Lección 3.10 : Aplicaciones de la transformada de Laplace.

Corolario (propiedades de la transformada)

Transformadas típicas de Laplace :

	$f(t)$	$(\mathcal{L}f)(s)$	
(1)	$t^n, \; n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}, \; s>0$	Tarea
(2)	1	$\frac{1}{s}, \; s>0$	✓
(3)	e^{at}	$\frac{1}{s-a}, \; s>a$	✓
(4)	$\sin at$	$\frac{a}{s^2+a^2}, \; s>0$	✓
(5)	$\cos at$	$\frac{s}{s^2+a^2}, \; s>0$	Tarea
(6)	$\cosh(at)$	$\frac{s}{s^2-a^2}, \; s>\lvert a\rvert$	
(7)	$\sinh(at)$	$\frac{a}{s^2-a^2}, \; s>\lvert a\rvert$	
(8)	$t \sin at$	$\frac{2as}{(s^2+a^2)^2}, \; s>0$	
(9)	te^{at}	$\frac{1}{(s-a)^2}, \; s>a$	
(10)	$e^{at}f(t)$	$(\mathcal{L}f)(s-a), \; s>a$	✓
(11)	$\frac{1}{t}\sin(at)$	$\arctan \frac{a}{s}, \; s>0$	

Demostración de (6), (7), (8), (9), (11) :

$$\cosh(at) = \frac{1}{2}\left(e^{at} + e^{-at}\right)$$

$$\Rightarrow \mathcal{L}(\cosh(at))(s) = \frac{1}{2}\mathcal{L}(e^{at})(s) + \frac{1}{2}\mathcal{L}(e^{-at})(s)$$

$$= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{s}{s^2-a^2}$$

$\sinh(at)$ es análogo (ejercicio).

$$\mathcal{L}(t\sin at) = -\frac{d}{ds}\left(\mathcal{L}(\sin at)(s)\right)$$

$$= -\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{2as}{(s^2+a^2)^2}$$

$$\mathcal{L}(te^{at}) = -\frac{d}{ds}\left(\mathcal{L}(e^{at})(s)\right)$$

$$= -\frac{d}{ds}\left(\frac{1}{s-a}\right)$$

$$= \frac{1}{(s-a)^2} \qquad s>a$$

$$\mathcal{L}\left(\frac{1}{t}\sin at\right)(s) \overset{(i)}{=} \int_s^\infty \mathcal{L}(\sin at)(y)\, dy$$

$$= \int_s^\infty \frac{a}{y^2+a^2}\, dy$$

$$= \frac{\pi}{2} - \operatorname{Arc\,tan}\frac{s}{a} = \operatorname{Arc\,tan}\frac{a}{s} \quad \square$$

Aplicaciones a ecuaciones diferenciales

Ejemplos:

(A) Consideremos el problema

$$\begin{cases} y'' + y = \cos \omega t, & \omega > 0,\ \omega^2 \neq 1 \\ y(0) = y_0 & \\ y'(0) = y_1 & y_j \in \mathbb{R} \text{ dados},\ j = 0,1 \end{cases}$$

Sea $Y(s) := (\mathcal{L}y)(s)$. Dado que

$$(\mathcal{L}f'')(s) = s^2 (\mathcal{L}f)(s) - sf(0) - f'(0)$$

tomamos la transformada de la ecuación

$$\mathcal{L}(y''+y)(s) = \mathcal{L}(\cos \omega t)(s)$$

$$\Rightarrow \quad s^2 \underbrace{Y(s)} - s \underbrace{y(0)}_{=y_0} - \underbrace{y'(0)}_{=y_1} + \underbrace{Y(s)} = \frac{s}{s^2+\omega^2}$$

$$\Rightarrow \quad Y(s)(1+s^2) = \frac{s}{s^2+\omega^2} + sy_0 + y_1$$

$$\Rightarrow \quad Y(s) = \frac{1}{1-\omega^2}\left[\underbrace{\frac{s}{s^2+\omega^2}}_{\mathcal{L}(\cos \omega t)(s)} - \underbrace{\frac{s}{s^2+1}}_{\mathcal{L}(\cos t)}\right] + \underbrace{\left(\frac{s}{s^2+1}\right)} y_0 + \underbrace{\frac{y_1}{s^2+1}}_{\mathcal{L}(\sin t)}$$

$$\Rightarrow (\mathcal{L}y)(s) = Y(s) =$$

$$= \frac{1}{1-\omega^2} \mathcal{L}(\cos \omega t)(s) - \frac{1}{1-\omega^2} \mathcal{L}(\cos t)(s)$$
$$+ y_0 \mathcal{L}(\cos t)(s) + y_1 \mathcal{L}(\sin t)(s)$$

$$\Rightarrow y(t) = \underbrace{\frac{1}{1-\omega^2} \cos \omega t - \frac{1}{1-\omega^2} \cos t}_{} + \underbrace{y_0 \cos t + y_1 \sin t}_{}$$

Ejercicio: comprobarlo.

