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An expansion method for non-linear Rayleigh waves

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Abstract

We use ideas from analytic bifurcation theory to develop expansions for periodic small amplitude traveling surface elastic waves of permanent form in the half-plane (Rayleigh waves). We focus on the case of hyperelastic materials where the traveling wave problem has a variational structure, and solve numerically the equations describing the lowest order approximation to the traveling wave solutions. For the materials considered, there is evidence for solutions describing elastic displacements that have discontinuous derivative at the boundary of the domain. © 2002 Published by Elsevier Science B.V.

1. Introduction

In this work, we develop a perturbation theory for periodic small amplitude non-linear surface elastic waves of permanent form in the half-plane, and present some new numerical results on the possible shape of such waves for a number of hyperelastic materials. One of our motivations for considering the problem is related to the common interpretation of traveling waves as solutions bifurcating from the trivial state. This viewpoint has been very fruitful in dispersive wave equations, especially in the simplest case of spatially periodic waves, where the problem is often reduced to solve a finite dimensional bifurcation equation. On the other hand, linear surface elastic waves in the half-plane are non-dispersive, and traveling waves that decay away from the surface would correspond to solutions bifurcating from an eigenvalue of infinite multiplicity.

To study the possibility of such a bifurcation, we develop a systematic perturbation theory that is formally analogous to solve the bifurcation and complementary equations appearing in Liapunov–Schmidt reduction order by order in a suitable small parameter. In this approach, the bifurcation equation is replaced by an infinite set of solvability conditions. The first of these conditions is a non-linear equation for the lowest order contribution, while the second and higher order solvability conditions have the same structure and involve the linearization of the first solvability equation around its solutions. The solvability equations are infinite dimensional, but one of the advantages of this approach is that they are equations for the boundary values of the elastic displacement, i.e. the displacement inside the domain can be at each order recovered from its boundary values.

The above scheme was developed in an effort to extend and understand earlier works on the problem. In particular, the first solvability condition coincides with the approximate equation derived and solved numerically by Lardner, and Parker and Talbot [1,2]. These authors were the first to obtain results on the possible shape of traveling surface elastic waves, and their argument has been simplified by Hunter [3], who derived an asymptotic evolution equation

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for the surface elastic displacement. This model equation is interesting in its own right, being a conservation law on the line with non-local flux, moreover, its traveling wave solutions are precisely the solutions of the first solvability condition.

Although the argument leading to the first solvability condition has been understood and applied to a variety of other non-dispersive systems in the half-plane, the existence of solutions to this equation has not been established for surface elastic waves, and numerical results have led to some controversy (see [4]). We thus start our analysis of the perturbation theory by studying the first solvability condition numerically. A new feature of our approach is the use of the constrained variational structure of the traveling wave problem for hyperelastic materials. This variational structure is also present in the solvability conditions, and is useful because the first solvability condition considered here has no intrinsic scale. By using the constraint to normalize the sequences of approximate solutions obtained numerically, we see that for all the non-linearities considered there exist sequences of numerical solutions that start to converge to non-trivial shapes. An interesting feature of these surface displacements obtained numerically is that they have discontinuities in their first derivative, specifically, cusps in the horizontal component of the surface elastic displacement. These features were observed for all the non-linearities considered. The second and higher order solvability conditions will be studied in a future work.

The paper is organized as follows. In Section 2, we state the traveling wave problem and fix the notation. In Section 3, we gather necessary information on the linear traveling wave problem. In Section 4, we describe the perturbation theory for traveling waves of small amplitude. We derive a sequence of solvability conditions (equations) for the terms in the expansion, and remark on the relation of our approach to earlier works and to bifurcation theory and Signorini's method. In Section 5, we study the basic structure of the solvability conditions showing that in the case of hyperelastic materials these equations are constrained variational problems. In Section 6, we present some numerical solutions of the first solvability condition for three hyperelastic materials, the St. Venant-Kirchhoff material and two toy-models with simpler structure.

2. The traveling wave problem

We consider $\mathbf{R}^2$ with the Cartesian coordinates (x_1, x_2) and an elastic medium occupying in its unstrained state the half-plane $H = \{x = (x_1, x_2) \in \mathbf{R}^2 : x_2 \geq 0\}$. We also let $u(x) = H \to \mathbf{R}^2$ denote the elastic displacement with Cartesian components $u_i(x_1, x_2)$, $i = 1, 2$. The density of the material will be denoted by ρ, and will be assumed to be constant. The internal elastic forces due to a displacement u can be obtained from the (first Piola–Kirchhoff) stress tensor $\tau(x) : H \to \mathbf{R}^2 \times \mathbf{R}^2$, thought of here as a 2×2 real matrix with components τ_{ij}, $i, j = 1, 2$. The stress tensor τ is a function of the derivative ∇u of the displacement, and the elastic response of the material is determined by the particular choice of the function $\tau(\nabla u)$. With this notation, the equations of motion of elasticity are

$$\rho \partial_{tt} u_i = \sum_{j=1}^{2} \partial_{x_j} \tau_{ij}, \quad i = 1, 2, \text{ in } H. \tag{2.1}$$

At the boundary $\partial H = \mathbf{R}$, we will impose the "zero-traction" boundary condition

$$\sum_{j=1}^{2} \tau_{ij} \hat{n}_j = 0 \quad \text{at } \partial H, \tag{2.2}$$

with $\hat{n} = [\hat{n}_1, \hat{n}_2] = [0, -1]$, the outward unit normal at ∂H. A convenient shorthand for (2.1) and (2.2) is

$$\rho u_{tt} = \nabla \cdot \tau \text{ in } H, \qquad \tau \cdot \hat{n} = 0 \quad \text{at } \partial H. \tag{2.3}$$

The above equations, possibly with additional boundary conditions, constitute the equations for free surface elastic waves, and are clearly meaningful for a variety of domains.

In the problem of traveling waves of permanent form in the half-plane we are seeking solutions of (2.1), (2.2) that have the form $u(x_1 - ct, x_2)$ for some constant $c \in \mathbf{R}$. Rewriting the equations in the system of coordinates $\tilde{x}_1 = x_1 - ct$, $\tilde{x}_2 = x_2$, $\tilde{t} = t$, and dropping the tilde from the notation, we are thus looking for displacements $u(x_1, x_2)$ satisfying

$$\rho c^2 \partial_{x_1}^2 u = \nabla \cdot \tau \quad \text{in} \quad H, \qquad \tau \cdot \hat{n} = 0 \quad \text{at} \quad \partial H. \tag{2.4}$$

In addition, we will impose the decay condition

$$\lim_{x_2 \to \infty} u(x_1, x_2) = 0, \quad \forall x_1 \in \mathbf{R}, \tag{2.5}$$

and periodicity in the horizontal direction, i.e.

$$u(x_1 + 2\pi, x_2) = u(x_1, x_2), \quad \forall (x_1, x_2) \in H. \tag{2.6}$$

In view of the periodicity condition (2.6), we can work in the half-cylinder D obtained by identifying the points $(x_1 + 2\pi, x_2)$ and (x_1, x_2) of the strip $\tilde{D} = \{(x_1, x_2) \in H : x_1 \in [-\pi, \pi]\}$. The formalism that follows is independent of the choice of the particular fundamental strip $\tilde{D}$.

In this work, we will consider the traveling wave problem for hyperelastic materials. We will thus assume that the stress tensor τ is given by

$$\tau_{ij} = \frac{\partial W(\nabla u)}{\partial u_{i,j}}, \quad \text{with} \ u_{i,j} = \partial_{x_i} u_j, \ \ i, j = 1, 2. \tag{2.7}$$

The real function W is the potential energy density.

Note that by Kirchhoff's variational formulation of hyperelasticity (see [5], Chapter 7) the traveling wave equations and boundary conditions (2.4)–(2.6) can also be written as the Euler–Lagrange equations for the Lagrangian L given by

$$L = \frac{1}{2} \rho c^2 \int_D \sum_{i=1}^{2} (\partial_{x_1} u_i)^2 - \int_D W(\nabla u). \tag{2.8}$$

The velocity c^2 plays the role of the Lagrange multiplier.

The potential energy density W will be decomposed into $W = W^{\mathrm{L}} + W^{\mathrm{NL}}$ with $W^{\mathrm{L}} = W^{\mathrm{L}}(u, u)$ quadratic and $W^{\mathrm{NL}} = W^{\mathrm{L}}(u, u, u)$ cubic in ∇u, respectively. The corresponding stresses obtained via (2.7) will be denoted by $\tau^{\mathrm{L}}(u)$ and $\tau^{\mathrm{NL}}(u, u)$, respectively. The quadratic potential energy density is standard for isotropic materials and is given by

$$W^{\mathrm{L}} = \tfrac{1}{2} \lambda (\operatorname{tr} \gamma)^2 + \mu \operatorname{tr}(\gamma^2), \quad \text{with} \quad \gamma = \tfrac{1}{2} [\nabla u + (\nabla u)^{\mathrm{T}}], \tag{2.9}$$

and $\lambda, \mu > 0$ the Lamé constants. Quadratic non-linearities will be specified by the real constant coefficients W^{NL}_{abcdef} and the expression

$$W^{\mathrm{NL}} = \sum_{a,b,c,d,e,f=1}^{2} W^{\mathrm{NL}}_{abcdef} u_{a,b} u_{c,d} u_{e,f}. \tag{2.10}$$

The corresponding stress is

$$\tau^{\mathrm{NL}}_{ij}(u, u) = \sum_{\kappa,\lambda,\mu,\nu=1}^{2} S^{\mathrm{NL}}_{ij\kappa\lambda\mu\nu} u_{\kappa,\lambda} u_{\mu,\nu}, \tag{2.11}$$

with

$$S^{\mathrm{NL}}_{ij\kappa\lambda\mu\nu} = \sum_{\phi,\chi,\psi,\omega=1}^{2} (W^{\mathrm{NL}}_{ij\phi\chi\psi\omega} + W^{\mathrm{NL}}_{\phi\chi ij\psi\omega} + W^{\mathrm{NL}}_{\phi\chi\psi\omega ij}). \tag{2.12}$$

The coefficients W^{NL}_{abcdef} are not uniquely determined for a given non-linearity, but once they are specified, they determine the $S^{\mathrm{NL}}_{ij\kappa\lambda\mu\nu}$ unambiguously through (2.12). It is not necessary at this point to specify the non-linearity.

The choice of quadratic non-linearity is for notational convenience. Physically relevant models typically include higher order non-linearities and a consistent theory must take them into account. On the other hand, as we will see the inclusion of higher order terms require minor modifications and the main features of the expansion remain the same. For instance, the leading order approximation to the shape of the traveling waves involves the quadratic non-linearity only, even in the presence of higher order terms.

3. Linear traveling waves

To develop a perturbation theory for the non-linear traveling wave problem, we first consider its linearization around the trivial solution $u \equiv 0$. Using the quadratic potential energy (2.9), and (2.7), the linear traveling wave equations become

$$(\lambda + \mu)\partial_{x_i}(\nabla \cdot u) + \mu \Delta u_i - \rho c^2 \partial^2_{x_1} u_i = 0, \quad i = 1, 2 \ \text{in} \ H, \tag{3.1}$$

and

$$-\lambda \delta_{i2}(\nabla \cdot u) - \mu(u_{i,2} + u_{2,i}) = 0, \quad i = 1, 2 \ \text{at} \ \partial H. \tag{3.2}$$

We also impose the decay and periodicity conditions (2.5) and (2.6), respectively.

