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Periodic oscillations of discrete NLS solitons in the presence of diffraction management

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Abstract
We consider the discrete NLS equation with a small-amplitude time-periodic diffraction coefficient which models diffraction management in nonlinear lattices. In the space of one dimension and at the zero-amplitude diffraction management, multi-peak localized modes (called discrete solitons or discrete breathers) are stationary solutions of the discrete NLS equation which are uniquely continued from the anti-continuum limit, where they are compactly supported on finitely many non-zero nodes. We prove that the multi-peak localized modes are uniquely continued to the time-periodic space-localized solutions for small-amplitude diffraction management if the period of the diffraction coefficient is not multiple to the period of the stationary solution. The same result is extended to multi-peaked localized modes in the space of two and three dimensions (which include discrete vortices) under additional non-degeneracy assumptions on the stationary solutions in the anti-continuum limit.

Mathematics Subject Classification: 37K60, 35Q55, 70K42, 34C25, 47H14, 58C15

1. Introduction

Diffraction management of discrete localized modes in nonlinear lattices was proposed in experiments with arrays of coupled optical waveguides [ESMA00, GSK07]. The experiments were modelled theoretically in [AM01] by using a non-local averaged equation derived from a discrete NLS equation with a time-periodic diffraction coefficient. The approximation error and well-posedness of the averaged lattice equation were considered in the rigorous work [M05].

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The existence of a single-peak localized solution was proved in [P05] using the variational method. Multi-peak localized modes of the averaged equation were constructed analytically and numerically in [P06] from the anti-continuum limit of the nonlinear lattice and at the small-amplitude diffraction management.

We shall consider here the same limit of small coupling between lattice nodes and small amplitude of the diffraction management but we will address the discrete NLS equation with a time-periodic diffraction coefficient avoiding passage to the averaged lattice equation. We shall ask if the stationary localized solutions of the discrete NLS equation at the zero amplitude of diffraction management *persist* for small non-zero values of the time-periodic diffraction coefficient.

In a similar problem in the context of dispersion management in the continuous NLS equation, it is known that the main question has a *negative* answer in the sense that the stationary localized solution decays in the time-evolution dynamics due to parametric resonance with the time-periodic dispersion term. This phenomenon was modelled analytically and numerically for the strong dispersion management in [YK01] and for the weak dispersion management in [PY04]. Rigorous analysis of radiative decay of small-amplitude localized solutions due to parametric resonance with the time-periodic cubic nonlinear term is developed in [CKP06] in the context of a three-dimensional NLS equation with a space-localized potential.

Our main goal here is to show that the main question has a *positive* answer in the context of small-amplitude diffraction management in the discrete NLS equation provided that the period of the time-periodic diffraction term is not a multiple of the period of a stationary localized mode. Note that this 'non-resonance condition' is violated in the continuous systems. The difference between the continuous and discrete systems comes from the fact that the continuous Laplacian is unbounded while its discrete analogue is bounded.

The main result of this paper was recently reported in [P08] for a one-peak discrete soliton. Our current consideration is more general and includes multi-peak localized modes in one [PKF05a], two [PKF05b] and three [LPK08] dimensions. Moreover, we use a different analytical technique, which simplifies the proofs of [P08]. The idea of [P08] was to view the periodic oscillations of discrete solitons under diffraction management as a 2-torus of a Hamiltonian dynamical system with symmetries and to apply a theorem of Nekhoroshev on persistence of invariant tori in Hamiltonian systems with additional conserved quantities. The latter theorem was proved for finite-dimensional systems in [BG02] but the method of the proof can be easily extended to some infinite-dimensional systems (e.g. for quasiperiodic breathers in Hamiltonian lattices with symmetries [BV02]).

