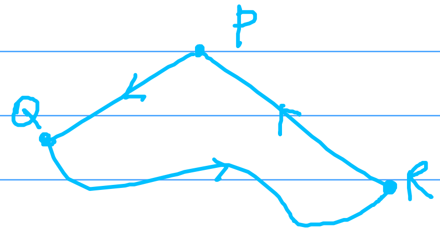
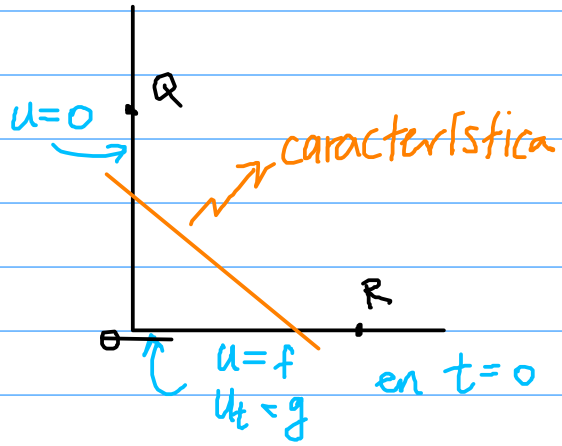
Lección 2.4: Unicidad. Método de energía.

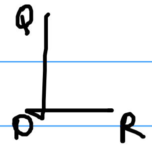


$$u(P) = \frac{1}{2}u(Q) + \frac{1}{2}u(R) + \frac{1}{2c}\int_{\widehat{QR}} u_t\,dx + c^2 u_x\,dt + \frac{1}{2c}\int_{QPR} h(x,t)\,dx\,dt$$

Fórmula de Green-Lagrange $\Leftarrow$ $u_{tt} - c^2 u_{xx} = h$.



No se puede prescribir arbitrariamente u, $\partial_\nu u$ sobre



QOR corta a rectas características en 2 puntos.

Curvas espaciales y temporales

Sea una curva $\mathcal{C}$ en el espacio-tiempo

$$\mathcal{C} = \{ (\hat{x}(s), \hat{t}(s)) \; : \; s \in I \subseteq \mathbb{R} \}, \qquad \hat{x}, \hat{t} \in C^1(I)$$

Idea: resolver $\Box u = h$, con datos sobre $\mathcal{C}$: $u, \partial_\nu u$ sobre $\mathcal{C}$.

$\hat{x}'^2 + \hat{t}'^2 \neq 0 \quad \Rightarrow \quad \hat{\tau}, \hat{\nu}$ tangente/normal sobre $\mathcal{C}$.

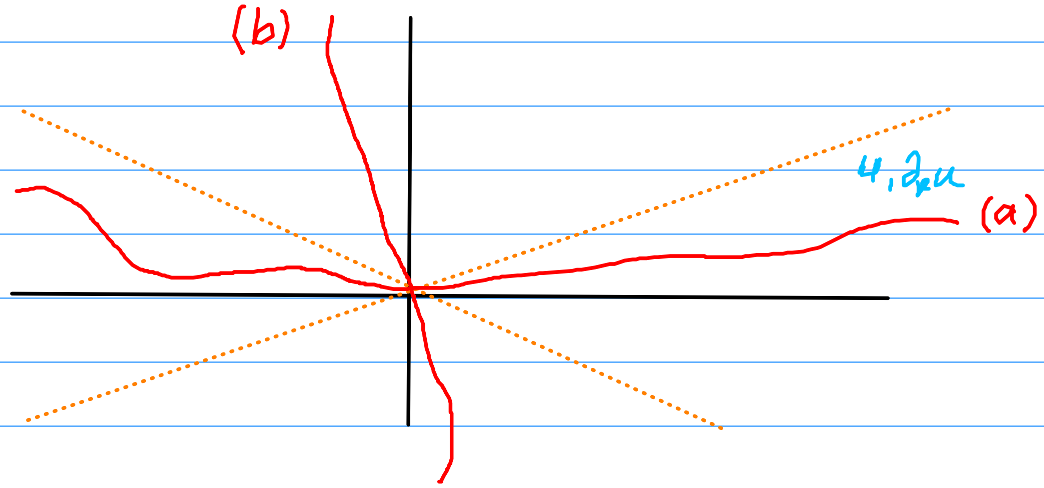
$\mathcal{C}$ nunca intersecta a una recta caracte-rística más de una vez si se cumple