To solve this equation we expand the displacement in Fourier series as $u_i(x_1, x_2) = \sum_{k\in \mathbf{Z}} \mathrm{e}^{\mathrm{i}kx_1} \hat{v}_i(k, x_2)$, $i, j = 1, 2$. For each $k \in \mathbf{Z}$, we obtain a linear homogeneous system of two second-order ordinary differential equations for $\hat{v}_1(k, x_2)$, $\hat{v}_2(k, x_2)$, with boundary conditions at $x_2 = 0$ and at infinity. The system also involves the velocity c^2 as a parameter. We find that, in order for solutions to exist, the velocity c^2 must satisfy

$$\frac{(A^2+1)^2}{4AB} = 1, \quad \text{where} \ A^2 = 1 - \frac{\rho c^2}{\mu}, \qquad \text{and} \qquad B^2 = 1 - \frac{\rho c^2}{\lambda + 2\mu}. \tag{3.3}$$

Then, letting c_0^2 be a solution of (3.3) and $A = A(c_0^2)$, $B = B(c_0^2)$ from now on, the solutions are

$$\hat{v}_1(k, x_2) = ia_k \frac{k}{|k|}\left(-A\,\mathrm{e}^{-|k|Ax_2} + \frac{2A}{A^2+1}\mathrm{e}^{-|k|Bx_2}\right), \quad k \in \mathbf{Z} \setminus \{0\}, \tag{3.4}$$

$$\hat{v}_2(k, x_2) = a_k \left(\mathrm{e}^{-|k|Ax_2} - \frac{2AB}{A^2+1}\mathrm{e}^{-|k|Bx_2}\right), \quad k \in \mathbf{Z} \setminus \{0\}, \tag{3.5}$$

with $a_k \in \mathbf{C}$ arbitrary ($a_{-k} = \bar{a}_k$ for real displacements). For $k = 0$, we have the trivial solution $\hat{v}_i(0, x_2) \equiv 0$, $i = 1, 2$.

Eq. (3.3) is a cubic equation for c^2 and has only one real solution c_0^2, known as the Rayleigh speed (see e.g. [5], Chapter 8). Eq. (3.3) also plays the role of the dispersion relation, and since it does not involve k, linear waves are dispersionless. From (3.4) and (3.5), the boundary value $u_1(x_1, 0)$ of the horizontal displacement is arbitrary and completely determines $u_2(x_1, 0)$ and the displacement $u(x_1, x_2)$ in the rest of the domain (alternatively, we may specify $u_2(x_1, 0)$). All possible displacements $u(x_1, x_2)$ corresponding to the different boundary values of $u_1(x_1, 0)$ travel with the same speed c_0. The above construction also give us solutions in $H^1(D, \mathbf{R}^2)$ with boundary displacements in $L^2(S^1, \mathbf{R}^2)$.

Remark 3.1. The lack of dispersion in linear free surface elastic waves on the half-plane can also be deduced from the scale invariance of the equations of motion and the domain (see e.g. [6]). Examples of domains leading to dispersive free surface elastic waves are the strip (e.g. with $x_2 \in [0, 1]$), and the half-plane occupied by two materials of different density (e.g. density ρ_1 for $x_2 \in [0, 1)$, and density ρ_2 for $x_2 \in [1, \infty)$). Stokes waves and their modulation for the second domain were studied in [7].

We will also need information on the inhomogeneous linear traveling wave problem

$$\nabla \cdot \tau^{\mathrm{L}}(u) - \rho c_0^2 \partial_{x_1}^2 u = F \quad \text{in } H, \qquad \tau^{\mathrm{L}}(u) \cdot \hat{n} = f \quad \text{at } \partial H, \tag{3.6}$$

with the decay and periodicity conditions (2.5) and (2.6), respectively. The functions $F = [F_1, F_2] : H \to \mathbf{R}^2$ and $f = [f_1, f_2] : H \to \mathbf{R}^2$ are assumed to be 2π-periodic in x_1.

To solve the inhomogeneous equation, we expand the displacement in Fourier series as before with $F_i(x_1, x_2) = \sum_{k\in\mathbf{Z}} \mathrm{e}^{\mathrm{i}kx_1} \hat{F}_i(k, x_2)$ and $f_i(x_1, x_2) = \sum_{k\in\mathbf{Z}} \mathrm{e}^{\mathrm{i}kx_1} \hat{f}_i(k, x_2)$, $i, j = 1, 2$. We similarly obtain for each $k \in \mathbf{Z}$ the inhomogeneous version of the linear system encountered previously. The general solution has the form $\hat{u}_i(k, x_2) = \hat{w}_i(k, x_2) + c_k \hat{v}_i(k, x_2)$, $i = 1, 2$, with $\hat{v}_i(k, x_2)$, $i = 1, 2$ the solutions of the homogeneous system given in (3.4), (3.5), $c_k \in \mathbf{C}$ arbitrary, and $\hat{w}_i(k, x_2)$ a solution of the inhomogeneous equations given in Appendix A.

A necessary condition for solutions of the inhomogeneous equations to exist is that F and f must satisfy

$$\int_D u^*(x_1, x_2) \cdot F(x_1, x_2) - \int_{\partial D} u^*(x_1, 0) \cdot f(x_1) = 0, \tag{3.7}$$

for every solution $u(x_1, x_2)$ of the homogeneous problem (3.1). This solvability condition is derived straightforwardly by multiplying (3.6) by $u^*(x_1, x_2)$ and integrating by parts. Alternatively, writing an arbitrary solution u of the homogeneous problem as $u(x_1, x_2) = \sum_{\mathbf{Z}\setminus\{0\}} a_k v(k)$ with $v_i(k) = \mathrm{e}^{\mathrm{i}kx_1} \hat{v}_i(k, x_2)$, $i = 1, 2$, see (3.4) and (3.5), we may write (3.7) as

$$\int_0^\infty \hat{v}^*(k, x_2) \cdot \hat{F}(k, x_2)\,\mathrm{d}x_2 - \hat{v}^*(k, 0) \cdot \hat{f}(k) = 0, \quad \forall k \in \mathbf{Z} \setminus \{0\}, \tag{3.8}$$

and

$$\int_0^\infty \hat{F}_i(0, x_2)\,\mathrm{d}x_2 - \hat{f}_i(0) = 0, \quad i = 1, 2 \tag{3.9}$$

for $k = 0$.

Remark 3.2. In the special case where $F = \nabla \cdot g$ and $f = g|_{\partial H}$ for some tensor $g : H \to \mathbf{R}^2 \times \mathbf{R}^2$, i.e. 2π-periodic in the horizontal direction and decays at infinity, condition (3.9) is satisfied identically.

4. Small amplitude non-linear traveling waves

We now develop a perturbation theory for the traveling wave equations (2.4)–(2.6). We assume that the displacement u and the difference $c^2 - c_0^2$ from the Rayleigh speed can be expanded in powers of a small parameter α, so that

$$u = \alpha u^{[0]} + \alpha^2 u^{[1]} + \alpha^3 u^{[2]} + \cdots, \tag{4.1}$$

and

$$c^2 - c_0^2 = \alpha\lambda_1 + \alpha^2\lambda_2 + \alpha^3\lambda_3 + \cdots. \tag{4.2}$$

The $u^{[i]}$ should satisfy the periodicity and decay conditions (2.5) and (2.6), respectively. Physically, α can be the ratio of the boundary displacement (u_1 or u_2) to the horizontal period. Inserting the expansions into (2.4)–(2.6) and matching powers of α, we have at order α^1

$$\nabla \cdot \tau^{\mathrm{L}}(u^{[0]}) - \rho c^2 \partial_{x_1}^2 u^{[0]} = 0, \quad \text{in } H, \qquad \tau^{\mathrm{L}}(u^{[0]}) \cdot \hat{n} = 0 \quad \text{at } \partial H, \tag{4.3}$$

at order α^2

$$\nabla \cdot \tau^{\mathrm{L}}(u^{[1]}) - \rho c_0^2 \partial_{x_1}^2 u^{[1]} = -\nabla \cdot \tau^{\mathrm{NL}}(u^{[0]}, u^{[0]}) + \lambda_1 \rho \partial_{x_1}^2 u^{[0]} \quad \text{in } H, \tag{4.4}$$

$$\tau^{\mathrm{L}}(u^{[1]})\cdot\hat{n} = -\tau^{\mathrm{NL}}(u^{[0]},u^{[0]})\cdot\hat{n} \quad \text{at } \partial H, \tag{4.5}$$

and at order α^r, $r\geq 2$

$$\nabla\cdot\tau^{\mathrm{L}}(u^{[r-1]}) - \rho c_0^2\partial_{x_1}^2 u^{[r-1]} = \sum_{\substack{i+j=r,\\ i,j\geq 1}}(-\nabla\cdot\tau^{\mathrm{NL}}(u^{[i-1]},u^{[j-1]}) + \lambda_i\rho\partial_{x_1}^2 u^{[j-1]}) \quad \text{in } H, \tag{4.6}$$

$$\tau^{\mathrm{L}}(u^{[r-1]})\cdot\hat{n} = -\sum_{\substack{i+j=r,\\ i,j\geq 1}}\tau^{\mathrm{NL}}(u^{[i-1]},u^{[j-1]})\cdot\hat{n} \quad \text{at } \partial H. \tag{4.7}$$

Thus, $u^{[0]}$ satisfies Eqs. (3.1) and (3.2) for linear traveling waves, while the $u^{[r-1]}$ with $r\geq 2$ satisfy the inhomogeneous linear equation (3.6), with the inhomogeneous part depending on $u^{[0]},\dots,u^{[r-2]}$ and $\lambda_1,\dots,\lambda_{r-1}$. In the inhomogeneous equations, we are looking for pairs $u^{[r-1]}$, λ_{r-1}. The way to proceed is standard: first we let $u^{[0]}=v^{[0]}$ be a solution of the homogeneous system (4.3). To solve the order α^2 equations (4.4) and (4.5) for $u^{[1]}$, the inhomogeneous part must satisfy the solvability condition (3.7). This is an equation for $v^{[0]}$ and λ_1. Assuming that solutions $v^{[0]}$, λ_1 exist and that the solvability condition is also sufficient, we have a solution $u^{[1]}=w^{[1]}+v^{[1]}$ of (4.4) and (4.5) with $w^{[1]}$ given by the expressions of Appendix A, and $v^{[1]}$ an arbitrary solution of the homogeneous linear equation (4.3). We can continue this formal procedure to higher order, i.e. determining $v^{[1]}$ and λ_2 from the solvability condition for the equation for $u^{[2]}$, decomposing the solution using the particular solution of Appendix A and so on. Assuming that the solvability conditions have solutions, and that they are sufficient for solving the inhomogeneous equation at each order, we may thus write the solution as

$$u = \alpha v^{[0]} + \alpha^2(v^{[1]}+w^{[1]}) + \alpha^3(v^{[2]}+w^{[2]}) + \cdots, \tag{4.8}$$

where each $v^{[r-2]}$, $r\geq 2$, is a solution of the homogeneous linear traveling wave equation and is found from the solvability condition for the order α^r equation for $u^{[r-1]}$, and each $w^{[r-1]}$, $r\geq 2$, is the particular solution of the order α^r inhomogeneous equation for $u^{[r-1]}$, given in Appendix A.

