Lección 2.6: Ecuación de Euler-Poisson-Darboux. Fórmulas de Kirchhoff. Principio de Huygens.

$f \in L^1_{loc}(\mathbb{R}^n)$

$$F(x,r) := \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(x)} f(y)\, dS_y \qquad \text{media esférica de } f$$

$$f \in C^2 \Rightarrow \quad F_{rr} + \frac{(n-1)}{r} F_r - \Delta_x F = 0 \quad \cdots \quad (1)$$

ec. Darboux

$$F_r = \frac{r}{\omega_n} \Delta_x \int_{B_1(0)} f(x+r\eta)\, d\eta \quad \cdots \quad (*)$$

Demostración del lema:

$u \in C^2(\mathbb{R}^n \times (0,\infty)) \cap C^1(\mathbb{R}^n \times [0,\infty))$ es sol. de

$$(2) \cdots \begin{cases} u_{tt} - c^2 \Delta_x u = 0, & x \in \mathbb{R}^n, \ t > 0 \\ u(x,0) = f(x), & x \in \mathbb{R}^n \\ u_t(x,0) = g(x) & \end{cases}$$

Medias esféricas de $\begin{Bmatrix} u(x,t) \\ f(x) \\ g(x) \end{Bmatrix} : \begin{Bmatrix} U(x,r,t) \\ F(x,r) \\ G(x,r) \end{Bmatrix}$

Por $(*)$:

$$U_r = \frac{r}{\omega_n} \Delta_x \left(\int_{B_1(0)} u(x+r\eta, t)\, d\eta \right)$$

$\Rightarrow U_r, \ \dfrac{U_r}{r}$ son funciones continuas de $r > 0$ y $t > 0$ para cada $x \in \mathbb{R}^n$ fijo.

Por la ec. de Darboux (1) :

$$U_{rr} = \Delta_x U - \frac{(n-1)}{r} U_r$$

Calculando:

$$\Delta_x U = \frac{1}{r^{n-1}} \Delta_x \left(\int_{|y|=r} u(x+y,t)\, dS_y \right)$$

$$= \Delta_x \left(\int_{|\eta|=1} u(x+r\eta, t)\, dS_\eta \right)$$

$\therefore$ $\Delta_x U$ es continua en $r>0$, $t>0$, $\forall x \in \mathbb{R}^n$ fijo.

$\therefore$ U_{rr} es continua " " ".

Análogamente $U_{tt} = \frac{1}{\omega_n} \int_{|\eta|=1} u_{tt}(x+r\eta, t)\, dS_\eta$ es continua en $r>0$, $t>0$, $\forall x \in \mathbb{R}^n$ fijo.

Concluimos $U \in C^2$ en $r>0$, $t>0$.

Calculamos:

$$U_{tt} - c^2 U_{rr} - c^2 \frac{(n-1)}{r} U_r \underset{(1)}{=} U_{tt} - c^2 \Delta_x U$$

$$= \frac{1}{\omega_n} \int_{|\eta|=1} \underbrace{(u_{tt} - c^2 \Delta_x u)}_{=0} (x+r\eta, t)\, dS_\eta$$

$$\Rightarrow U_{tt} - c^2 U_{rr} - c^2 \frac{(n-1)}{r} U_r = 0 \quad \cdots (3)$$

Euler-Poisson-Darboux.

Finalmente:

$$U(x,r,0) = \lim_{t\to 0^+} U(x,r,t) = \lim_{t\to 0^+} \frac{1}{\omega_n} \int_{|\eta|=1} u(x+r\eta, t)\, dS_\eta$$

$$= \frac{1}{\omega_n} \int_{|\eta|=1} \underbrace{u(x+r\eta, 0)}_{=f(x+r\eta)}\, dS_\eta = F(x,r)$$

Igualmente $\quad U_t(x,r,0) = G(x,r) \qquad \square$

Prop. media esférica: $\displaystyle\lim_{r\to 0^+} U(x,r,t) = u(x,t)$.

Idea: resolver (3) y tomar lim $r\to 0^+$.
Podemos hacerlo en dim <u>impar</u>.

<u>Problema de Cauchy en $\mathbb{R}^3$: fórmula de Kirchhoff</u>

$$(1). \begin{cases} u_{tt} - c^2 \Delta u = 0, & x\in\mathbb{R}^3,\ t>0 \\ u(x,0) = f(x), & x\in\mathbb{R}^3 \\ u_t(x,0) = g(x) & \end{cases}$$

U – media esférica de u, satisface Euler–Poisson–Darboux.