The idea of our work is more closely related to the Lyapunov theorem on persistence of periodic orbits in the Hamiltonian dynamical systems [L92]. (See [MH92] for the proof of the Lyapunov theorem in the finite-dimensional problems.) We rewrite the problem of existence of time-periodic space-localized solutions as a fixed-point problem. Then, the implicit function theorem is invoked for continuations of stationary solutions from the anti-continuum limit and for continuations of the time-periodic solutions from the stationary solutions. Our consideration differs from the Lyapunov theorem in autonomous systems [L92] by the fact that the period of the time-periodic solution is defined by the time-periodic coefficient of the non-autonomous system and it is hence fixed.

Our paper is constructed as follows. The main results in one dimension are formulated in section 2. Continuations of stationary and time-periodic multi-peak localized solutions are constructed in section 3. Generalizations for localized modes in higher dimensions are developed in section 4.

2. Formulation of the main results

We consider the discrete nonlinear Schrödinger (NLS) equation in the form

$$\mathrm{i}\dot{u}_n = |u_n|^2 u_n + \delta \Delta u_n + \epsilon D(t) \Delta u_n, \qquad \forall n \in \mathbb{Z}, \tag{2.1}$$

where $u_n(t) : \mathbb{R} \mapsto \mathbb{C}$ satisfies $\lim_{|n|\to\infty} u_n(t) = 0$, $\Delta u_n = u_{n+1} - 2u_n + u_{n-1}$ is the discrete Laplacian, δ and ϵ are two small real-valued parameters and $D(t)$ is a real-valued, bounded and T-periodic function with zero mean. Parameter δ represents the mean value of the diffraction coefficient, while ϵ represents the standard deviation of the varying part of the diffraction coefficient. We shall use $\Omega = 2\pi/T$ for frequency of the time-periodic coefficient $D(t)$.

Let us consider the initial value problem for the discrete NLS equation (2.1) in the space of square-summable complex-valued functions on $\mathbb{Z}$ with the norm $\|\boldsymbol{u}(t)\|_{l^2}^2 = \sum_{n\in\mathbb{Z}} |u_n(t)|^2$. By the gauge invariance of the discrete NLS equation (2.1) with respect to the transformation $\boldsymbol{u}(t) \mapsto \mathrm{e}^{\mathrm{i}\theta}\boldsymbol{u}(t)$ for any $\theta \in \mathbb{R}$, the l^2-norm is constant in time $\|\boldsymbol{u}(t)\|_{l^2}^2 = \|\boldsymbol{u}(t_0)\|_{l^2}^2$ for all $t, t_0 \in \mathbb{R}$ provided that $\lim_{|n|\to\infty} u_n(t) = 0$. To generalize our consideration, we shall use a weighted l^2-space with the norm

$$\|\boldsymbol{u}(t)\|_{l_p^2}^2 = \sum_{n\in\mathbb{Z}} (1+n^2)^p |u_n(t)|^2, \qquad p \geqslant 0. \tag{2.2}$$

For any $p \geqslant 0$, we have $||u||_{l^\infty} \leqslant ||u||_{l_p^2}$, hence l_p^2 is a Banach algebra with respect to pointwise multiplication. The existence of solutions is proved in the following theorem.

Theorem 2.1. *Assume that $D(t)$ is a real-valued, bounded, piecewise-continuous, and T-periodic function on $t \in \mathbb{R}$. Then, for any $t_0 \in \mathbb{R}$, $p \geqslant 0$, and $\boldsymbol{u}_0 \in l_p^2(\mathbb{Z})$, there exists a unique solution $\boldsymbol{u}(t) \in l_p^2(\mathbb{Z})$ for all $t \in \mathbb{R}$, such that $\boldsymbol{u}(t_0) = \boldsymbol{u}_0$. Moreover, the solution $\boldsymbol{u}(t)$ is piecewise continuously differentiable in $t \in \mathbb{R}$.*