$$c\,|\hat{t}'(s)| \neq |\hat{x}'(s)| \quad \forall s \in I$$

<u>Definición</u> (a) Si $c|\hat{t}'(s)| < |\hat{x}'(s)| \ \forall s \in I$ decimos que la curva $\mathcal{C}$ es espacial o "como espacio".
(b) Si $c|\hat{t}'(s)| > |\hat{x}'(s)| \ \forall s \in I$ entonces $\mathcal{C}$ es temporal o "como tiempo".

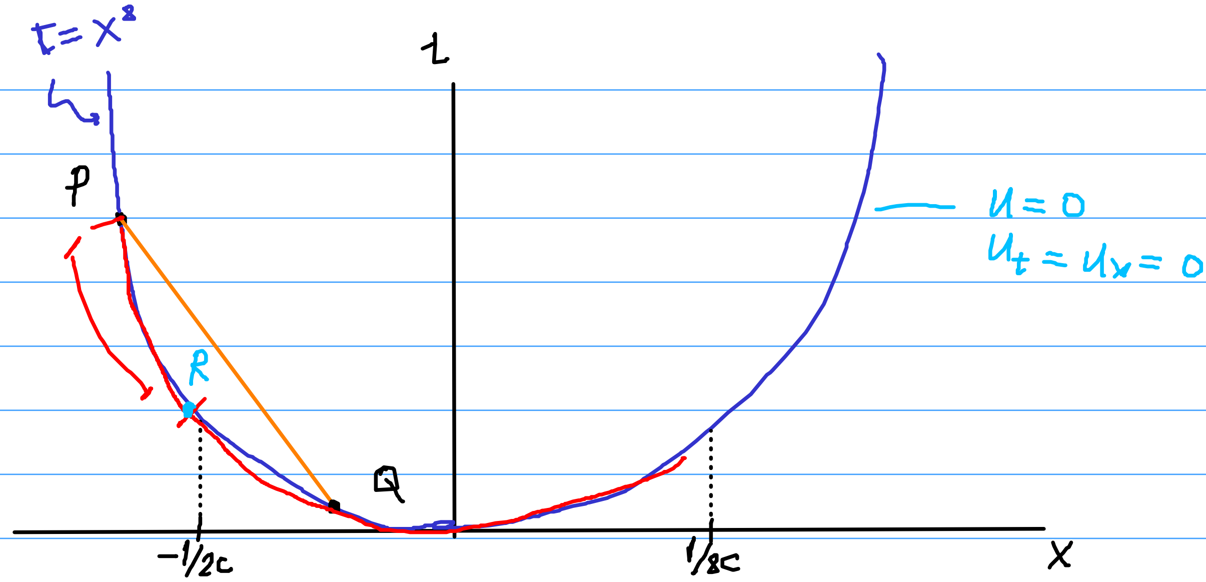
Observación: (i) El eje t es una curva temporal. $\hat{t}(s) = s$, $\hat{x}(s) \equiv 0$.

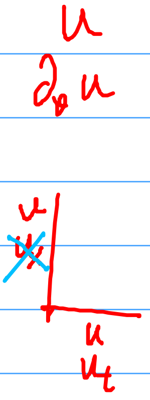
(ii) El eje x es una curva espacial $\hat{t}(s) \equiv 0$, $\hat{x}(s) = s$.



Ejemplo: $u_{tt} - c^2 u_{xx} = 1$

Datos $0 = u = u_x = u_t$ sobre $t = x^2$.





$$\mathcal{C} = \{ (x, \hat{t}(x)) = (x, x^2) \ : \ x \in I \}$$

$\mathcal{C}$ es espacial sólo en el segmento

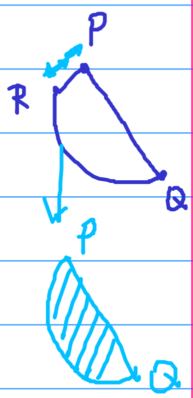
$$-\frac{1}{2c} < x < \frac{1}{2c}$$

$\overline{PQ}$ = recta característica.