Using (4.3)–(4.7) and the decomposition (4.8) of the displacement u, the solvability condition determining $v^{[r-2]}$ and λ_{r-1}, $r\geq 2$ can be written as

$$\int_D \hat{v}^*(k)\cdot F^{[r-1]} - \int_{\partial D}\hat{v}^*(k)\cdot f^{[r-1]} = 0, \quad \forall k\in\mathbf{Z}\setminus\{0\}, \tag{4.9}$$

where the $\hat{v}_i(k)=\mathrm{e}^{\mathrm{i}kx_1}\hat{v}_i(k,x_2)$, i.e. as in (3.4), (3.5), and

$$\begin{aligned} F^{[r-1]} = -\sum_{\substack{i+j=r,\\ i,j\geq 1}}&(\nabla\cdot\tau^{\mathrm{NL}}(v^{[i-1]},v^{[j-1]}) + \nabla\cdot\tau^{\mathrm{NL}}(v^{[i-1]},w^{[j-1]}) + \nabla\cdot\tau^{\mathrm{NL}}(w^{[i-1]},v^{[j-1]}) \\ &+\nabla\cdot\tau^{\mathrm{NL}}(w^{[j-1]},w^{[j-1]})) + \sum_{\substack{i+j=r,\\ i,j\geq 1}}\lambda_i\partial_{x_1}^2(v^{[i-1]}+w^{[j-1]}), \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} f^{[r-1]} = -\sum_{\substack{i+j=r,\\ i,j\geq 1}}&(\tau^{\mathrm{NL}}(v^{[i-1]},v^{[j-1]})\cdot\hat{n} + \tau^{\mathrm{NL}}(v^{[i-1]},w^{[j-1]})\cdot\hat{n} + \tau^{\mathrm{NL}}(w^{[i-1]},v^{[j-1]})\cdot\hat{n} \\ &+\tau^{\mathrm{NL}}(w^{[i-1]},w^{[j-1]})\cdot\hat{n}). \end{aligned} \tag{4.11}$$

Here, we have set $w^{[0]}\equiv 0$. From (4.10) and (4.11), the solvability condition for the inhomogeneous equation for $u^{[1]}$ is a non-linear equation for $v^{[0]}$ and λ_1, while the solvability condition for the inhomogeneous equation for $u^{[r-1]}$, $r\geq 3$ is a linear equation for $v^{[r-2]}$ and λ_{r-1} involving the functions $v^{[0]},\dots,v^{[r-3]}$, $w^{[1]},\dots,w^{[r-2]}$ and numbers $\lambda_1, \dots, \lambda_{r-2}$ determined at the previous stages. Once the solvability condition for the equation for $u^{[r-1]}$, $r \geq 3$ is satisfied, the functions $v^{[0]}, \dots, v^{[r-3]}, v^{[r-2]}, w^{[1]}, \dots, w^{[r-2]}$ and numbers $\lambda_1, \dots, \lambda_{r-2}, \lambda_{r-1}$ appearing in the inhomogeneous part of the equation are known, and we let $u^{[r-1]} = w^{[r-1]} + u^{[r-1]}$ as described above.

Remark 4.1. Expanding $F^{[r-1]}$ and $f^{[r-1]}$ in their Fourier series, the solvability conditions take the form of (3.8). Note that the solvability condition (3.9) for $k = 0$ is satisfied automatically, since the zeroth Fourier coefficient of the $\partial_{x_1}^2(v^{[i-1]} + w^{[j-1]})$ vanishes, while the zeroth Fourier coefficient of the remaining terms satisfy (3.9) by Remark 3.2.

The formalism we have described does not require that the material be hyperelastic and can be applied to the question of traveling waves in a variety of non-dispersive, non-linear wave equations in the half-plane. Some examples are piezoelectricity (see e.g. [8]), non-linear optics, and the class of hyperbolic conservation laws considered by Hunter [3]. The first solvability condition was essentially derived by Lardner, and Parker and Talbot [1,2]. The higher order theory here follows a standard procedure for continuing solutions perturbatively (e.g. in the presence of bifurcation). A closely related method for constructing higher order corrections was originally given by Parker [9], and alternative derivations can be obtained by extending the multiple-scales argument of Hunter [3] (see also [10]) to higher orders (see e.g. [11]).

The formal connection to (analytic) bifurcation theory and Liapunov–Schmidt reduction is through the decomposition of the displacement $u^{[i]}$ at each order into a part $v^{[i]}$ in the kernel of the linear traveling wave operator, and a part belonging to the image of the operator. Instead of projections to the image and co-image of the linear traveling wave operator, we here use the solution given in Appendix A, and the procedure above amounts to solving the bifurcation and complementary equation perturbatively, order by order. The analog of the bifurcation equation is the infinite set of solvability conditions and one of the difficulties here is that these equations are infinite dimensional (see [12,13] for rigorous results on bifurcations from an eigenvalue of infinite multiplicity). The expansion method we described is also similar to Signorini's method in static elasticity (see e.g. [14], where the formal relation to Liapunov–Schmidt reduction is also pointed out).

5. Variational structure of the solvability conditions

In this section we show that the solvability conditions for hyperelastic materials are constrained variational problems. Also we see that the second and higher order solvability conditions have a common structure.

In order to solve (4.4) and (4.5) with the appropriate boundary conditions we must satisfy the first solvability condition, which by (4.9) may be written as

$$S_k(c^{[0]}, \lambda_1) = -\int_D \hat{v}^*(k) \cdot (\nabla \cdot \tau^{\mathrm{NL}}(v^{[0]}, v^{[0]})) + \int_{\partial D} \hat{v}^*(k) \cdot (\tau^{\mathrm{NL}}(v^{[0]}, v^{[0]}) \cdot \hat{n}) + \lambda_1 \rho \int_D \hat{v}^*(k) \cdot \partial_{x_1}^2 v^{[0]} = 0,$$
$$\forall k \in \mathbf{Z} \setminus \{0\}, \tag{5.1}$$

with $\hat{v}_i(k) = \mathrm{e}^{\mathrm{i}kx_2} \hat{v}_i(k, x_2)$, i.e. see (3.4) and (3.5). Writing

$$v^{[0]} = \sum_{k \in \mathbf{Z} \setminus \{0\}} c_k^{[0]} \hat{v}(k), \tag{5.2}$$

and letting $c^{[0]}$ be the vector with the components $c_k^{[0]} \in \mathbf{C}$, $k \in \mathbf{Z} \setminus \{0\}$ ($c_{-k} = \bar{c}_k$ for real $v^{[0]}$), (5.1) is an infinite system of quadratic equations indexed by $k \in \mathbf{Z} \setminus \{0\}$, from which we want to determine $c^{[0]}$ and λ_1. We can also consider the Galerkin projections of (5.1) by choosing finite subsets $\mathcal{J} \subset \mathbf{Z} \setminus \{0\}$ and solving the finite set of equations $S_k(c_{\mathcal{J}}^{[0]}, \lambda_1) = 0$, $k \in \mathcal{J}$, with $c_{\mathcal{J}}^{[0]}$ vectors $c^{[0]}$ for which $c_k^{[0]} = 0$, $\forall k \notin \mathcal{J}$.

Remark 5.1. For higher order non-linearities, the first solvability condition will also have the form (5.1), only involving the quadratic non-linearity.

An important property of Eq. (5.1) and its Galerkin approximations is homogeneity in $c^{[0]}$ and λ_1, if $v^{[0]} = v$, $\lambda_1 = \lambda$ is a solution of (5.1), then so is $\epsilon v, \epsilon\lambda, \forall\epsilon \in \mathbf{C}$. Also, for hyperelastic materials, (5.1) describes a constrained variational problem:

Proposition 5.1.

1. *The system of Eq.* (5.1) *is equivalent to*

$$\nabla_{\bar{c}^{[0]}} V(c^{[0]}) = \lambda_1 \nabla_{\bar{c}^{[0]}} I(c^{[0]}), \tag{5.3}$$

where the kth component of $\nabla_{\bar{c}^{[0]}}$ *is* $\partial_{\bar{c}_k^{[0]}}$, $k \in \mathbf{Z} \setminus \{0\}$, *and*

$$V(c^{[0]}) = \int_D W^{\mathrm{NL}}(v^{[0]}, v^{[0]}, v^{[0]}), \qquad I(c^{[0]}) = \frac{1}{2}\rho\int_D \sum_{i=1}^{2}(\partial_{x_1} v_i^{[0]})^2. \tag{5.4}$$

2. *The Galerkin projections of* (5.1) *to finite sets* $\mathcal{J} \subset \mathbf{Z} \setminus \{0\}$ *are equivalent to*

$$\nabla_{\bar{c}_{\mathcal{J}}^{[0]}} V_{\mathcal{J}}(c_{\mathcal{J}}^{[0]}) = \lambda_1 \nabla_{\bar{c}_{\mathcal{J}}^{[0]}} I_{\mathcal{J}}(c_J^{[0]}), \tag{5.5}$$

with $V_{\mathcal{J}}$, $I_{\mathcal{J}}$ *as in* (5.4) *with* $c^{[0]}$, $v^{[0]}$ *replaced by* $c_{\mathcal{J}}^{[0]}$, $v_{\mathcal{J}}^{[0]}$, *respectively*.

Proof.

1. From the definition of V and (5.2) we compute for $p \in \mathbf{Z} \setminus \{0\}$

$$\frac{\partial V}{\partial \bar{c}_p^{[0]}} = \int_D \sum_{a,b,c,d,e,f=1}^{2} V_{abcdef}^{\mathrm{NL}}((\partial_{x_b}\hat{v}_a^*(p))u_{c,d}u_{e,f} + u_{a,b}(\partial_{x_d}\hat{v}_c^*(p))u_{e,f} + u_{a,b}u_{c,d}(\partial_{x_f}\hat{v}^*(p))).$$

Passing the derivative to the other side in each of the three terms, we have

$$\begin{aligned}\frac{\partial V}{\partial \bar{c}_p^{[0]}} = &-\int_D \sum_{i=1}^{2}\hat{v}^*(p)\sum_{j=1}^{2}\partial_{x_j}\sum_{\phi,\chi,\psi,\omega=1}^{2}(W_{ij\phi\chi\psi\omega}^{\mathrm{NL}} + W_{\phi\chi ij\psi\omega}^{\mathrm{NL}} + W_{\phi\chi\psi\omega ij}^{\mathrm{NL}})u_{\phi,\chi}u_{\psi,\omega}\\ &+\int_{\partial D}\sum_{i=1}^{2}\hat{v}^*(p)\sum_{j=1}^{2}\sum_{\phi,\chi,\psi,\omega=1}^{2}(W_{ij\phi\chi\psi\omega}^{\mathrm{NL}} + W_{\phi\chi ij\psi\omega}^{\mathrm{NL}} + W_{\phi\chi\psi\omega ij}^{\mathrm{NL}})u_{\phi,\chi}u_{\psi,\omega}\hat{n}_j,\end{aligned}$$

which by (2.12) yields

$$\frac{\partial V}{\partial \bar{c}_p^{[0]}} = -\int_D \hat{v}^*(p)\cdot(\nabla\cdot\tau^{\mathrm{NL}}(v^{[0]}, v^{[0]})) + \int_{\partial D}\hat{v}^*(k)\cdot(\tau^{\mathrm{NL}}(v^{[0]}, v^{[0]})\cdot\hat{n}).$$

Similarly,

$$\frac{\partial I}{\partial \bar{c}_p^{[0]}} = \rho\int_D \sum_{i=1}^{2}(\partial_{x_i}\hat{v}_i^*(p))v_{i,1}^{[0]} = -\rho\int_D \hat{v}_i^*(p)\cdot\partial_{x_1}^2 v^{[0]},$$

so that adding the two terms we have (5.1).