(B) Hallar la solución de

$$\left.\begin{aligned} y'' - 3y' + 2y &= e^{3t} \\ y(0) &= 1 \\ y'(0) &= 0 \end{aligned}\right\}$$

Sea $Y(s) = (\mathcal{L}y)(s)$. Tomando $\mathcal{L}$ de la ecuación

$$s^2 Y(s) - s\underbrace{y(0)}_{=1} - \cancel{y'(0)}^{0} +$$
$$- 3\left[sY(s) - \underbrace{y(0)}_{=1} \right] + 2Y(s) = \frac{1}{s-3}$$

$$\Rightarrow \quad Y(s) = \underbrace{\frac{1}{(s-3)(s^2-3s+2)}}_{=:\, I_1(s)} + \underbrace{\frac{s-3}{s^2-3s+2}}_{=:\, I_2(s)}$$

$$I_1(s) = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$ (fracciones parciales)

$$\Rightarrow \quad A = \frac{1}{2}, \quad B = -1, \quad C = \frac{1}{2}$$

$$\Rightarrow \quad I_1(s) = \frac{1}{2}\frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}$$

Análogamente,

$$I_2(s) = \frac{D}{s-1} + \frac{E}{s-2}$$

$$\Rightarrow \quad D = 2, \quad E = -1$$

$$\Rightarrow \quad I_2(s) = \frac{2}{s-1} - \frac{1}{s-2}$$

Así,

$$Y(s) = \frac{5}{2}\frac{1}{s-1} - \frac{2}{s-2} + \frac{1}{2}\frac{1}{s-3}$$

$$= \mathcal{L}\left(\frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}\right)$$

$$\Rightarrow \quad y(t) = \frac{5}{2}e^{t} - 2e^{2t} + \frac{1}{2}e^{3t}$$

Ejercicio: comprobarlo.

Función de Heaviside:

$$H_c(t) := \begin{cases} 1, & t \geq c \\ 0, & t < c \end{cases} \quad \cdots \quad (1)$$

$c > 0$

Es de orden exponencial.

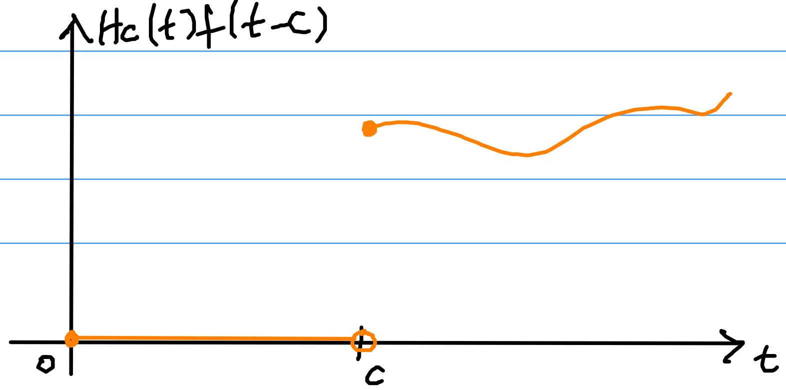
$$\Rightarrow \quad (\mathcal{L} H_c)(s) = \int_0^\infty e^{-st} H_c(t)\, dt$$

$$= \int_c^\infty e^{-st}\, dt$$

$$= \frac{e^{-cs}}{s}, \quad s > 0.$$

Sea $f = f(t)$ continua en $t \in (0, \infty)$, sea $c > 0$ y consideremos la traslación $t \longmapsto f(t-c)$.

La función $H_c(t) f(t-c)$ es la traslación si $t \geq c$ y es cero si $0 < t < c$.



Lema 1 Si $(\mathcal{L}f)(s)$ existe para $s > a \geq 0$ y tomando $c > 0$, entonces

$$(2) \cdots \quad \mathcal{L}\left(H_c(t) f(t-c)\right)(s) = e^{-cs} (\mathcal{L}f)(s)$$

$s > a$

Demostración:

$$\mathcal{L}\left(H_c(t) f(t-c)\right)(s) = \int_0^\infty e^{-st} H_c(t) f(t-c)\, dt$$

$$= \int_c^\infty e^{-st} f(t-c)\, dt$$

$$= \int_0^\infty e^{-s(\xi + c)} f(\xi)\, d\xi$$

$\xi = t - c$

$$= e^{-sc} \int_0^\infty e^{-s\xi} f(\xi)\, d\xi$$

$$= e^{-sc} (\mathcal{L}f)(s), \qquad \forall\, s > a$$

$\square$

Aplicación: hallar la función f tal que

$$(\mathcal{L}f)(s) = \frac{1 - e^{-as}}{s^2}, \qquad a > 0$$

Por linealidad de $\mathcal{L}$:

$$(\mathcal{L}f)(s) = \frac{1}{s^2} - \frac{e^{-as}}{s^2}$$

$\mathcal{L}(t^n)(s) = \dfrac{n!}{s^{n+1}}$

$$= \mathcal{L}(t)(s) - e^{-as}\mathcal{L}(t)(s)$$

$$= \mathcal{L}(t)(s) - \mathcal{L}(H_a(t)(t-a))(s)$$

lema 1

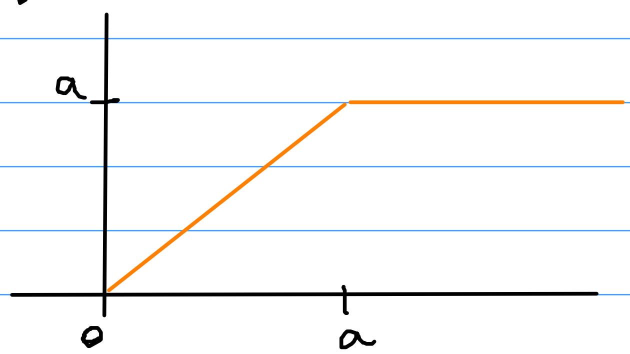
$$= \mathcal{L}\big(t - H_a(t)(t-a)\big)(s)$$

La función buscada es

$$f(t) = t - H_a(t)(t-a)$$

$$= \begin{cases} a, & t \geq a \\ t, & 0 \leq t < a \end{cases}$$

$=$



Lema 2 Si $(\mathcal{L}f)(s)$ existe $\forall s > a \geq 0$ y $c \in \mathbb{R}$ es constante entonces

(3) ... $\mathcal{L}(e^{ct}f(t))(s) = (\mathcal{L}f)(s-c)$

$\forall s > a + c$

Ya se probó: lema de propiedades inciso (d).

Aplicación: hallar transformada inversa de

$$F(s) = \frac{1}{s^2 - 2s + 2}$$

Dado que $\mathcal{L}(\sin t)(s) = \frac{1}{s^2+1}$

escribimos

$$F(s) = \frac{1}{(s-1)^2+1}$$

Por lema 2: $F(s) = \mathcal{L}(\sin t)(s-1)$

$= \mathcal{L}(e^t \sin t)(s)$

lema 2

$$\Rightarrow \quad F(s) = \mathcal{L}(e^t \sin t)(s).$$

Ejemplo: forzamiento discontinuo

Sea el problema:

$$y'' + y = h(t), \quad t > 0 \quad \cdots (4)$$

donde $h(t) = \begin{cases} 1, & 0 \leq t \leq \pi/2 \\ 0, & t > \pi/2 \end{cases}$

con condiciones $y(0) = 0, \quad y'(0) = 1.$

Podemos escribir

$$h(t) = 1 - H_{\pi/2}(t)$$

Tomando $\mathcal{L}$:

$$Y(s) = (\mathcal{L}y)(s)$$

$$\therefore \quad \mathcal{L}(y'' + y) = s^2 Y(s) - 1 + Y(s)$$

$$= \mathcal{L}(1 - H_{\pi/2}(t))(s)$$

$$= \frac{1}{s} - \frac{e^{-\pi/2\, s}}{s}$$

$$\Rightarrow Y(s) = \frac{1}{s^2+1} + \frac{1}{s(s^2+1)} - \frac{e^{-\pi/2\, s}}{s(s^2+1)}$$

$$= \frac{1}{s^2+1} + \overbrace{\frac{1}{s} - \frac{s}{s^2+1}} - \frac{e^{-\pi/2\, s}}{s} + \frac{s e^{-\frac{\pi}{2}s}}{s^2+1}$$

$$= \mathcal{L}(\sin t)(s) + \mathcal{L}(1)(s) +$$

$$- \mathcal{L}(\cos t)(s) - e^{-\frac{\pi}{2}s}\mathcal{L}(1)(s)$$

$$+ e^{-\pi/2\, s}\mathcal{L}(\cos t)(s)$$

Lema 1

$$= \mathcal{L}\Big(\sin t + 1 - \cos t - H_{\pi/2}(t) +$$

$$+ H_{\pi/2}(t)\cos\left(t - \tfrac{\pi}{2}\right)\Big)$$

La solución es:

$$Y(t) = \sin t + 1 - \cos t +$$
$$+ H_{\pi/2}(t)(\sin t - 1).$$

Ejercicio: resolverlo directamente.

Resolver:

(a) $y_1'' + y_1 = 1$ $\quad y_1(0) = 0$

en $t \in [0, \pi/2)$ $\quad y_1'(0) = 1$

(b) $y_2'' + y_2 = 0$, en $t \in (\frac{\pi}{2}, \infty)$

con $y_2(\pi/2) = \lim\limits_{t \to \frac{\pi}{2}^-} y_1(t)$

$y_2'(\pi/2) = \lim\limits_{t \to \frac{\pi}{2}^-} y_1'(t)$