Green-Lagrange :

$$\underset{0}{u(P)} = \frac{1}{2}\overset{0}{u(P)} + \frac{1}{2}\overset{0}{u(Q)} + \frac{1}{2c}\int_{P \to Q} c^2 \overset{0}{u_x}\, dt + \overset{0}{u_t}\, dx$$

$$+ \frac{1}{2c}\int h(x,t)\ dx\,dt$$



$$\Rightarrow \quad \frac{1}{2c}\text{ Área}\left(\ \right) = 0 \quad ! \quad \text{contradicción.}$$

<u>Teorema 1</u> (unicidad)
Sean $f \in C^2(\mathbb{R})$, $g \in C^1(\mathbb{R})$, $h \in C^1(\mathbb{R}\times(0,\infty))$
Entonces la solución $u \in C^2(\mathbb{R}\times(0,\infty)) \cap C^1(\mathbb{R}\times[0,\infty))$ del problema de Cauchy

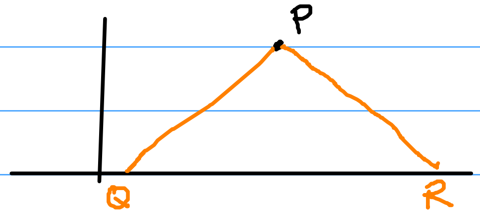
$$(1)\begin{cases} u_{tt} - c^2 u_{xx} = h, & x \in \mathbb{R},\ t>0 \\ u(x,0) = f(x) & x \in \mathbb{R} \\ u_t(x,0) = g(x), & \end{cases}$$

es <u>única</u>.

<u>Dem.</u> Sean u_1, u_2 dos soluciones. Entonces $u = u_1 - u_2$ es solución de

$$u_{tt} - c^2 u_{xx} = 0$$
$$u(x,0) = u_t(x,0) = 0$$

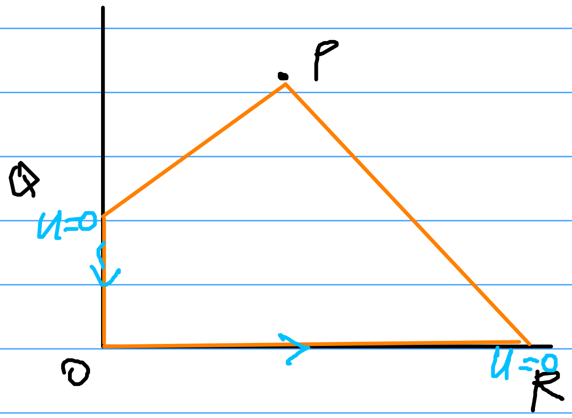
Sea $P = (x,t) \in \mathbb{R} \times (0,\infty)$ arbitrario.
$Q = (x-ct, 0)$, $R = (x+ct, 0)$.



$$GL \Rightarrow u(P) = u(x,t) = \frac{1}{2}\overbrace{u(x-ct,0)}^{u(Q)} + \frac{1}{2}\underbrace{\overbrace{u(x+ct,0)}^{u(R)}}_{=0} \quad 0 =$$
$$+ \frac{1}{2c}\int_{\overrightarrow{QR}} \underbrace{u_t}_{=0}\,dx + \underbrace{c^2 u_x}_{0}\,dt$$
$$+ \frac{1}{2c}\int_{PQR} \underbrace{(0)\,dx\,dt}_{=0}$$

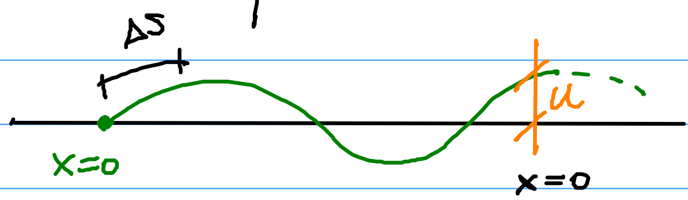
$$\Rightarrow u(x,t) \equiv 0 \quad \forall (x,t) \in \mathbb{R}\times(0,\infty) \qquad \square$$