2. The variational formulation for the Galerkin projections follows from the same calculation after replacing $v^{[0]}$ by $v^{[0]}_{\mathcal{J}}$. □

Corollary 5.2. *The Galerkin projections of the first solvability condition* (5.1) *have non-trivial solutions.*

Proof. Let $c_{-k} = \bar{c}_k$. We observe that the set $I_{\mathcal{J}}(c^{[0]}_{\mathcal{J}}) = h > 0$ is an ellipsoid in $\mathbf{R}^{2|\mathcal{J}|}$, $|\mathcal{J}| = \mathrm{card}(\mathcal{J})$. Since $V_{\mathcal{J}}(c^{[0]}_{\mathcal{J}})$ is a smooth real valued function, it will attain its extrema at some points on the ellipsoid and satisfy the Galerkin projection of the solvability condition (5.3) for for appropriate reals $\lambda_1^{\mathcal{J}}$. □

Remark 5.2. The corollary does not guarantee the existence of Galerkin solutions with $\lambda_1 = 0$.

By the corollary, one way to approach the problem of finding solutions of (5.1) is to try to understand limits of sequences of solutions of Galerkin projection of increasing size $|\mathcal{J}| \to \infty$. In view of the invariance of the Galerkin projections of (5.1) under rescaling of $c^{[0]}_{\mathcal{J}}$ and $\lambda_1^{\mathcal{J}}$ by an arbitrary constant, it appears that we may choose quite arbitrary sequences of Galerkin solutions by changing the scaling factor as we increase the size of the projections. Although this may result useful, in the next section we use the variational interpretation of (5.1) to consider sequences of solutions belonging to $I(c^{[0]}_{\mathcal{J}}) = h$, with h fixed as we increase $|\mathcal{J}|$.

Although the perturbation theory of the previous section produces an infinite number of solvability conditions, we now see that the second and higher solvability conditions have the same structure. We may use (4.9)–(4.11) to write the solvability conditions for the inhomogeneous equation obtained at order α^r, $r > 2$, as

$$\begin{aligned}
&-\int_D \hat{v}^*(k) \cdot (\nabla \cdot \tau^{\mathrm{NL}}(v^{[r-2]}, v^{[0]}) + \nabla \cdot \tau^{\mathrm{NL}}(v^{[0]}, v^{[r-2]})) \\
&+\int_{\partial D} \hat{v}^*(k) \cdot (\tau^{\mathrm{NL}}(v^{[r-2]}, v^{[0]}) \cdot \hat{n} + \tau^{\mathrm{NL}}(v^{[0]}, v^{[r-2]}) \cdot \hat{n}) + \rho\lambda_1 \int_D \hat{v}^*(k) \cdot \partial^2_{x_1} v^{[r-2]} \\
&+\rho\lambda_{r-1} \int_D v^*(k) \cdot \partial^2_{x_1} v^{[0]} = G_k^{[r-1]}(v^{[0]}, \dots, v^{[r-3]}, w^{[1]}, \dots, w^{[r-2]}, \lambda_1, \dots, \lambda_{r-2})
\end{aligned} \tag{5.6}$$

for all $k \in \mathbf{Z} \setminus \{0\}$. The $G_k^{[r-1]}$ are known since it is assumed that we have solved the previous solvability conditions and inhomogeneous equations. The left-hand side of (5.6) involves the linearization of the first solvability condition (5.1) around a solution $v^{[0]}$, λ_1, applied to the unknown $v^{[r-2]}$, and we may also write (5.6) as

$$[(\mathrm{D}_1 S^{[1]}(c^{[0]}, \lambda_1))v^{[r-2]}]_k + \rho\lambda_{r-1} \int_D \hat{v}^*(k) \cdot \partial^2_{x_1} v^{[0]} = G_k^{[r-1]}, \quad k \in \mathbf{Z} \setminus \{0\} \tag{5.7}$$

with D_1 the derivative with respect to the first variable, $[\cdot]_k$ the kth row. Thus to solve the higher order solvability conditions we must invert the linearization of the first solvability condition (5.1) around its solution $c^{[0]}$. The presence of the λ_{r-1}, $r > 3$, allows for one null direction. For instance, if we can find a non-trivial solution $v^{[0]}$ of (5.1) with $\lambda_1 = 0$, the homogeneity of (5.1) gives us a line $\epsilon v^{[0]}$, $\epsilon \in \mathbf{R}$, of solutions. However, it may still be possible to adjust the λ_{r-1}, $r > 3$, so that the linearization is invertible in the complementary subspace.

Remark 5.3. Note that the addition of higher order non-linear terms in the traveling wave equation will not alter the general structure of (5.6). Cubic and higher order terms will involve terms $v^{[j]}$, $w^{[j]}$ with j at most $r - 3$, and will thus be absorbed in the right-hand side of $G_k^{[r-1]}$.

The variational structure of the higher solvability conditions is expressed by the following statement.

Proposition 5.2.

1. *Let $c^{[r-2]}$ be the vector of the coefficients $c_k^{[r-2]} \in \mathbf{C}$, $k \in \mathbf{Z} \setminus \{0\}$ of $v^{[r-2]} = \sum_{k\in\mathbf{Z}\setminus\{0\}} c_k^{[r-2]} v(k)$. Then the solvability condition for the order α^r, $r > 3$ inhomogeneous Eq.* (4.6), (4.7) *is equivalent to*

$$\nabla_{\bar{c}^{[r-2]}} V^{[r-2]}(c^{[r-2]}) = \lambda_{r-1} \nabla_{\bar{c}^{[r-2]}} I^{[r-2]}(c^{[r-2]}), \tag{5.8}$$

where the gradient $\nabla_{\bar{c}^{[r-2]}}$ has components $\partial_{\bar{c}_k^{[r-2]}}$, $k \in \mathbf{Z} \setminus \{0\}$, and

$$I^{[r-2]} = \rho \int_D \sum_{i=1}^{2} (\partial_{x_1} v_i^{[r-2]})(\partial_{x_1} v_i^{[0]}). \tag{5.9}$$

For the potential energy $V^{[r-2]}$, each pair of terms $\nabla \cdot \tau^{\mathrm{NL}}(f, g)$, $\tau^{\mathrm{NL}}(f, g) \cdot \hat{n}$ of $F^{[r-1]}$, $f^{[r-1]}$ in (4.10), (4.11), *respectively, is the gradient $\nabla_{\bar{c}^{[r-2]}}$ of*

$$V_{f,g} = \int_D (V^{\mathrm{NL}}(v^{[r-2]}, f, g) + V^{\mathrm{NL}}(f, v^{[r-2]}, g) + V^{\mathrm{NL}}(f, g, v^{[r-2]})), \tag{5.10}$$

also, each term $\lambda_i \partial_{x_1}^2 f$ in (4.10) *is the gradient $\nabla_{\bar{c}^{[r-2]}}$ of $\frac{1}{2}\lambda_i \tilde{I}_f^{[r-2]}$ if $f = v^{[r-2]}$, and $\lambda_i \tilde{I}_f^{[r-2]}$ if $f = v^{[0]}, \dots, v^{[r-3]}, w^{[1]}, \dots, w^{[r-2]}$, where*

$$\tilde{I}_f^{[r-2]} = -\rho \int_D \sum_{i=1}^{2} (\partial_{x_1} v_i^{[r-2]})(\partial_{x_1} f). \tag{5.11}$$

2. *The variational structure of the Galerkin projections is given by (1), with $V^{[r-2]}$, $I^{[r-2]}$ replaced by $V^{[r-2]}(c_{\mathcal{J}}^{[r-2]})$, $I^{[r-2]}(c_{\mathcal{J}}^{[r-2]})$, respectively.*

Proof. The proposition follows by a calculation similar to the one used in showing Proposition 5.1, i.e. we integrate by parts to transfer derivatives on $\hat{v}^*(k)$ to the other terms of the area integrals. □

In contrast to the first solvability condition, here the variational structure does not automatically guarantee the existence of solutions for the Galerkin projections. The constraint $I^{[r-2]} = h \in \mathbf{R}$ is a hyperplane, while the function $V^{[r-2]}$ is quadratic in $v^{[r-2]}$ (see e.g. (5.10) with $f = v^{[0]}$, $g = v^{[r-2]}$), and there is no a priori reason to expect that $V^{[r-2]}$ has a convex quadratic part. In the case where the first solvability condition has solutions with $\lambda_1 = 0$, the constraint $I^{[r-2]} = h$ eliminates a null direction.

6. Numerical study of the first solvability condition

In this section, we study the first solvability condition numerically for three different hyperelastic materials, namely the St. Venant-Kirchhoff material, and two simpler but less physical models.

According to the theory of elasticity, the assumptions of material frame indifference and isotropy (see e.g. [15], Chapter 4) which imply that the potential energy density W of a two-dimensional hyperelastic material is a function of the trace of the matrices $\mathcal{E}$, $\mathcal{E}^2$ and $\mathcal{E}^3$, or

$$W = W(\operatorname{tr}\mathcal{E}, \operatorname{tr}\mathcal{E}^2, \operatorname{tr}\mathcal{E}^3), \quad \text{where } \mathcal{E} = \tfrac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}} + (\nabla u)^{\mathrm{T}} \nabla u) \tag{6.1}$$

is the strain tensor. We may also assume that the potential energy density does not involve terms that are linear in the displacement u. This way the trivial displacement $u \equiv 0$ is a solution of the equations of motion.

The first non-linearity we choose corresponds to one of the simplest models satisfying the above constraints, and is specified by

$$W = \tfrac{1}{2}\lambda(\mathrm{tr}\,\mathcal{E})^2 + \mu\,\mathrm{tr}\,\mathcal{E}^2 \tag{6.2}$$

(this is the St. Venant-Kirchhoff material, see [15], Chapter 4). The quadratic terms in (6.2) give us the quadratic potential energy density W^{L} of (2.9), while the first solvability condition involves the cubic terms in (6.2). The simpler models we will consider correspond to the potential energy densities $W = W^{\mathrm{L}} + W^{\mathrm{NL}}$ with W^{L} as in (2.9) and

$$W^{\mathrm{NL}} = \tfrac{1}{4}(\lambda+\mu)u_{1,1}u_{1,2}^2, \quad \text{and} \quad W^{\mathrm{NL}} = \tfrac{1}{4}(\lambda+\mu)(u_{1,1}u_{1,2}^2 + u_{2,2}u_{2,1}^2), \tag{6.3}$$

respectively. Note that these models are not consistent with isotropy.