Proof. We consider the case where $D(t)$ is continuous, the piecewise case follows by a slight modification of the arguments below. Let $p \geqslant 0$. To prove local existence we write (2.1) as

$$u_n(t) = u_n(t_0) - \mathrm{i} \int_{t_0}^{t} \left[|u_n(t')|^2 u_n(t') + \left(\delta + \epsilon D(t')\right) \Delta u_n(t') \right] \mathrm{d}t'. \tag{2.3}$$

For any $\tau, \rho > 0$, using the fact that the l^∞ norm is bounded by the l_p^2 norm, we see that the right-hand side of (2.3) defines a Lipschitz continuous map in $C^0([t_0, t_0+\tau], \mathcal{B}_\rho(u(t_0)))$, where $\mathcal{B}_\rho(v)$ is a ball of radius ρ around the point v in $l_p^2(\mathbb{Z})$. The Lipschitz constant depends on τ, ρ, δ, and the supremum of ϵD (and can be made independent of t, t_0). Furthermore, choosing τ sufficiently small we can also make the Lipschitz constant strictly smaller than unity. We thus have a contraction in $C^0([t_0, t_0+\tau], \mathcal{B}_\rho(u(t_0)))$, and a unique fixed point. Using the continuity of $u_n(t)$ we check that $u_n(t)$ is differentiable in t directly from (2.3).

To extend τ to infinity it is sufficient to bound the l_p^2-norm of the solution. Multiplying (2.1) by $\bar{u}_n$ and subtracting the complex conjugate equation, we eliminate the nonlinear term in the equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|u_n|^2 = (\delta + \epsilon D(t)) \left(\bar{u}_n \Delta u_n - u_n \Delta \bar{u}_n\right).$$

We write this equation as

$$|u_n(t)|^2 = |u_n(t_0)|^2 - \mathrm{i} \times \int_{t_0}^{t} \left(\delta + \epsilon D(t')\right) \left[\bar{u}_n(t')(u_{n+1}(t') + u_{n-1}(t')) - u_n(t')(\bar{u}_{n+1}(t') + \bar{u}_{n-1}(t'))\right] \mathrm{d}t'. \tag{2.4}$$

By the Cauchy–Schwarz inequality, we obtain $\sum_{n\in\mathbb{Z}}(1+n^2)^p|u_n||u_{n+1}| \leqslant C_p\|\boldsymbol{u}\|^2_{l^2_p}$ for some $C_p > 0$. Since $|\delta + \epsilon D(t)| \leqslant C_D$ on $\mathbb{R}$ for some $C_D > 0$, we obtain from (2.4) that

$$\|\boldsymbol{u}(t)\|^2_{l^2_p} \leqslant \|\boldsymbol{u}(t_0)\|^2_{l^2_p} + C\int_{t_0}^{t}\|\boldsymbol{u}(t')\|^2_{l^2_p}\,\mathrm{d}t',$$

for $C = 4C_pC_D > 0$ and $p \geqslant 0$. The global bound on the l^2_p-norm of the solution,

$$\|\boldsymbol{u}(t)\|^2_{l^2_p} \leqslant \|\boldsymbol{u}(t_0)\|^2_{l^2_p} e^{C|t-t_0|}, \qquad \forall t \in \mathbb{R},$$

follows from Gronwall's inequality. □

Let us look for solutions of the discrete NLS equation in the form $u_n(t) = \mathrm{e}^{-\mathrm{i}\lambda t + \mathrm{i}\theta}v_n(t)$, where $\lambda \in \mathbb{R}$ and $\theta \in \mathbb{R}$ are arbitrary parameters and $v_n(t)$ satisfies the lattice differential equation

$$\mathrm{i}\dot{v}_n + \left(\lambda - |v_n|^2\right)v_n = \delta\Delta v_n + \epsilon D(t)\Delta v_n, \qquad \forall n \in \mathbb{Z}, \tag{2.5}$$

where $v_n(t) : \mathbb{R} \mapsto \mathbb{C}$ satisfies $\lim_{|n|\to\infty} v_n(t) = 0$. We note that the parameter θ does not enter the lattice equation (2.5) due to gauge invariance of the discrete NLS equation (2.1).