Definimos
$$\left.\begin{aligned} \tilde U(x,r,t) &:= r\, U(x,r,t) \\ \tilde F(x,r) &:= r\, F(x,r) \\ \tilde G(x,r) &:= r\, G(x,r) \end{aligned}\right\} (2)$$

Calculamos:

$$\tilde{U}_{tt} = r\bar{U}_{tt} \overset{(EPD),\ n=3}{=} r\left(c^2\bar{U}_{rr} + \frac{2c^2}{r}\bar{U}_r\right)$$
$$= c^2\left(r\bar{U}_{rr} + 2\bar{U}_r\right)$$
$$= c^2(r\bar{U})_{rr} = c^2\tilde{U}_{rr}$$
$$\forall\ r>0,\ t>0.$$

$$\tilde{U}(x,0,t) = 0 \qquad \left(u(x,t) = \lim_{r\to 0^+} \bar{U}(x,r,t)\ \exists\right)$$

$$\tilde{U}(x,r,0) = \tilde{F}(x,r) \qquad \forall\ r\geq 0$$
$$\tilde{U}_t(x,r,0) = \tilde{G}(x,r)$$

$\therefore$ $\forall\ x\in\mathbb{R}^3$ fijo, $\tilde{U}$ es solución de la ec. de onda unidimensional en $r>0,\ t>0$ + condiciones iniciales:

$$(3)\ldots\begin{cases} \tilde{U}_{tt} - c^2\tilde{U}_{rr} = 0, & r>0,\ t>0 \\ \tilde{U}(x,r,0) = \tilde{F}(x,r), & r>0 \\ \tilde{U}_t(x,r,0) = \tilde{G}(x,r) & \\ \tilde{U}(x,0,t) = 0, & t>0 \end{cases}$$

con $\tilde{F}(x,0) = \tilde{G}(x,0) = 0$.

La solución de (3) en la región $0\leq r<ct$ es:

$$\tilde{U}(x,r,t) = \frac{1}{2}\tilde{F}(x,r+ct) - \frac{1}{2}\tilde{F}(x,ct-r) + \frac{1}{2c}\int_{ct-r}^{ct+r} \tilde{G}(x,\rho)\, d\rho$$

Así,

$$\begin{aligned}
U(x,r,t) &= \frac{1}{r}\tilde{U}(x,r,t) \\
&= \frac{1}{2r}\tilde{F}(x,r+ct) - \frac{1}{2r}\tilde{F}(x,ct-r) + \frac{1}{2cr}\int_{ct-r}^{ct+r} \tilde{G}(x,\rho)\, d\rho \\
&= \left(\frac{ct+r}{2r}\right) F(x,r+ct) - \left(\frac{ct-r}{2r}\right) F(x,ct-r) + \frac{1}{2cr}\int_{ct-r}^{ct+r} \rho\, G(x,\rho)\, d\rho \\
&= \frac{1}{2}F(x,r+ct) + \frac{1}{2}F(x,ct-r) + \frac{ct}{2r}\Big[F(x,ct+r) - F(x,ct-r)\Big] + \frac{1}{c}\cdot\frac{1}{2r}\int_{ct-r}^{ct+r} \rho\, G(x,\rho)\, d\rho
\end{aligned}$$

Pero
$$\lim_{r\to 0^+} \frac{F(x,ct+r) - F(x,ct-r)}{2r} = F_r(x,ct)$$
$$\lim_{r\to 0^+} \frac{1}{2r}\int_{ct-r}^{ct+r} \rho\, G\, d\rho = ct\, G(x,ct)$$

$$\begin{aligned}
\therefore \lim_{r\to 0^+} U(x,r,t) &= \overbrace{F(x,ct) + ct\, F_r(x,ct)} + t\, G(x,ct) \\
&= \frac{\partial}{\partial t}\Big(t\, F(x,ct)\Big) + t\, G(x,ct)
\end{aligned}$$

Resultado: fórmula de Kirchhoff

$$u(x,t) = \frac{\partial}{\partial t}\left(t\,F(x,ct)\right) + t\,G(x,ct) \quad \cdots (4).$$

Expandiendo:

$$u(x,t) = \frac{\partial}{\partial t}\left[\frac{t}{4\pi}\int_{|\eta|=1} f(x+ct\eta)\,dS_\eta\right] + \frac{t}{4\pi}\int_{|\eta|=1} g(x+ct\eta)\,dS_\eta$$

$$\underset{\substack{y = x+ct\eta \\ dS_y = c^2t^2\,dS_\eta}}{=} \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{|x-y|=ct} f(y)\,dS_y\right] + \frac{1}{4\pi c^2 t}\int_{|x-y|=ct} g(y)\,dS_y \quad \cdots (5)$$