To simplify the first solvability conditions, we will look for traveling wave solutions of (2.4) and (2.5) satisfying

$$u_1(-x_1, x_2) = -u_1(x_1, x_2), \quad \text{and} \quad u_2(-x_1, x_2) = u_2(x_1, x_2), \;\; \forall (x_1, x_2) \in \tilde{D}. \tag{6.4}$$

These parities are specific to the fundamental domain $\tilde{D}$ of Section 2. We easily check that the traveling wave equation (2.4) for the non-linearities we are considering, as well as the linear equations of the perturbative scheme of Section 4 are compatible with the parities of (6.4). The periodicity and decay conditions are (2.5) and (2.6), respectively. Solutions of the linear traveling wave equation with the parities (6.4) will have the form

$$v_1^{[0]}(x_1, x_2) = \sum_{p=1}^{\infty} a_p \sin px_1 \hat{\mathcal{A}}(p, x_2), \qquad v_2^{[0]}(x_1, x_2) = \sum_{p=1}^{\infty} a_p \cos px_1 \hat{\mathcal{B}}(p, x_2),$$
$$\text{where} \quad \hat{\mathcal{A}}(p, x_2) = -i\hat{v}_1(p, x_2), \qquad \hat{\mathcal{B}}(p, x_2) = -\hat{v}_2(p, x_2), \tag{6.5}$$

i.e. see (3.4) and (3.5), and the coefficients $a_p \in \mathbf{R}$, $\forall p \in \mathbf{Z}^+$. The solvability condition for the inhomogeneous linear equations is (3.7), with $\hat{v}^*(k, x_2)$ replaced by $[\hat{\mathcal{A}}(k, x_2), -\hat{\mathcal{B}}(k, x_2)]$ and the $\hat{F}_i(k, x_2)$, $\hat{f}_i(k, x_2)$ replaced by the sine ($i = 1$) and cosine ($i = 2$) transforms of F_i, f_i, respectively. Furthermore, the first solvability condition has the variational formulation

$$\partial_{a_p} V(a) = \lambda_1 \partial_{a_p} I(a), \quad p \in \mathbf{Z}^+, \tag{6.6}$$

where the components a_p, $p \in \mathbf{Z}^+$ of the vector a are as in (6.6). The functions $V(a)$ and $I(a)$ are given by (5.4), with the linear solutions $v^{[0]}$ as in (6.5).

With the above information we can write the solvability condition in spectral form in a straightforward manner by evaluating $V(a)$, $I(a)$ and the variational equation (6.6). The potential energy $V(a)$ for a general cubic potential energy density W^{NL} of the form given in (2.10) will be

$$\begin{aligned} V(a) = &\sum_{p_2 > p_3 \geq 1}^{\infty} (p_2 - p_3) p_2 p_3 a_{p_2 - p_3} a_{p_2} a_{p_3} \frac{\pi}{2} C_-(p_2 - p_3, p_2, p_3) \\ &+ \sum_{p_2, p_3 \geq 1}^{\infty} (p_2 + p_3) p_2 p_3 a_{p_2 + p_3} a_{p_2} a_{p_3} \frac{\pi}{2} C_+(p_2 + p_3, p_2, p_3). \end{aligned} \tag{6.7}$$

For instance, for the first model non-linearity of (6.3) we have $C_+(q, r, s) = C_-(q, r, s)$ with

$$C_-(q, r, s) = \frac{\lambda+\mu}{4} I_{11'1'}(q, r, s) \equiv \frac{\lambda+\mu}{4rs} \int_0^{\infty} \hat{\mathcal{A}}(q, x_2) \hat{\mathcal{A}}'(r, x_2) \hat{\mathcal{A}}'(s, x_2)\, \mathrm{d}x_2, \tag{6.8}$$

$q, r, s \in \mathbf{Z}^+$ (derivatives are with respect to x_2). The coefficients $I_{11'1'}(q, r, s)$ are given in Appendix B.

For the kinetic energy part $I(a)$ we have

$$I(a) = \frac{M}{2}\sum_{p=1}^{\infty} pa_p^2 \tag{6.9}$$

(the constant M is in Appendix B), so that by (6.7) the solvability condition (6.6) takes the form

$$G_p(a, \lambda_1) = \sum_{p_1=1}^{p-1} pp_1(p-p_1)a_{p_1}a_{p-p_1}K_{p,p_1} + \sum_{p_1=1}^{\infty} pp_1(p+p_1)a_{p_1}a_{p+p_1}\Lambda_{p,p_1} - \lambda_1 Mpa_p = 0,$$
$$p \in \mathbf{Z}^+ \tag{6.10}$$

For the first non-linearity of (6.2), we have

$$K_{p,p_1} = \frac{(\lambda+\mu)\pi}{8}(I_{11'1'}(p-p_1, p, p_1) - I_{11'1'}(p, p-p_1, p_1)), \tag{6.11}$$

$$\Lambda_{p,p_1} = \frac{(\lambda+\mu)\pi}{8}(I_{11'1'}(p, p+p_1, p_1) + I_{11'1'}(p_1, p+p_1, p) - I_{11'1'}(p+p_1, p, p_1) - I_{11'1'}(p+p_1, p_1, p)). \tag{6.12}$$

The coefficients $C_\pm(q, r, s)$ for the other non-linearities are in Appendix B. They similarly involve triple integrals of the functions $\hat{\mathcal{A}}(q, x_2)$, $\hat{\mathcal{B}}(q, x_2)$ of (6.5) and their derivatives, and are sums of terms $\alpha/(\beta q + \gamma r + \delta s)$, with $\alpha, \beta, \gamma, \delta \in \mathbf{R}$ constants depending on A and B. The denominators in the coefficients K_{p,p_1}, Λ_{p,p_1} are bounded away from zero uniformly in the allowed integers p, p_1, so that the coefficients are well-defined. These observations also apply to the coefficients $C_\pm(q, r, s)$, K_{p,p_1}, and Λ_{p,p_1} obtained for general non-linearities that have the form of (2.10). We can see this by counting the number of derivatives in the cubic potential energy terms.

To study the first solvability condition (6.10) numerically, we consider the Galerkin approximations of (6.7) and (6.10) for the vectors $a^N = [a_1^N, a_2^N, \dots, a_N^N] \in \mathbf{R}^N$, with the necessary modifications in the summations. It is also convenient to introduce the vectors $\tilde{a}^N$ with components $\tilde{a}_p^N = pa_p^N$, $p = 1, \dots, N$. Expressing the Lagrangian $L(a^N) = V(a^N) - \lambda_1^N I(a^N)$ and the Galerkin projection $G_p^N(a^N, \lambda_1^N) = 0$, $p = 1, \dots, N$ of (6.10) in the tilde variables we have

$$G_p^N(a^N, \lambda_1^N) = \partial_{a_p} L^N(a^N, \lambda_1^N) = p\partial_{\tilde{a}_p} L^N(\tilde{a}^N, \lambda_1^N) \equiv p\tilde{G}_p^N(\tilde{a}^N, \lambda_1^N) = 0, \tag{6.13}$$

$p = 1, \dots, N$, and it is sufficient to solve numerically the equation $\tilde{G}_p^N(\tilde{a}^N, \lambda_1^N) = 0$, $p = 1, \dots, N$. By Corollary 5.2, the Galerkin projections of the first solvability condition have non-trivial solutions and to find them we set $\lambda_1^N = 1$ and solve numerically

$$\tilde{G}_p^N(\tilde{a}^N, 1) = 0, \quad p = 1, \dots, N. \tag{6.14}$$

Letting $\tilde{\beta}^N = [\tilde{\beta}_1^N, \dots, \tilde{\beta}_N^N] \in \mathbf{R}^N$ be a non-trivial solution of (6.14), the vector γ^N with components

$$\gamma_p^N = \left(\frac{M}{2}\sum_{p=1}^{N}\frac{1}{p}(\tilde{\beta}_p^N)^2\right)^{-1/2} p^{-1}\tilde{\beta}_p^N, \quad p = 1, \dots, N \tag{6.15}$$

satisfies the Galerkin projection of (6.10) with N modes, i.e. we have

$$G_p^N(\gamma^N, \lambda_1^N) = 0, \quad p = 1, \dots, N, \quad \text{with} \quad \lambda_1^N = \left(\frac{M}{2}\sum_{p=1}^{N}\frac{1}{p}(\tilde{\beta}_p^N)^2\right)^{-1/2}. \tag{6.16}$$

Moreover,

$$I^N(\gamma^N) = \frac{M}{2} \sum_{p=1}^{N} p(\gamma_p^N)^2 = 1. \tag{6.17}$$

Remark 6.1. The choice of the constraint $I^N(a^N) = 1$ used here is arbitrary, we can also work with $I^N(a^N) = h$ for any $h > 0$. We can rescale γ^N and λ_1^N by h to also obtain a solution of the N-mode Galerkin projection of the first solvability condition. Even though the first solvability condition has no intrinsic scale, it is natural to impose a scale on the lowest order displacement $u^{[0]}$. This way the assumption on the smallness of the parameter α becomes meaningful (the scale of γ^N and λ_1^N can be absorbed in α).

For the purpose of the numerical study, the choice of scale is useful for comparing solutions of the different Galerkin projections. To find candidates for solutions of (6.10), we consider Galerkin projections with $N_1 < N_2 < \cdots < N_i < \cdots$ modes and construct sequences of numerical solutions γ^{N_i}, $\lambda_1^{N_i}$ by solving (6.14) and rescaling as above. Since each Galerkin projection has many solutions, to produce sequences that have a chance of converging in a reasonably strong sense, we start by considering a low order Galerkin projection, e.g. with $N_1 = 4$ modes

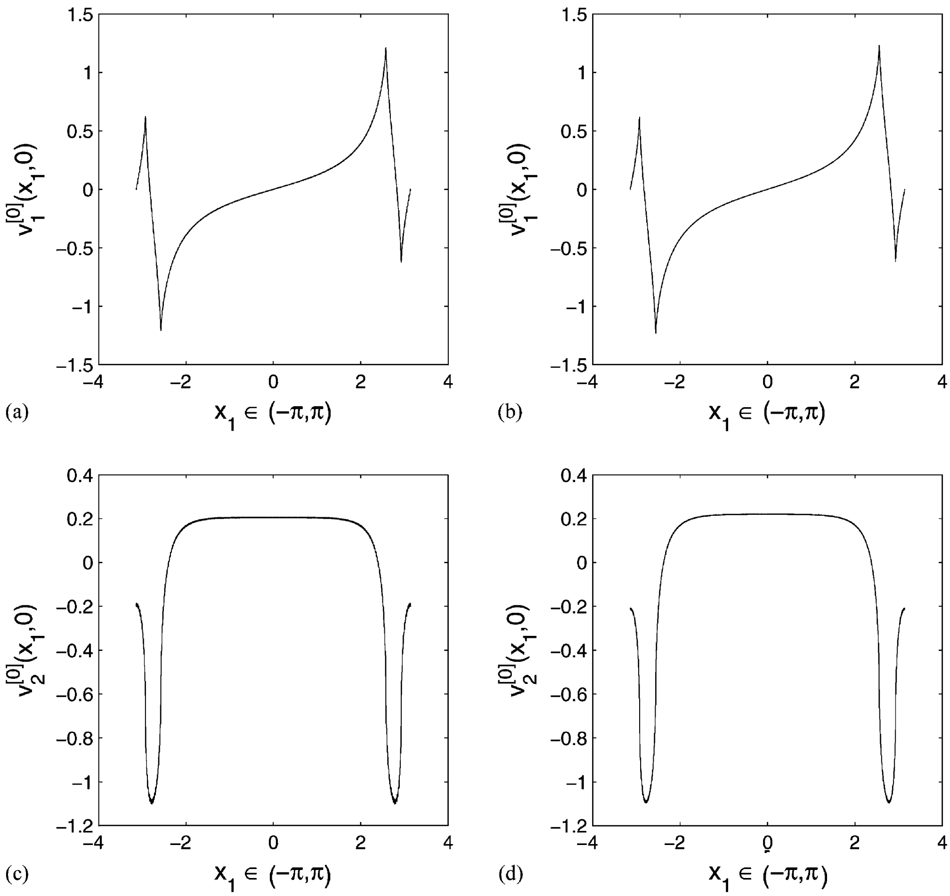

Fig. 1. (a) and (b) shows the surface displacement and horizontal component $v_1^{[0]}(x_1, 0)$ of 260, 500 modes, respectively, for first non-linearity of (6.3); (c) and (d) shows the surface displacement and vertical component $v_2^{[0]}(x_1, 0)$ of 260, 500 modes, respectively, for first non-linearity of (6.3) (multiply by 1.47).

and find numerically a solution γ^{N_1}, $\lambda_1^{N_1}$. Then to increase the number of modes from N_i to N_{i+1}, we apply Newton's iteration for the equation $\tilde{G}_p^{N_{i+1}}(\tilde{a}^{N_{i+1}}, 1) = 0$, $p = 1, \ldots, N_{i+1}$ using as initial condition the vector $[\gamma_i^{N_1}, \ldots, \gamma_{N_i}^{N_i}, 0, \ldots, 0] \in \mathbf{R}^{N_{i+1}}$, and obtain (after rescaling the numerical result) $\gamma^{N_{i+1}}$ and $\lambda_1^{N_{i+1}}$. Although we cannot control the dynamics of Newton's iteration, it is reasonable to expect that γ^{N_i}, $\lambda_1^{N_i}$ and $\gamma^{N_{i+1}}$, $\lambda_1^{N_{i+1}}$ will be getting closer as we increase N_i.

