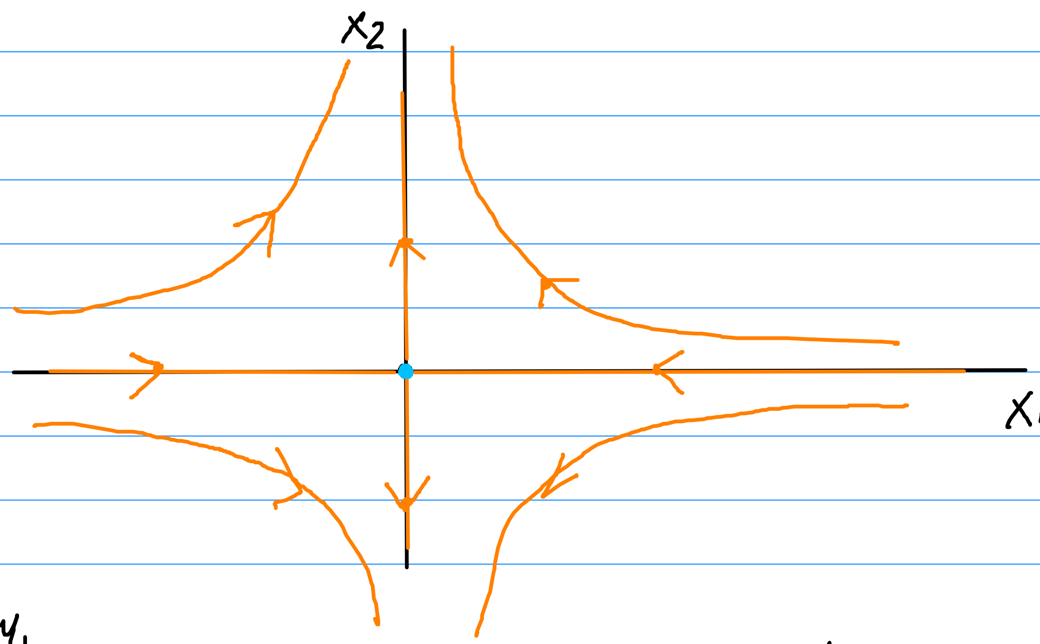
Caso I: [punto silla]

$B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ caso (a)

pero con $\lambda_1 < 0 < \lambda_2$

$(\det A \neq 0)$

Origen es un punto silla



Caso II :

$\left[\text{nodo}\right] \leftrightarrow \left\{\begin{array}{l}\text{estable}\\ \text{inestable}\end{array}\right\}$

Estable : (a) $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\quad \lambda_1 \leq \lambda_2 < 0$

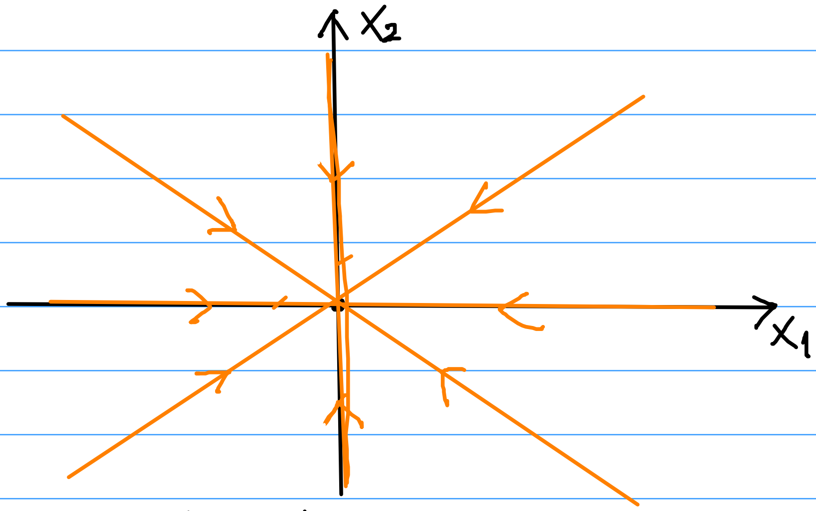
(b) $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ $\quad \lambda < 0$