Kirchhoff.

observación: (4) (5) $\Rightarrow$

$$u(x,t) = \frac{1}{4\pi}\int_{|\eta|=1}\left(f(x+ct\eta) + t\,g(x+ct\eta)\right)dS_\eta + \frac{t}{4\pi}\int_{|\eta|=1}\sum_{j=1}^{3} c\eta_j\, f_{x_j}(x+ct\eta)\,dS_\eta$$

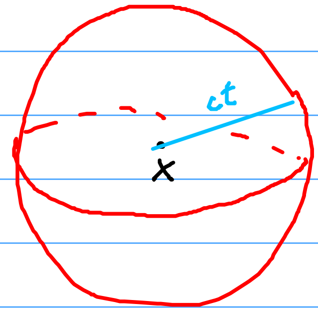
$\therefore$ u es menos regular que sus datos iniciales

$f \in C^k,\ g \in C^{k-1} \Rightarrow u \in C^{k-1},\ u_t \in C^{k-2} \quad t>0$

Esto no sucede en dimensión $n=1$.

(B) En dim $n=3$ el dominio de dependencia de la solución es:

$$\partial B_{ct}(x) = \{ y \in \mathbb{R}^3 : |x-y| = ct \} =$$



Lema Sean $f \in C^3(\mathbb{R}^3)$, $g \in C^2(\mathbb{R}^3)$. Entonces $u = u(x,t)$ definida por la fórmula de Kirchhoff (5) es de clase $C^2(\mathbb{R}^3 \times (0,\infty))$ y es la solución al problema de Cauchy (1).

Demostración:

$$u(x,0) = \lim_{t\to 0^+} u(x,t) \underset{(4)}{=} \lim_{t\to 0^+} \left(\partial_t (t F(x,ct)) + t G(x,ct) \right)$$

$$= \lim_{t\to 0^+} \Big(\underbrace{F(x,ct)}_{\to F(x,0)} + \underbrace{ct F_r(x,ct) + t G(x,ct)}_{\to 0} \Big)$$

$$= f(x)$$

Análogamente $u_t(x,0) = g(x)$ (ejercicio).

Calculando:

$$(t G(x,ct))_{tt} = \left(G(x,ct) + ct G_r(x,ct) \right)_t$$

$$= 2c G_r(x,ct) + c^2 t G_{rr}(x,ct)$$

Darboux : $G_{rr} + \frac{2}{r} G_r = \Delta_x G$

$$\Rightarrow \quad \big(t\, G(x,ct)\big)_{tt} = 2c\, G_r(x,ct) + c^2 t \left[\Delta_x G - \frac{2}{ct} G_r(x,ct) \right]$$

$$= c^2 \Delta_x \big(t\, G(x,ct) \big).$$

$$\therefore \quad \Box\big(t\, G(x,ct)\big) = 0 \qquad \Box := \partial_t^2 - c^2 \Delta_x$$

Lo mismo para $t\, F(x,ct)$ y con $\big(t\, F(x,ct)\big)_t$

$$\therefore \quad \Box u = \partial_t^2 u - c^2 \Delta_x u = 0 .$$

$u \in C^2$ por propiedades de las medias esféricas

$\Box$

Principio de Huygens

Para $(x,t) \in \mathbb{R}^3 \times (0,\infty)$, fijo, la solución u en (x,t) depende únicamente de los valores que toman f y g sobre la superficie $\partial B_{ct}(x) = \{ y \in \mathbb{R}^3 : |x-y| = ct \}$.

Supongamos que $\operatorname{supp} f, \operatorname{supp} g \subset K \subset \mathbb{R}^3$, K compacto.

Sea $x \notin K$.

Entonces $u(x,t) \neq 0$ sólo en un intervalo de tiempo

$$0 < t_{min} \leq t \leq t_{max} < \infty$$

$t_{min} = \frac{1}{c} \inf_{y \in K} |y-x| > 0$ es el tiempo
$\downarrow x \notin K$
a partir del cual $K \cap \partial B_{ct}(x) \neq \phi$.

$t_{max} = \frac{1}{c} \sup_{y \in K} |y-x| > t_{min}$ es el tiempo
a partir del cual $K \cap \partial B_{ct}(x) = \phi$.

$K \cap \partial B_{ct}(x) \neq \phi$ sólo cuando $t_{min} \leq t \leq t_{max}$.