If $\epsilon = 0$, there exist stationary solutions $\boldsymbol{v}(t) = \boldsymbol{\phi}$ of the second-order difference equation

$$\epsilon = 0: \qquad \left(\lambda - |\phi_n|^2\right)\phi_n = \delta\Delta\phi_n, \qquad \forall n \in \mathbb{Z}. \tag{2.6}$$

Localized solutions of the difference equation (2.6) with $\lim_{|n|\to\infty}\phi_n = 0$ are uniquely continued for small $\delta \neq 0$ from the compact solutions with finitely many non-zero nodes at $\delta = 0$ [MA94] (other results on existence of localized solutions can be found in [BBJ00, PKF05a, W99]). These localized solutions are referred to both as *discrete breathers* and *discrete solitons*. The former name is due to the fact that they are periodic solutions of the discrete NLS equation (2.1) with the period $\tau = 2\pi/\lambda$, while the latter name is due to the fact that they are stationary solutions of the discrete NLS equation (2.5) with a fixed parameter λ.

The main result on construction of discrete solitons is described in the following theorem.

Theorem 2.2. *Let U_+, U_- be finite subsets of $\mathbb{Z}$ and let $U_0 = \mathbb{Z}\setminus\{U_+\cup U_-\}$. Assume that $\lambda > 0$ and consider a solution $\boldsymbol{\phi}_0$ of the difference equation (2.6) with $\delta = 0$*

$$(\phi_0)_n = 0, \quad \forall n \in U_0; \qquad (\phi_0)_n = \pm\sqrt{\lambda}, \quad \forall n \in U_\pm. \tag{2.7}$$

Let $p \geqslant 0$. Then, there exists $\delta_0 > 0$ such that for $|\delta| < \delta_0$ the difference equation (2.6) has a unique solution $\boldsymbol{\phi}_\delta \in l^2_p(\mathbb{Z})$ satisfying $(\phi_\delta)_{n_0} \in \mathbb{R}\backslash\{0\}$ for any fixed $n_0 \in U_+\cup U_-$ and

$$\forall 0 \leqslant \delta < \delta_0: \quad \|\boldsymbol{\phi}_\delta - \boldsymbol{\phi}_0\|_{l^2_p} \leqslant C\delta, \tag{2.8}$$

for some $C > 0$. Moreover, $\boldsymbol{\phi}_\delta$ is real and the dependence of $\boldsymbol{\phi}_\delta$ on δ is real analytic.

Remark 2.3. We shall call the localized solutions of theorem 2.2 the k-peak discrete solitons, where $k = \dim(U_+) + \dim(U_-)$. The proof of theorem 2.2 follows from [MA94] (see also [P06]). The real analyticity statement is added in [PKF05a] as it follows from the real analytic version of the implicit function theorem. The constant δ_0 depends in general on U_+, U_- and λ.

Our problem is now reformulated as a continuation of the time-independent solution $\boldsymbol{v}(t) = \boldsymbol{\phi}_\delta$ of the differential lattice equation (2.5) at $\epsilon = 0$ to time-periodic space-localized solutions $\boldsymbol{v}(t)$ of the same equation for small $\epsilon \neq 0$, where the values of $\lambda > 0$ and $0 < \delta < \delta_0$ are fixed. The main result of this paper is formulated in the following theorem.