Se puede usar GL para probar la unicidad de
$$\begin{cases} \square u = h(x,t) & x \in (0,L) \\ u = f,\ u_t = g \\ u = 0 & x = L \end{cases} \qquad \begin{cases} \square u = 0 \\ u = f,\ u_t = g \\ u = 0, & x = 0, L \end{cases}$$



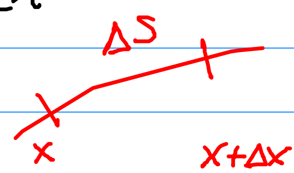
Energía y unicidad

Cuerda vibrante densidad $\rho_0 > 0$ constante por unidad de longitud.



$u = u(x,t)$

$$\frac{1}{2}\int_0^L \rho_0 u_t(x,t)^2\, dx = \text{energía cinética}$$



$$\Delta S - \Delta x = \int_x^{x+\Delta x} \left(\sqrt{1+u_x^2} - 1\right) dx$$

$$\approx \frac{1}{2} u_x^2 \Delta x \qquad |u_x| \ll 1$$

Tarea 2.

$$\Delta W \approx \frac{1}{2} T_0 u_x^2 \Delta x \qquad T_0 - \text{tensión constante}$$

trabajo $\qquad \frac{1}{2}\int_0^L T_0 u_x^2\, dx$ = energía potencial

$$\Rightarrow \quad \frac{1}{2}\int_0^L \left(\rho_0 u_t^2 + T_0 u_x^2\right) dx\, dt$$

energía total

$$c^2 := \frac{T_0}{\rho_0}$$

$\square u = u_{tt} - c^2 u_{xx} = h$

Se define :

$$E(t) := \frac{1}{2}\int_0^L (u_t^2 + c^2 u_x^2)(x,t)\, dx$$

energía.

consideramos el problema :

$$(2) \begin{cases} u_{tt} - c^2 u_{xx} = h, & x \in [0,L],\ t>0 \\ u(x,0) = f(x) & x \in [0,L] \\ u_t(x,0) = g(x), & \\ u(0,t) = a(t) & t>0 \\ u(L,t) = b(t) & \end{cases}$$

f, g, h, a, b conocidas.

<u>Lema</u> Sea $u \in C^2((0,L)\times(0,\infty)) \cap C^1([0,L]\times[0,\infty))$ una solución de $u_{tt} - c^2 u_{xx} = 0$. Entonces $\forall\, t_1 \geq t_0$, u satisface :

$$(3) \cdot\cdot\quad E(t_1) - E(t_0) = c^2 \int_{t_0}^{t_1} (u_x u_t)(L,t) - (u_x u_t)(0,t)\, dt.$$

<u>Dem</u>. $0 = u_{tt} - c^2 u_{xx} = \frac{1}{2}(u_t^2 + c^2 u_x^2)_t + {}$
$- c^2 (u_x u_t)_x$

Para $t_1 \geq t_0$:

$$E(t_1)-E(t_0)=\int_{t_0}^{t_1}\frac{dE}{dt}\,dt$$
$$=\int_{t_0}^{t_1}\int_0^L \tfrac{1}{2}\left(u_t^2+c^2u_x^2\right)_t\,dx\,dt$$
$$=c^2\int_{t_0}^{t_1}\int_0^L (u_xu_t)_x\,dx\,dt$$
$$=c^2\int_{t_0}^{t_1}(u_xu_t)(L,t)-(u_xu_t)(0,t)\,dt$$
$\square$

Lema Si $\exists\, u \in C^2([0,L)\times(0,\infty))\cap C^1([0,L]\times(0,\infty))$ solución al problema (2) entonces es única.