Remark 6.2. The sequence of Fourier coefficient vectors $\beta^{N_i} = [\gamma_1^{N_i}, \ldots, \gamma_{N_i}^{N_i}, 0, \ldots]$ obtained from the Galerkin solutions are bounded in ℓ_2 since $\sum_{p=1}^{\infty} p(\beta_p^{N_i})^2 = 1, \forall N_i$. Therefore, there will be subsequences of $\{\beta^{N_i}\}_{i=1}^{\infty}$ that converge weakly in ℓ_2. Also the corresponding boundary values $v_1^{[0]}$, $v_2^{[0]}$ of the displacement given by (6.5) will converge weakly in $L^2(S^1)$, and $H^{1/2}(S^1)$. However, these notions of convergence do not necessarily imply that the limits are non-trivial.

Remark 6.3. We will not seek solutions with $\lambda_1 = 0$ in this work. The non-invertibility of the linearization of the function $G(a, 0)$ in (6.10) around such solutions will likely require that $\lambda_i \neq 0$ for $i > 1$, and we plan to consider this case in the future.

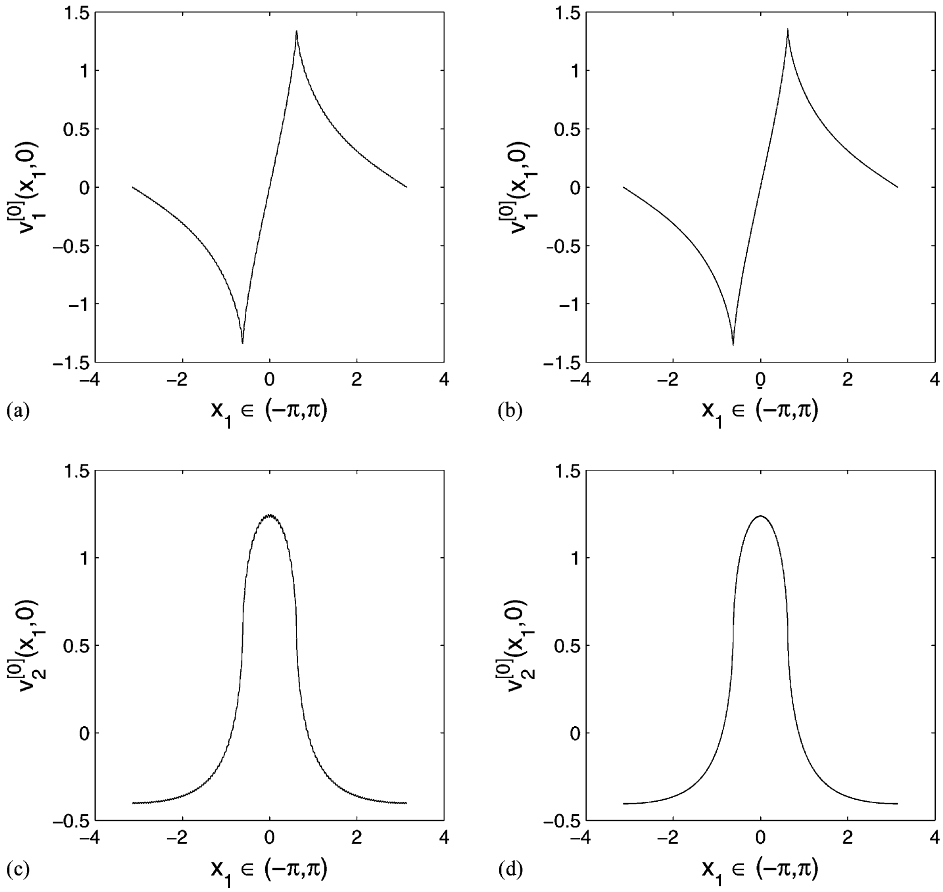

Fig. 2. (a) and (b) shows the surface displacement and horizontal component $v_1^{[0]}(x_1, 0)$ of 120, 500 modes, respectively, for second non-linearity of (6.3); (c) and (d) shows the surface displacement and vertical component $v_2^{[0]}(x_1, 0)$ of 120, 500 modes, respectively, for second non-linearity of (6.3) (multiply by 1.47).

The numerical results that follow were obtained following the above procedure, and solving (6.14) using the NAG library implementation of the hybrid Newton–Raphson method of [16]. The computed Fourier coefficients of the surface displacement are normalized as in (6.15). By (6.10), to obtain numerical values for the surface displacement and λ_1 we need to specify the Poisson's ratio $\nu = \frac{1}{2}(\lambda/(\lambda+\mu))$, and we set $\nu = \frac{1}{4}$ (e.g. $\nu = 0.28$ for glass, $\nu = 0.28$ for iron, see [15], p. 129). The Rayleigh speed for $\nu = \frac{1}{4}$ is approximately $c_0^2 = 0.845(\mu/\rho)$.

We found two types of sequences of numerical solutions. First, we see sequences where the computed surface displacements $v_1^{[0]}(x_1, 0)$, $v_2^{[0]}(x_1, 0)$ approach definite nontrivial shapes as we increase the number of modes. The corresponding sequences $\lambda_1^{N_i}$ also seem to approach a limit. The conjectured limits are candidates for solutions of the first solvability condition. The shapes of some of the surface displacements obtained numerically for the first and second non-linearities of (6.3), and for the the St. Venant-Kirchhoff material of (6.2) are shown in Figs. 1–3. Also, the values of λ_1 corresponding to the solutions of Figs. 1–3 are shown in Fig. 6(a)–(c). Evidently, there are sequences of surface displacements and λ_1 with possible non-trivial limits for all three non-linearities considered. An interesting feature of the numerical solutions is the appearance of well-defined cusps in $v_1^{[0]}(x_1, 0)$ for all the non-linearities considered. On the other hand, $v_2^{[0]}(x_1, 0)$ appears to be differentiable. The numerical solutions also

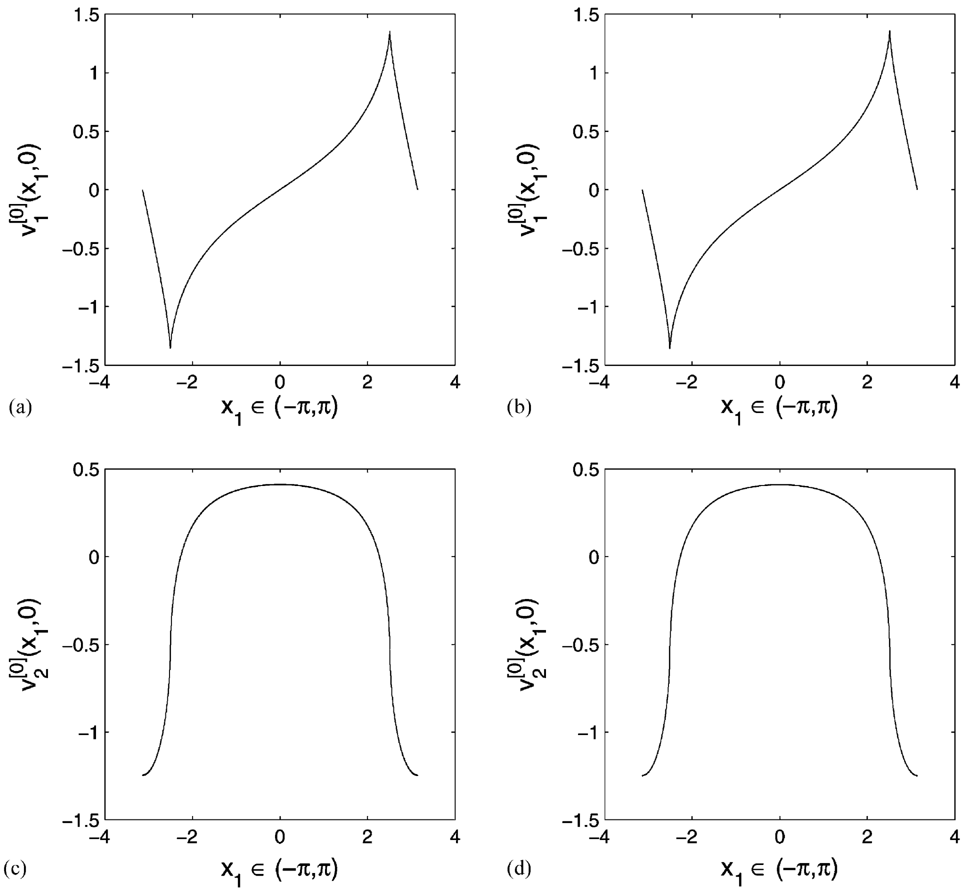

Fig. 3. (a) and (b) shows surface displacement and horizontal component $v_1^{[0]}(x_1, 0)$ of 300, 500 modes, respectively, St. Venant-Kirchhoff material of (6.2); (c) and (d) shows surface displacement and vertical component $v_1^{[0]}(x_1, 0)$ of 300, 500 modes, respectively, St. Venant-Kirchoff material of (6.2) (multiply by 1.47).