Theorem 2.4. *Fix U_+, U_-, $\lambda > 0$ and $0 < \delta < \delta_0$. Let ϕ_δ be a solution of the difference equation (2.6) in $l^2_p(\mathbb{Z})$ for a fixed $p \geqslant 0$ and $n_0 \in U_+ \cup U_-$. Assume that $D(t)$ is a real-valued T-periodic function in $H^s([0,T])$ for any fixed $s > \frac{1}{2}$. Fix $\Omega = 2\pi/T$ such that $\lambda \neq m\Omega$ for all $m \in \mathbb{N}$. Then, there exist $\epsilon_0 > 0$ which depends on δ and $\min_{m\in\mathbb{N}} |\lambda - m\Omega|$ such that the differential lattice equation (2.5) has a unique solution $\boldsymbol{v}_\epsilon(t)$ in $X_{s,p} = H^s([0,T], l^2_p(\mathbb{Z}))$ satisfying $\boldsymbol{v}_\epsilon(t+T) = \boldsymbol{v}_\epsilon(t)$ on $t \in \mathbb{R}$, $(\boldsymbol{v}_\epsilon)_{n_0}(0) \in \mathbb{R}\backslash\{0\}$, and*

$$\forall 0 \leqslant \epsilon < \epsilon_0 : \quad \|\boldsymbol{v}_\epsilon(t) - \phi_\delta\|_{X_{s,p}} \leqslant C\epsilon, \tag{2.9}$$

for some $C > 0$. Moreover, the dependence of $\boldsymbol{v}_\epsilon(t)$ on ϵ is real analytic.

Remark 2.5. If $k = \dim(U_+) + \dim(U_-) = 1$, the conclusion of theorem 2.4 holds in the limit $\delta \to 0$. However, if $k \geqslant 2$, the δ-dependent bound ϵ_0 shrinks to zero as $\delta \to 0$. Therefore, for $k \geqslant 2$, one cannot combine theorems 2.2 and 2.4 into one statement which would claim a unique continuation of the limiting solution (2.7) with respect to both independent parameters δ and ϵ.

Finally, the stability of periodic solutions $\boldsymbol{v}_\epsilon(t)$ in the time evolution of the differential lattice equation (2.5) for sufficiently small $\epsilon > 0$ is inherited from stability of stationary solutions ϕ_δ for $\epsilon = 0$, according to the following theorem.

Theorem 2.6. *Let $\delta \neq 0$ be sufficiently small. Let $m\Omega \neq 2\lambda$ for all $m \in \mathbb{N}$. Periodic solutions $\boldsymbol{v}_\epsilon(t)$ of theorem 2.4 for sufficiently small ϵ are spectrally stable or unstable with the same number of unstable eigenvalues as the stationary solutions ϕ_δ of theorem 2.2 for $\epsilon = 0$.*

Remark 2.7. If $m\Omega = 2\lambda$ for an odd m, the solution $\boldsymbol{v}_\epsilon(t)$ is uniquely continued in ϵ by theorem 2.4. However, stability of the solution $\boldsymbol{v}_\epsilon(t)$ may be different from stability of the solution ϕ_δ of theorem 2.2 due to bifurcations of Floquet multipliers μ near the point $\mu = -1$.

3. Continuations of stationary and periodic multi-peak solutions

We shall prove theorems 2.2 and 2.4 by using the same analytical technique based on the implicit function theorem. We will use the explicit information about eigenvalues of the linearized operators associated with non-trivial solutions of the differential lattice equation (2.5) in the anti-continuum limit $\delta = \epsilon = 0$. To develop our analysis, we represent the periodic functions $D(t)$ and $\boldsymbol{v}(t)$ by the Fourier series

$$D(t) = \sum_{m\in\mathbb{Z}} D_m e^{im\Omega t}, \qquad \boldsymbol{v}(t) = \sum_{m\in\mathbb{Z}} \boldsymbol{V}_m e^{im\Omega t}. \tag{3.1}$$