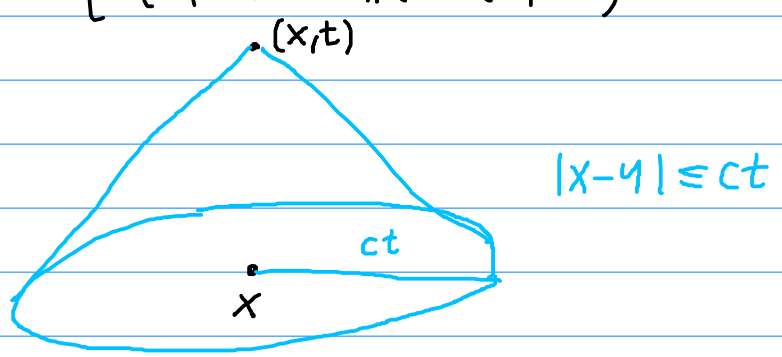
Si $t > t_{max}$ entonces $u(x,t) \equiv 0$.

"Principio de Huygens" en dim $n=3$ (impar).

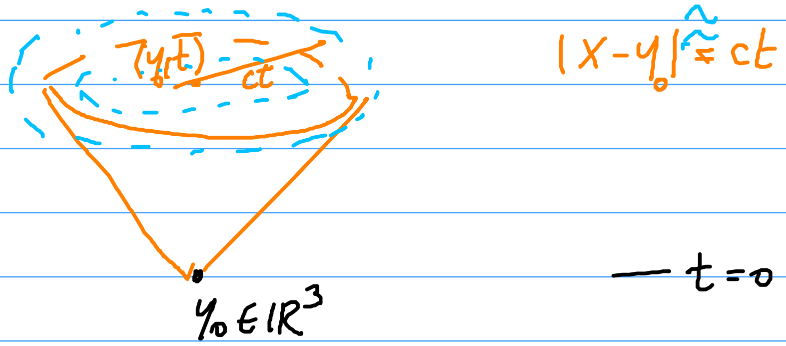
Dom. de dependencia $\partial B_{ct}(x)$.

Cono de luz:

$$\mathcal{C} = \left\{ (y,t) \in \mathbb{R}^3 \times (0,\infty) : \begin{array}{l} |x-y| \leq ct, \\ t > 0 \end{array} \right\}$$

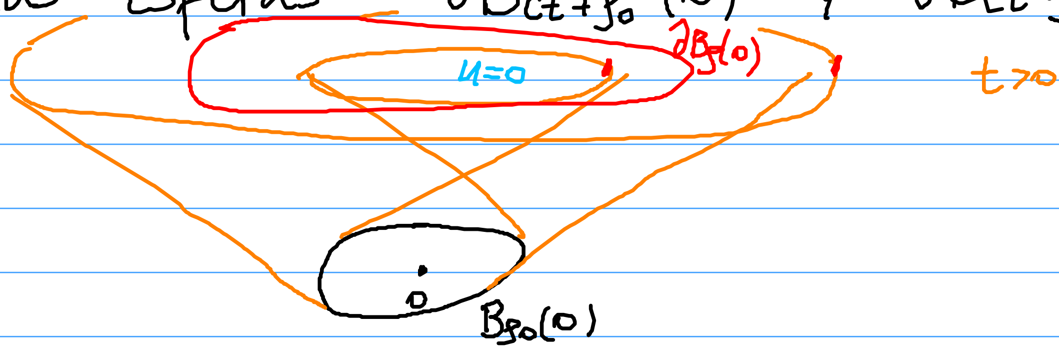


Los datos iniciales "fig cerca de un punto" $y_0 \in \mathbb{R}^3$ en $t=0$ sólo afectan la solución $u = u(x,t)$ en puntos cerca del cono $|x-y| = ct$



Ejemplo: sea $K = \overline{B_{\rho_0}(0)}$, $\operatorname{supp} f, g \subset K$
K compacto, $\rho_0 > 0$.

$\partial B_{ct}(x)$ intersecta $\overline{B_{\rho_0}(0)}$ para $ct > \rho_0$ sólo si $x \in \partial B_\rho(0)$ acotada por las esferas $\partial B_{ct+\rho_0}(0)$ y $\partial B_{ct-\rho_0}(0)$



Para cualquier $x \in \mathbb{R}^3$ y $t>0$ suficientemente grande, $t > \frac{|x|+\rho_0}{c}$, $u(x,t) \equiv 0$.