Inestable : (a) $\lambda_2 > \lambda_1 > 0$

(b) $\lambda > 0$

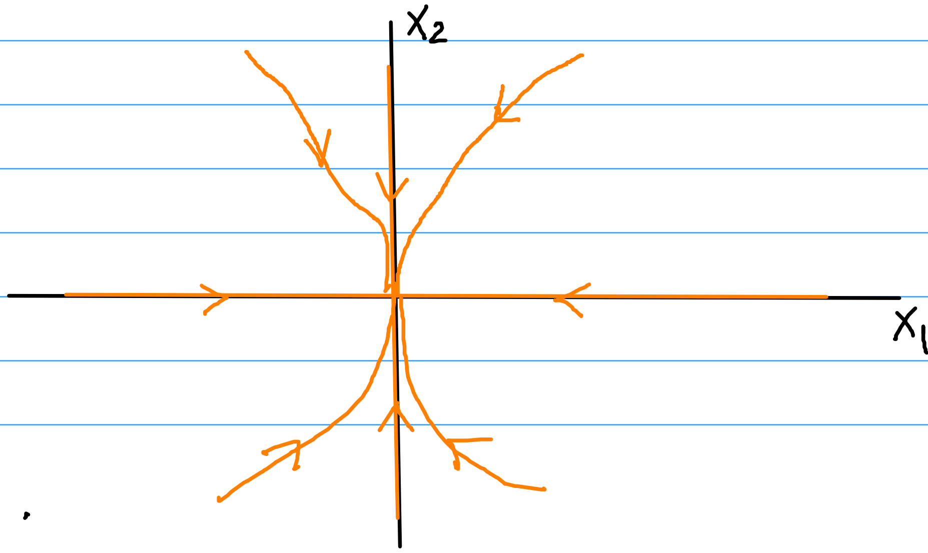
$\left[\begin{array}{l}\text{Nodo}\\ \text{estable}\end{array}\right]$:

(a)

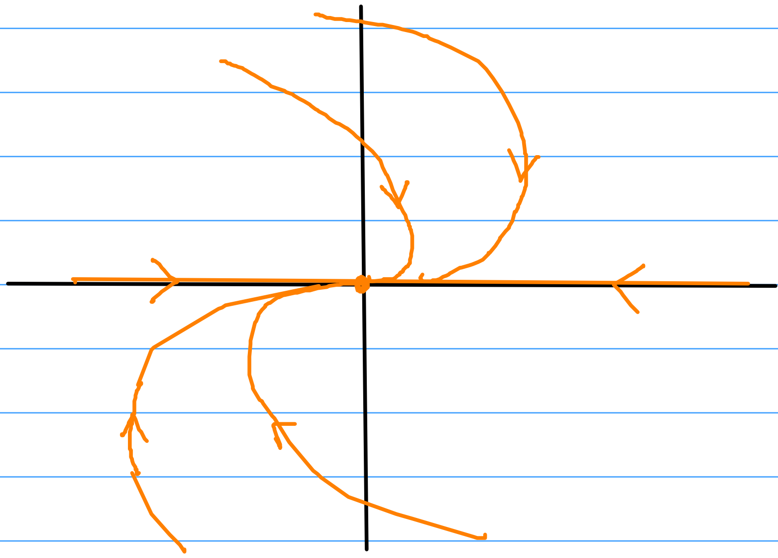


$\lambda_1 = \lambda_2 < 0$

$\lambda_1 < \lambda_2 < 0$



(b) $\lambda < 0$



$\begin{bmatrix} \text{Nodo} \\ \text{inestable} \end{bmatrix}$: cambia el sentido de las flechas.

Caso III :

$[\text{foco}] \rightarrow \begin{Bmatrix} \text{estable} \\ \text{inestable} \end{Bmatrix}$

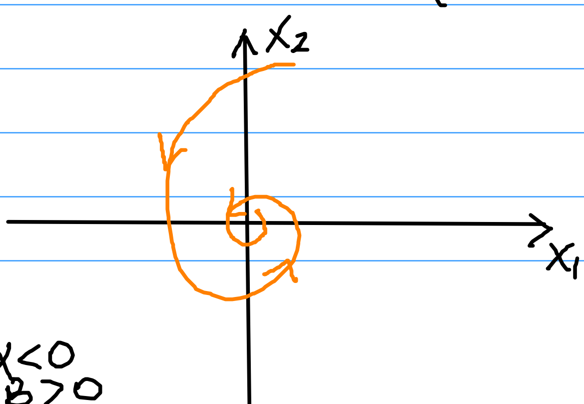
(c) $B = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$ $\beta \neq 0$