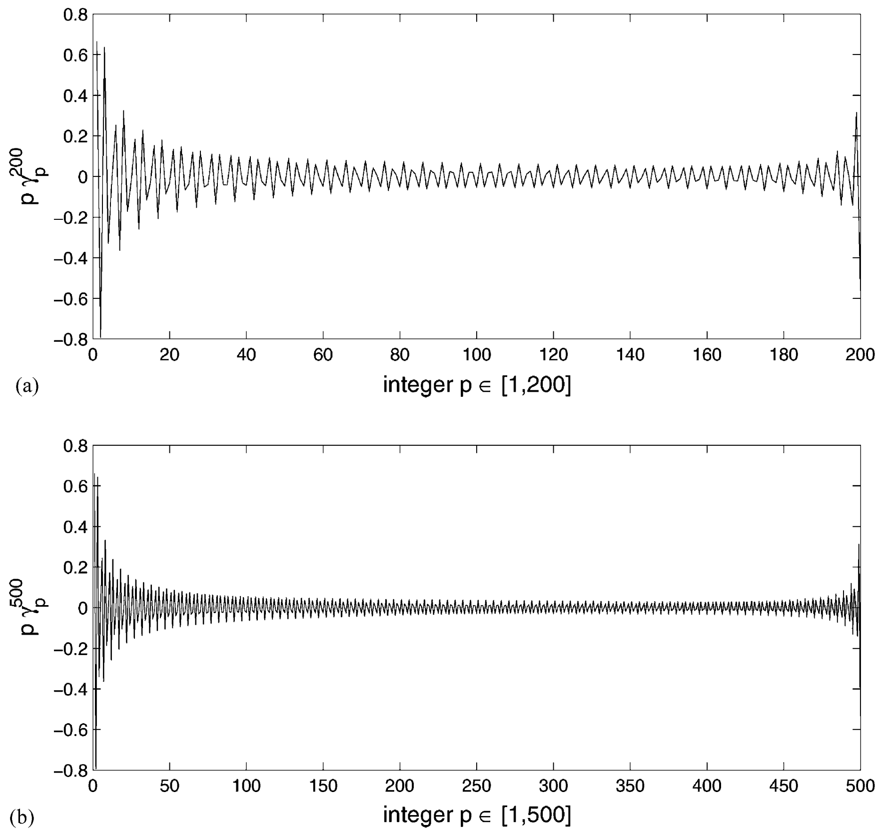

Fig. 4. (a) and (b) shows coefficients $p\gamma_p^{N_i}$, $N_i = 200, 500$ respectively, solution of Fig. 3.

exhibit small oscillations that decrease in scale and amplitude as we increase the size of the Galerkin truncation. This oscillatory behavior can be better appreciated by looking at Fig. 4(a) and (b) where we plot p vs. $p\gamma_p^{N_i}$ for three N_i-mode truncations. The $\gamma_p^{N_i}$ in Fig. 4(a) and (b) are the Fourier coefficients for the non-linearity of Fig. 3(a)–(d). The tails in $p\gamma_p^{N_i}$ are apparently moving to the right with almost constant amplitude as we increase the number of modes N_i, so that $\gamma_{N_i}^{N_i} \sim p^{-1}$. These features are common to the Fourier coefficients of all the numerical solutions of this first type, and are evidence that the weak L^2 limits of the surface displacements are non-trivial. It is also possible that we have stronger convergence, e.g. in $L^2(S^1)$, and it is also interesting to see whether there is a way to filter out the small-scale oscillations. A more qualitative comparison of the surface displacements shown is also possible. For instance, evaluating the surface displacements at 2500 uniformly distributed points in $[-\pi, \pi]$ we see that the difference between the horizontal and vertical displacements obtained using 400 and 500 modes is bounded by 2×10^{-2} for the solutions of Fig. 1(a)–(d), and by 2×10^{-3} for the solutions of Figs. 2 and 3(a)–(d). The pointwise difference is in all cases oscillatory and its integral over $[-\pi, \pi]$ is in the range 10^{-6}–10^{-5}.

Remark 6.4. The surface displacements we found are reminiscent of those reported in [PT] for a different non-linearity. The number of modes used in that work was smaller (~ 25), and the (possible) cusps were not resolved. It also appears that, at least for the models considered here, the main qualitative features of the numerical solutions do not depend on the details of the non-linearity.

Remark 6.5. Despite the presence of the cusps at the boundary, (6.15) implies that the lowest order elastic displacement $v^{[0]}(x_1, x_2)$ we obtain from the computed surface displacement is smooth inside the domain.

We have also found a second type of sequences of Galerkin solutions that seem to converge weakly in $L^2(S^1)$ to the trivial solution. A typical example is shown in Fig. 5(a)–(d). The particular example was found for the St. Venant-Kirchhoff material, but similar numerical solutions were also found for the other non-linearities. In these sequences, the surface displacement has small-scale oscillations that become finer decreasing slowly in amplitude, as well as spikes that become more and more concentrated as we increase the number of modes. Also, for this type of numerical solutions the $\lambda_1^{N_i}$ do not seem to approach any limit. The $\lambda_1^{N_i}$ corresponding to the sequence of solutions of Fig. 5 is shown in Fig. 6(d).

The accuracy of the numerical solutions presented is indicated by the vector of residuals $\tilde{G}_p^N(x_c^N, 1),\ p = 1, \dots, N$ of the numerical solutions x_c^N of (6.15). Since, however, $\lim_{x\to 0}\tilde{G}^N(x, 1) = 0$, it is more meaningful to consider the vector of relative residuals $[\tilde{G}_p^N(\tilde{x}_c^N, 1)]^{-1}\tilde{G}_p^N(x_c^N, 1),\ p = 1, \dots, N$, where $\tilde{x}_c^N$ has the same size as x_c^N, e.g. in practice we get $\tilde{x}_c^N$ by changing the sign of a few components of x_c^N. For the solutions corresponding to all the figures shown, all the components of the relative residuals were bounded by 10^{-7}–10^{-6}. These numbers

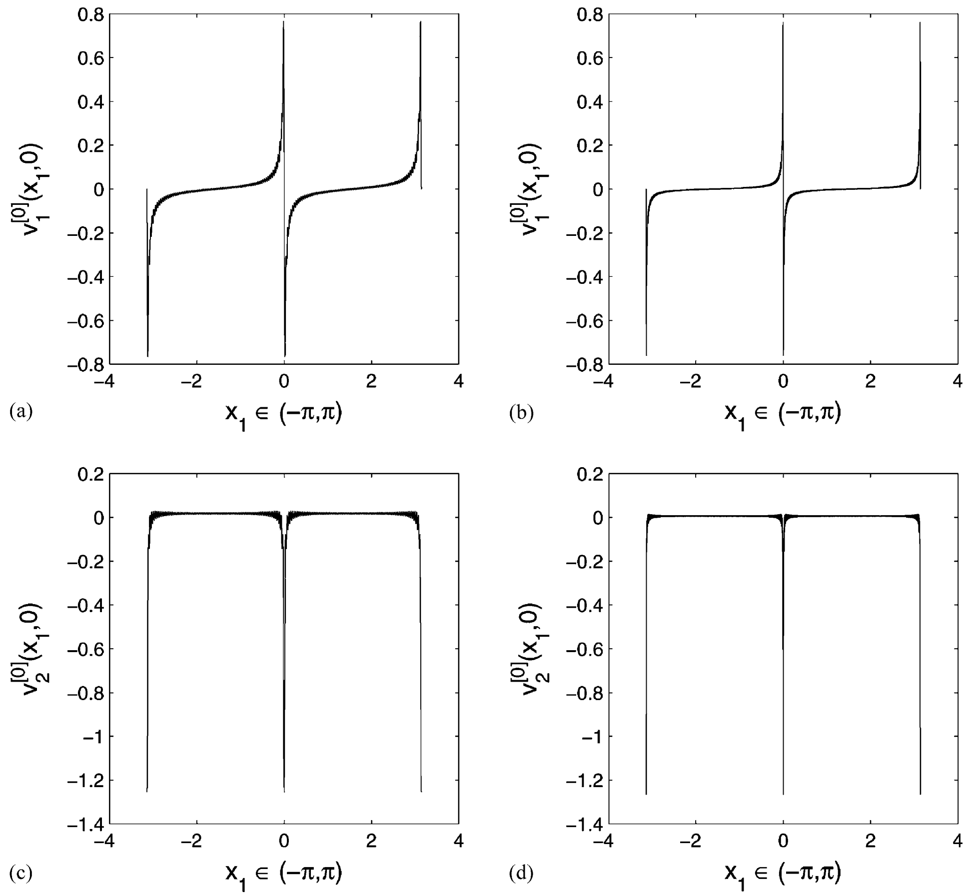

Fig. 5. (a) and (b) shows surface displacement and horizontal component $v_1^{[0]}(x_1, 0)$ of 160, 500 modes, respectively, St. Venant-Kirchhoff material of (6.2); (c) and (d) shows surface displacement and vertical component $v_1^{[0]}(x_1, 0)$ of 160, 500 modes, respectively, St. Venant-Kirchhoff material of (6.2) (multiply by 1.47).

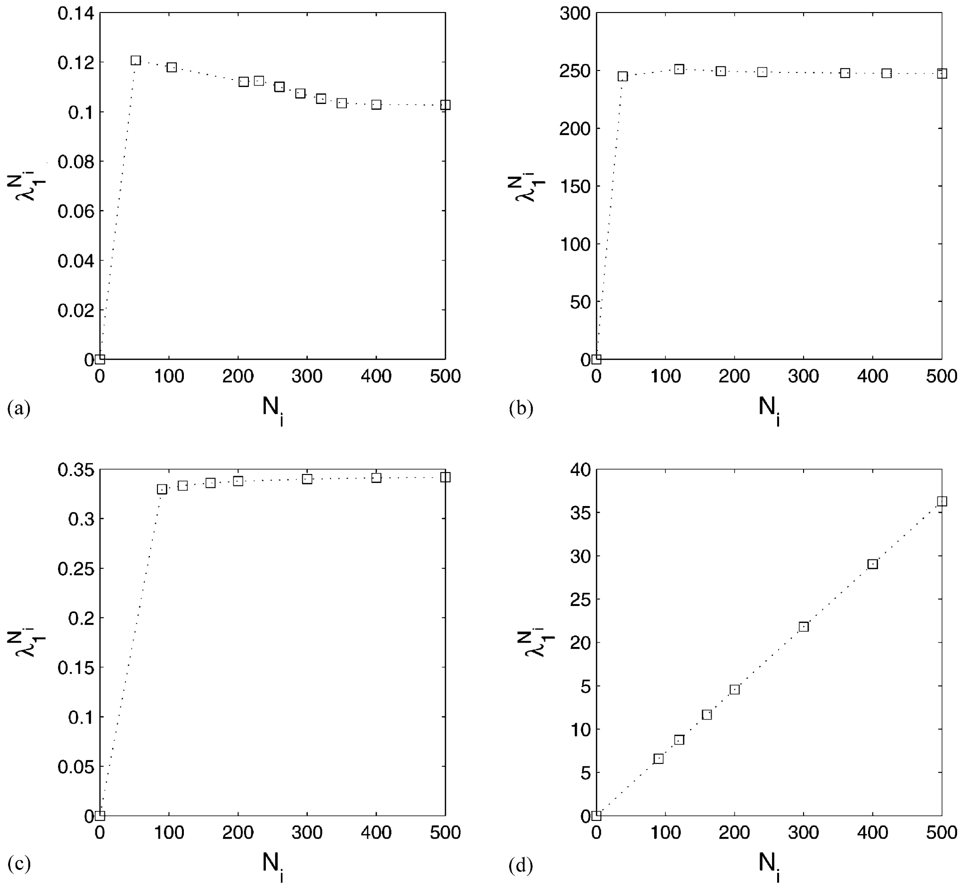

Fig. 6. Values of $\lambda_1^{N_i}$ for Galerkin approximations with N_i modes: (a) solution of Fig. 1; (b) solution of Fig. 2; (c) solution of Fig. 3; (d) solution of Fig. 5.

also agree with the condition for terminating the iteration successfully, i.e. with the relative residuals smaller than the square root of the machine accuracy.

7. Discussion

We have considered the problem of traveling surface elastic waves in the half-plane and extended the perturbative approach of Parker and Talbot, and Hunter [2,3] to higher order. Our extended scheme is formally equivalent to a perturbative implementation of the Liapunov–Schmidt method, and is also related to Signorini's method in static elasticity. The analog of the bifurcation equation is an infinite set of solvability conditions that involve the values of the displacement at the surface, and we also observed that for hyperelastic materials the solvability conditions are constrained variational problems. In this work, we focused on the first solvability condition. We noted that the Galerkin approximations of the first solvability condition must posses solutions, and our numerical results suggest that there exist sequences of Galerkin solutions with non-trivial limits; such limits are candidates for non-trivial solutions of the first solvability condition. The Galerkin solutions obtained numerically rather quickly tend to surface displacements of a well-defined shape, although the conjectured convergence is rather slow. Thus, although we consider that our work gives stronger numerical evidence for the existence of non-trivial solutions to the first solvability condition than earlier studies, we believe that further improvements are possible and that the existence question should also be understood theoretically. It would also be useful (and possibly related to the existence question) to see whether we can devise a numerical scheme filtering out the small-scale oscillations we saw.