Dem. Sean u_1, u_2 dos soluciones.
Sea $u := u_1 - u_2$.

u es solución de:

$$\begin{cases} u_{tt}-c^2u_{xx}=0, & x\in[0,L],\ t>0 \\ u(x,0)=u_t(x,0)=0, & x\in[0,L] \\ u(L,t)=u(0,t)=0, & t>0 \end{cases}$$

Por lema anterior: $t=t_1$, $t_0=0$

$$0=E(t)-E(0)=c^2\int_0^t\left((u_xu_t)(L,\tilde t)-(u_xu_t)(0,\tilde t)\right)d\tilde t$$

$u_t(L,t)=0$
$u_t(0,t)=0$

La energía es constante: $E(t) = E(0)$ $\forall t > 0$.

pero $u_t(x,0) = 0$, $u_x(x,0) = 0$

$\downarrow$

$u(x,0) = 0$ $\forall x$

$\therefore$ $E(0) = 0$.

$\Rightarrow E(t) = 0$ $\forall t > 0$.

$\Rightarrow$ $u_t = 0$, $u_x = 0$ en $[0,L] \times (0,\infty)$

$\Rightarrow$ $u_t^2 + c^2 u_x^2 \equiv 0$ en $[0,L] \times (0,\infty)$

u es constante, $u = 0$ en $t = 0$

$\Rightarrow u(x,t) \equiv 0$ en $[0,L] \times (0,\infty)$

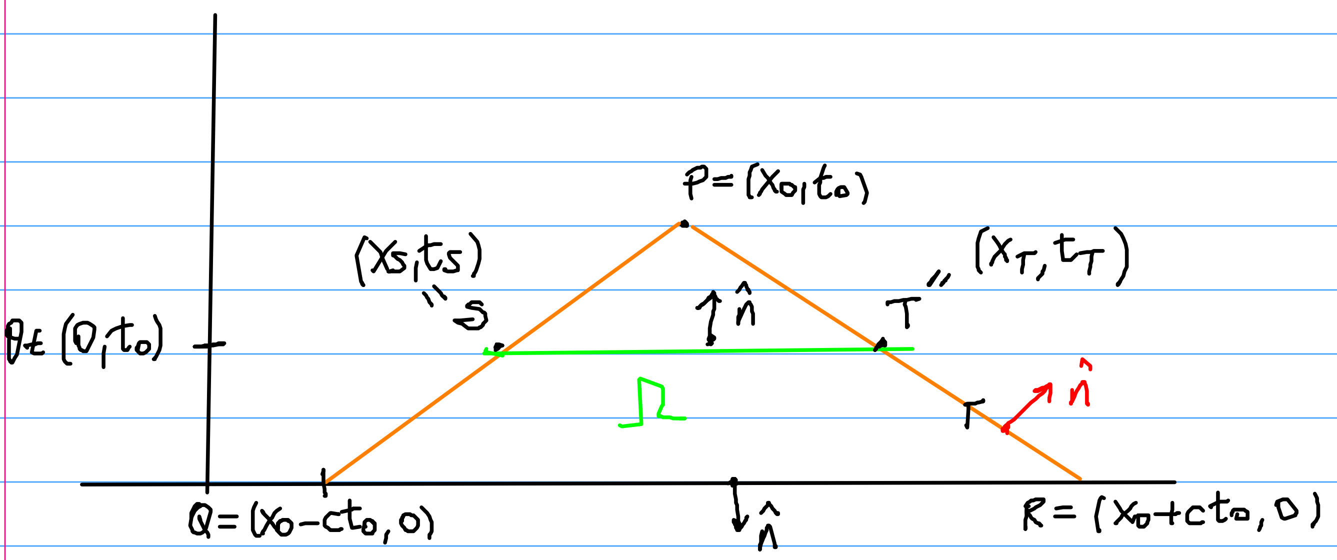
$\square$

Demostración alternativa del Teorema 1:

Sean u_1, u_2 dos soluciones de (1).
$u = u_1 - u_2$ es solución de

$$(4) \begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_t(x,0) = 0, & x \in \mathbb{R} \end{cases}$$

Sea $P=(x_0,t_0) \in \mathbb{R}\times(0,\infty)$ arbitrario.