$\alpha \neq 0$

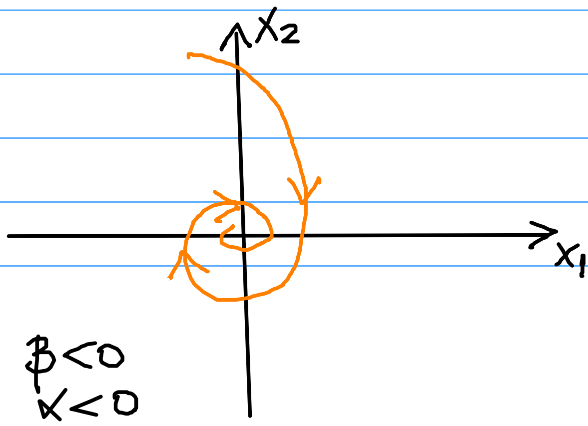
Estable : si $\alpha < 0$

Instable : si $\alpha > 0$

Foco estable $(\alpha < 0)$:



$\alpha < 0$
$\beta > 0$

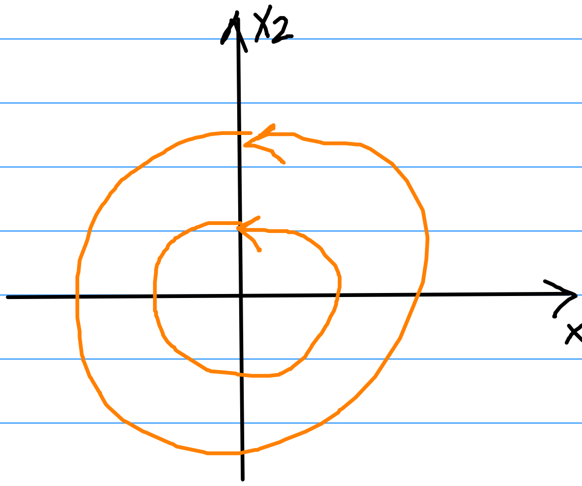


$\beta < 0$
$\alpha < 0$

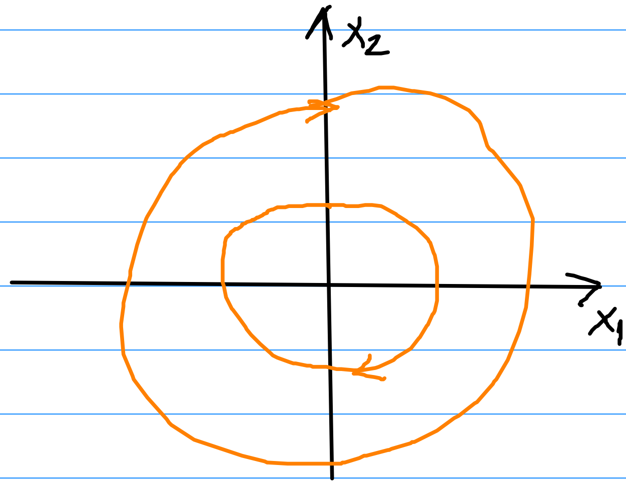
Caso IV : centro.

Caso (c) pero con $\alpha = 0$, $\beta \neq 0$.

$$B = \begin{pmatrix} 0 & -\beta \\ \beta & 0 \end{pmatrix}$$



$\beta > 0$



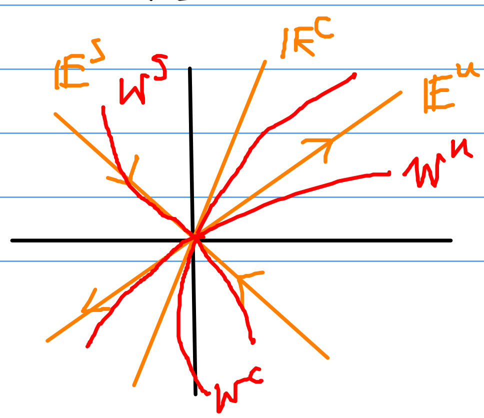
$\beta < 0$

$A \in \mathcal{L}(\mathbb{R}^n) \longrightarrow \mathbb{R}^n \supset \mathbb{E}^s$ espacio estable $\operatorname{Re}\lambda < 0$

$\mathbb{R}^n \supset \mathbb{E}^u$ espacio inestable $\operatorname{Re}\lambda > 0$

$\mathbb{R}^n \supset \mathbb{E}^c$ espacio central $\operatorname{Re}\lambda = 0$

$$\mathbb{E}^s \oplus \mathbb{E}^u \oplus \mathbb{E}^c = \mathbb{R}^n$$



$y' = f(y)$, $0 = y_*$ punto de equ.

$A = Df(0)$ $W^s(0)$

$W^u(0)$

$W^c(0)$