The second step in pursuing the expansion method we described will be to numerically examine the invertibility of the first solvability condition around its solutions, and the possibility of constructing higher order corrections to the approximate solutions presented here. A positive result would give further evidence for the existence of traveling wave solutions, although proving existence using the present constructive approach seems difficult at this point, and other strategies may be more practical.

Further, dynamical questions can be addressed by considering the asymptotic evolution equation derived by Hunter and Parker [3,9]. For instance, one may ask whether the conjectured traveling wave solutions are stable. Instabilities of several types are possible, e.g. to shocks or to radiation (in an extended framework where other modes are included). Another question is whether arbitrary smooth initial data lead to the formation of cusps and then shocks. In the neighborhood of a cusp one expects very large forces, so that such loss of regularity phenomena may have an interesting physical interpretation, relating non-linear effects to the appearance of small-scale cracks near the surface of solids.

Acknowledgements

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Appendix A

Assuming that the solvability conditions (3.8) and (3.9) are satisfied, the general solution of the inhomogeneous linear system for the Fourier coefficients $\hat{u}_i(k, x_2)$, $k \in \mathbf{Z}$, of the displacement is

$$\hat{u}_i(k, x_2) = \hat{w}_i(k, x_2) + c_k \hat{v}_i(k, x_2), \quad i = 1, 2, \tag{A.1}$$

where $\hat{v}_i(k, x_2), i = 1, 2$ are the solutions of the homogeneous system given by (3.4) and (3.5), $c_k \in \mathbf{C}$, and $\hat{w}_i(k, x_2)$ is a solution of the inhomogeneous system given for $k \in \mathbf{Z}^+ \setminus \{0\}$ by

$$\begin{aligned}
\hat{w}_1(k, x_2) = {} & -iA\,\mathrm{e}^{-kAx_2} \int_0^{x_2} \frac{\mathrm{e}^{kAs}}{2k} \left(-\frac{i\mu^{-1}}{A^2-1} \hat{F}_1(k,s) + \frac{(\lambda+2\mu)^{-1}}{A(B^2-1)} \hat{F}_2(k,s) \right) \mathrm{d}s \\
& -iA\,\mathrm{e}^{kAx_2} \int_{x_2}^{\infty} \frac{\mathrm{e}^{-kAs}}{2k} \left(-\frac{i\mu^{-1}}{A^2-1} \hat{F}_1(k,s) - \frac{(\lambda+2\mu)^{-1}}{A(B^2-1)} \hat{F}_2(k,s) \right) \mathrm{d}s \\
& +i\,\mathrm{e}^{-kBx_2} \left[\int_0^{x_2} \frac{\mathrm{e}^{kBs}}{2k} \left(\frac{i\mu^{-1}}{B(A^2-1)} \hat{F}_1(k,s) - \frac{(\lambda+2\mu)^{-1}}{B^2-1} \hat{F}_2(k,s) \right) \mathrm{d}s \right. \\
& - \frac{A^2+1}{2B} \int_0^{\infty} \frac{\mathrm{e}^{-kAx_2}}{2k} \left(\frac{i\mu^{-1}}{A^2-1} \hat{F}_1(k,s) + \frac{(\lambda+2\mu)^{-1}}{A(B^2-1)} \hat{F}_2(k,s) \right) \mathrm{d}s \\
& \left. - \int_0^{\infty} \frac{\mathrm{e}^{-kBx_2}}{2k} \left(-\frac{i\mu^{-1}}{A^2-1} \hat{F}_1(k,s) + \frac{(\lambda+2\mu)^{-1}}{A(B^2-1)} \hat{F}_2(k,s) \right) \mathrm{d}s - \frac{i\mu^{-1}\hat{f}_1(k)}{2kB} \right] \\
& -i\,\mathrm{e}^{kBx_2} \int_{x_2}^{\infty} \frac{\mathrm{e}^{-|k|Bs}}{2k} \left(\frac{i\mu^{-1}}{A^2-1} \hat{F}_1(k,s) + \frac{(\lambda+2\mu)^{-1}}{B^2-1} \hat{F}_2(k,s) \right) \mathrm{d}s,
\end{aligned} \tag{A.2}$$

$$
\begin{aligned}
\hat{w}_2(k,x_2) = {} & \mathrm{e}^{-kAx_2}\int_0^{x_2}\frac{\mathrm{e}^{kAs}}{2k}\left(-\frac{i\mu^{-1}}{A^2-1}\hat{F}_1(k,s)+\frac{(\lambda+2\mu)^{-1}}{A(B^2-1)}\hat{F}_2(k,s)\right)\mathrm{d}s \\
& -\mathrm{e}^{kAx_2}\int_{x_2}^{\infty}\frac{\mathrm{e}^{-kAs}}{2k}\left(-\frac{i\mu^{-1}}{A^2-1}\hat{F}_1(k,s)-\frac{(\lambda+2\mu)^{-1}}{A(B^2-1)}\hat{F}_2(k,s)\right)\mathrm{d}s \\
& -B\,\mathrm{e}^{-kBx_2}\left[\int_0^{x_2}\frac{\mathrm{e}^{kBs}}{2k}\left(-\frac{i\mu^{-1}}{B(A^2-1)}\hat{F}_1(k,s)+\frac{(\lambda+2\mu)^{-1}}{B^2-1}\hat{F}_2(k,s)\right)\mathrm{d}s\right. \\
& -\frac{A^2+1}{2B}\int_0^{\infty}\frac{\mathrm{e}^{-kAx_2}}{2k}\left(-\frac{i\mu^{-1}}{A^2-1}\hat{F}_1(k,s)-\frac{(\lambda+2\mu)^{-1}}{A(B^2-1)}\hat{F}_2(k,s)\right)\mathrm{d}s \\
& \left.-\int_0^{\infty}\frac{\mathrm{e}^{-kBx_2}}{2k}\left(\frac{i\mu^{-1}}{A^2-1}\hat{F}_1(k,s)+\frac{(\lambda+2\mu)^{-1}}{A(B^2-1)}\hat{F}_2(k,s)\right)\mathrm{d}s-\frac{i\mu^{-1}\hat{f}_1(k)}{2kB}\right] \\
& -B\,\mathrm{e}^{kBx_2}\int_{x_2}^{\infty}\frac{\mathrm{e}^{-kBs}}{2k}\left(\frac{i\mu^{-1}}{A^2-1}\hat{F}_1(k,s)+\frac{(\lambda+2\mu)^{-1}}{B^2-1}\hat{F}_2(k,s)\right)\mathrm{d}s, \qquad \text{(A.3)}
\end{aligned}
$$

and for $k=0$ by

$$
\hat{w}_1(0,x_2) = -i\mu^{-1}\int_{x_2}^{\infty}\left[\int_t^{\infty}\hat{F}_1(0,s)\,\mathrm{d}s\right]\mathrm{d}t, \qquad \hat{w}_2(0,x_2) = (\lambda+2\mu)^{-1}\int_{x_2}^{\infty}\left[\int_t^{\infty}\hat{F}_2(0,s)\,\mathrm{d}s\right]\mathrm{d}t. \qquad \text{(A.4)}
$$

For $k\in\mathbf{Z}^-$, $\hat{w}_i(k,x_2)=w_i^*(k,x_2)$. If the $\hat{F}_i(k,x_2)$, $i=1,2$, $k\in\mathbf{Z}$ are continuous and decay at least polynomially, the above expressions give well-defined solutions of the inhomogeneous system.

Appendix B

The coefficients $I_{11'1'}(q,r,s)$ defined in (6.8) are given by

$$
\begin{aligned}
I_{11'1'}(q,r,s) = {} & \left(-A^4\frac{1}{q+r+s}+\frac{2A^4B}{(A^2+1)}\frac{1}{(q+r)A+sB}+\frac{2A^4B}{(A^2+1)}\frac{1}{(q+s)A+rB}\right. \\
& -\frac{4A^3B^2}{(A^2+1)^2}\frac{1}{qA+(r+s)B}+\frac{2A^5}{(A^2+1)}\frac{1}{(r+s)A+qB}-\frac{4A^4B}{(A^2+1)^2}\frac{1}{rA+(q+s)B} \\
& \left.-\frac{4A^4B}{(A^2+1)^2}\frac{1}{sA+(q+r)B}+\frac{8A^3B}{(A^2+1)^3}\frac{1}{q+r+s}\right). \qquad \text{(B.1)}
\end{aligned}
$$

The coefficients $C_\pm(q,r,s)$ in the potential energy V of (6.7) for the other non-linearities are as follows: for the second non-linearity of (6.3) we have

$$
C_-(q,r,s) = C_+(q,r,s) = \tfrac{1}{4}(\lambda+\mu)(I_{11'1'}(q,r,s)+I_{222'}(q,r,s)), \qquad \text{(B.2)}
$$

while for the St. Venant-Kirchhoff non-linearity of (6.2) we have

$$
\begin{aligned}
C_\mp(q,r,s) = {} & \tfrac{1}{2}(\lambda+2\mu)(I_{11'1'}(q,r,s)+I_{222'}(q,r,s)+I_{111}(q,r,s)+I_{2'2'2'}(q,r,s) \\
& \pm I_{122}(q,r,s)\mp I_{2'1'1'}(q,r,s)) \\
& +\tfrac{1}{2}\lambda(I_{112'}(q,r,s)+I_{12'2'}(q,r,s))+\tfrac{1}{2}\mu(\mp I_{11'2}(q,r,s)-I_{21'2'}(q,r,s)). \qquad \text{(B.3)}
\end{aligned}
$$

Letting the subscripts ϕ, χ, ψ range over the symbols $1, 1', 1'', 2, 2', 2''$, the triple integrals $I_{\phi\chi\psi}(q,r,s)$ are defined by

$$I_{\phi\chi\psi}(q,r,s) = \int_0^\infty \mathcal{C}_\phi(q,x_2)\mathcal{C}_\chi(r,x_2)\mathcal{C}_\psi(s,x_2)\,\mathrm{d}x_2,$$

where $\mathcal{C}_1(t,x_2) = \hat{\mathcal{A}}(t,x_2), \mathcal{C}_{1'}(t,x_2) = t^{-1}\hat{\mathcal{A}}'(t,x_2), \mathcal{C}_{1''}(t,x_2) = t^{-2}\hat{\mathcal{A}}''(t,x_2), \mathcal{C}_2(t,x_2) = \hat{\mathcal{B}}(t,x_2), \mathcal{C}_{2'}(t,x_2) = t^{-1}\hat{\mathcal{B}}'(t,x_2)$, $\mathcal{C}_{2''}(t,x_2) = t^{-2}\hat{\mathcal{B}}''(t,x_2)$, $t = q,r,s \in \mathbf{Z}^+$. Evaluation of the triple integrals is straightforward and we omit the results here.

Also, the constant M of (6.9) is

$$M = \frac{A}{2} + \frac{2A^2}{(A^2+1)B} - \frac{4A^2}{(A^2+1)(A+B)} + \frac{1}{2A} + \frac{2A^2B^2}{(A^2+1)^2} - \frac{4AB}{(A^2+1)(A+B)}.$$

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