Método del descenso de Hadamard

Problema global de Cauchy en $\mathbb{R}^2$:

$$(1)\cdots \begin{cases} u_{tt} - c^2 \Delta_x u = 0, & x \in \mathbb{R}^2,\ t>0 \\ u(x,0) = f(x), & \\ u_t(x,0) = g(x), & x \in \mathbb{R}^2 \end{cases}$$

$x = (x_1, x_2) \in \mathbb{R}^2, \quad t > 0, \qquad f \in C^3, \; g \in C^2.$

Si $u = u(x_1, x_2, t)$ es solución de (1) entonces

$$\tilde{u}(x_1, x_2, x_3, t) := u(x_1, x_2, t)$$

es solución del siguiente problema de Cauchy en $\mathbb{R}^3$:

$$(2) \cdot\cdot \begin{cases} u_{tt} - c^2 \Delta \tilde{u} = 0, & x \in \mathbb{R}^3, \; t > 0 \\ u(x,0) = \tilde{f}(x) := f(x_1, x_2), & x \in \mathbb{R}^3 \\ u_t(x,0) = \tilde{g}(x) := g(x_1, x_2) & \end{cases}$$

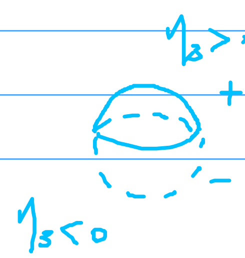
Por la fórmula de Kirchhoff:

$$\tilde{u}(x_1, x_2, x_3, t) = \frac{t}{4\pi} \int_{|\eta|=1} \tilde{g}(x + ct\eta)\, dS_\eta + \frac{\partial}{\partial t}\left(\frac{t}{4\pi} \int_{|\eta|=1} \tilde{f}(x + ct\eta)\, dS_\eta \right)$$

Aquí $\{|\eta| = 1\} = \{ \eta \in \mathbb{R}^3 : |\eta| = 1 \} = \partial B_1(0) \subset \mathbb{R}^3$.

$|\eta| = 1$

$$(\eta_1, \eta_2) \mapsto \Phi(\eta_1, \eta_2) = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \pm\sqrt{1 - (\eta_1^2 + \eta_2^2)} \end{pmatrix} \in \mathbb{R}^3$$



$$\eta_3 = +\sqrt{1-(\eta_1^2+\eta_2^2)} > 0 \qquad \text{(hemisferio norte)}.$$

$$dS_\eta = |X_{\eta_1} \times X_{\eta_2}| \, d\eta_1 d\eta_2$$

$$X_{\eta_1} = \begin{pmatrix} 1 \\ 0 \\ -\dfrac{\eta_1}{\sqrt{1-(\eta_1^2+\eta_2^2)}} \end{pmatrix}, \quad X_{\eta_2} = \begin{pmatrix} 0 \\ 1 \\ -\dfrac{\eta_2}{\sqrt{1-(\eta_1^2+\eta_2^2)}} \end{pmatrix}$$

$$|X_{\eta_1} \times X_{\eta_2}|^2 = \frac{1}{1-(\eta_1^2+\eta_2^2)}$$

$$\Rightarrow dS_\eta = \frac{d\eta_1 d\eta_2}{\sqrt{1-(\eta_1^2+\eta_2^2)}}$$

La int. de superficie se calcula integrando en un dominio (bola) en $\mathbb{R}^2$:

$$B_1(0) = \{\eta_1^2+\eta_2^2 < 1\} \subset \mathbb{R}^2.$$

$$\int_{\{|\eta|=1\} \subset \mathbb{R}^3} \tilde{g}(x+ct\eta)\, dS_\eta =$$

$$= 2 \int_{B_1(0) \subset \mathbb{R}^2} \frac{\tilde{g}(x_1+ct\eta_1, x_2+ct\eta_2)}{\sqrt{1-(\eta_1^2+\eta_2^2)}} \, d\eta_1 d\eta_2$$

$\tilde{g}$ no dep. de x_3

$x = (x_1, x_2) \in \mathbb{R}^2$

$$\Rightarrow \tilde{u}(x_1, x_2, x_3, t) = \frac{t}{2\pi} \int_{B_1(0)} \frac{g(x + ct\eta)}{\sqrt{1 - (\eta_1^2 + \eta_2^2)}} \, d\eta_1 \, d\eta_2$$

$$+ \frac{\partial}{\partial t} \left(\frac{t}{2\pi} \int_{B_1(0)} \frac{f(x + ct\eta)}{\sqrt{1 - (\eta_1^2 + \eta_2^2)}} \, d\eta_1 \, d\eta_2 \right)$$

$$= u(x_1, x_2, t)$$

Fórmula de Poisson. ($n = 2$).