Trapecio SQRT $=\Omega$

Mult. la ec. por u_t e integramos en Ω:

$$0=\int_\Omega u_t\,(u_{tt}-c^2u_{xx})\,dx\,dt$$

$$=\int_\Omega \frac{1}{2}\partial_t\left(u_t^2+c^2u_x^2\right)+\partial_x\left(-c^2u_xu_t\right)dx\,dt$$

$$\underset{\substack{\text{teo. de}\\ \text{la divergencia}}}{=}\int_{\partial\Omega}\begin{pmatrix}-c^2u_xu_t\\ \frac{1}{2}(u_t^2+c^2u_x^2)\end{pmatrix}\cdot\begin{pmatrix}\hat{n}_x\\ \hat{n}_t\end{pmatrix}d\sigma$$

$$u_tu_{tt}+\cancel{c^2u_xu_{xt}}-\cancel{c^2u_xu_{tx}}-c^2u_tu_{xx}$$

$\overline{QR}$: $t=0$, $\hat{n}_x=0$, $\hat{n}_t=-1$

$$\int_{\overrightarrow{QR}}(\,\ldots) = -\int_{x_Q}^{x_R}\frac{1}{2}\left(u_t^2+c^2u_x^2\right)dx = 0.$$

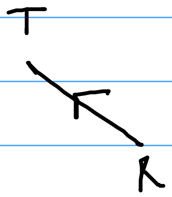
$u(x,0)=0 \Rightarrow u_x(x,0)=0$

$u_t(x,0)=0$

$\overline{TS}$: $\quad t=\theta, \qquad \hat{n}_x = 0, \quad \hat{n}_t = +1.$

$$\int_{S\leftarrow T} (\text{''}) = \int_{x_S}^{x_T} \frac{1}{2} \left(u_t^2 + c^2 u_x^2 \right) dx \geq 0$$

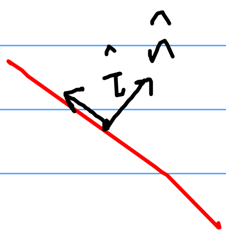
$x_S < x_T$

$\overline{RT}$:  $\quad \hat{x} = -cs + x_R, \quad \hat{t} = s$

$s \in (0, \theta)$

$\hat{x}'(s) = -c, \quad \hat{t}'(s) = 1, \quad \Rightarrow \quad \hat{n}_x = \frac{1}{\sqrt{1+c^2}}$

$\hat{n}_t = \frac{c}{\sqrt{1+c^2}}$



$\hat{\tau} = \begin{pmatrix} -c \\ 1 \end{pmatrix} \frac{1}{\sqrt{\ }}$

$\hat{n} = \begin{pmatrix} 1 \\ c \end{pmatrix} \cdot \frac{1}{\sqrt{\ }}$

$$\int_{T\leftarrow R} (\text{''}) = \int_0^\theta \frac{1}{\sqrt{1+c^2}} \left(\frac{c}{2} \left(u_t^2 + c^2 u_x^2 \right) - c^2 u_x u_t \right) ds$$

$$= \frac{c}{2\sqrt{1+c^2}} \int_0^\theta \underbrace{(u_t - c u_x)^2}_{u_t^2 - 2cu_xu_t + c^2u_x^2} ds \geq 0$$

Análogamente:

$$\int_{Q\leftarrow S} (\text{''}) = \frac{c}{2\sqrt{1+c^2}} \int_0^\theta (u_t + c u_x)^2 \, ds \geq 0$$

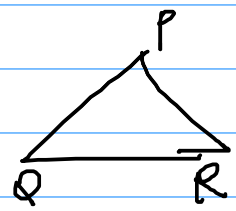
Sustituyendo:

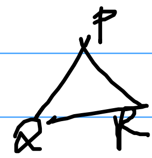
$$0 \leq \int_{S\leftarrow T} (\text{''}) = -\int_{T\leftarrow R} (\text{''}) - \int_{Q\leftarrow S} \leq 0$$

$$\Rightarrow \int_{x_S}^{x_T} \frac{1}{2}\left(u_t^2 + c^2 u_x^2\right)(x,\theta)\, dx = 0.$$

$$\Rightarrow \quad u_t^2 + c^2 u_x^2 = 0 \qquad \forall\ t=\theta \in (0, t_0)$$

arbitrario

$\Rightarrow \quad u_t^2 + c^2 u_x^2 = 0$ en 

Por condición inicial $u(x,t) \equiv 0$ en 

(x_0, t_0) arbitrario.

$\therefore$ la solución es única. $\square$