We will use vector notation $\boldsymbol{D}$ for $\{D_m\}_{m\in\mathbb{Z}}$ and $\boldsymbol{V}$ for $\{\boldsymbol{V}_m\}_{m\in\mathbb{Z}}$. If $D(t)$ is real-valued and has zero mean, then $D_0 = 0$ and $D_{-m} = \bar{D}_m$ for all $m \in \mathbb{Z}_+$. If $D(t)$ is a T-periodic function in $H^s([0,T])$ for any $s \geqslant 0$, then $\boldsymbol{D} \in l^2_s(\mathbb{Z})$. Components of the vector $\boldsymbol{V}_m$ for a fixed $m \in \mathbb{Z}$, denoted as $V_{m,n}$ for $n \in \mathbb{Z}$, satisfy the lattice equations

$$(\lambda - m\Omega)\, V_{m,n} = F_{m,n}, \qquad \forall (m,n) \in \mathbb{Z}^2, \tag{3.2}$$

where

$$F_{m,n} = \sum_{m_1\in\mathbb{Z}} \sum_{m_2\in\mathbb{Z}} V_{m_1,n} \bar{V}_{-m_2,n} V_{m-m_1-m_2,n} + \delta \Delta V_{m,n} + \epsilon \sum_{m_1\in\mathbb{Z}} D_{m_1} \Delta V_{m-m_1,n}. \tag{3.3}$$

Let us consider the vector space $X_{s,p} = l^2_s(\mathbb{Z}, l^2_p(\mathbb{Z}))$ for the solution $\boldsymbol{V}$ of the lattice problem (3.2) with the norm

$$\|\boldsymbol{V}\|^2_{X_{s,p}} = \sum_{m\in\mathbb{Z}} (1+m^2)^s \|\boldsymbol{V}_m\|^2_{l^2_p} = \sum_{m\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} (1+m^2)^s (1+n^2)^p |V_{m,n}|^2. \tag{3.4}$$

It is obvious that the vector space $\boldsymbol{V} \in X_{s,p}$ is equivalent to the function space of T-periodic vector-valued functions $\boldsymbol{v}(t)$ in $H^s([0,T], l_p^2(\mathbb{Z}))$ in the sense that $C_-\|\boldsymbol{V}\|_{X_{s,p}} \leqslant \|\boldsymbol{v}(t)\|_{H^s([0,T],l_p^2(\mathbb{Z}))} \leqslant C_+\|\boldsymbol{V}\|_{X_{s,p}}$ for some $0 < C_- \leqslant C_+ < \infty$.

Lemma 3.1. *Assume that $\boldsymbol{D} \in l_s^2(\mathbb{Z})$ for all $s > \frac{1}{2}$. The vector field $\boldsymbol{F}$ of the lattice equations (3.2) maps the vector space $X_{s,p}$ to itself for any fixed $s > \frac{1}{2}$ and $p \geqslant 0$.*

Proof. Since Δ is a bounded operator, we have $\Delta : l_p^2(\mathbb{Z}) \mapsto l_p^2(\mathbb{Z})$ for any $p \geqslant 0$. Let $\boldsymbol{N}(\boldsymbol{v})$ be defined by the elements $[\boldsymbol{N}(\boldsymbol{v})]_n = |v_n|^2 v_n$ for $n \in \mathbb{Z}$. In the discrete space, there exists $C > 0$ such that

$$\|\boldsymbol{N}(\boldsymbol{v})\|_{l_p^2}^2 \leqslant \|\boldsymbol{v}\|_{l^\infty}^4 \|\boldsymbol{v}\|_{l_p^2}^2 \leqslant C^4 \|\boldsymbol{v}\|_{l_p^2}^6, \qquad \forall p \geqslant 0.$$

Therefore, $\boldsymbol{N} : l_p^2(\mathbb{Z}) \mapsto l_p^2(\mathbb{Z})$ for any $p \geqslant 0$. Let $(\boldsymbol{U} \star \boldsymbol{V})$ be the vector for the convolution sum defined by the elements $(\boldsymbol{U} \star \boldsymbol{V})_m = \sum_{m_1 \in \mathbb{Z}} U_{m_1} V_{m-m_1}$, $\forall m \in \mathbb{Z}$. Since l_s^2 forms a Banach algebra with respect to the convolution sum for any $s > \frac{1}{2}$, there exists $C > 0$ such that

$$\forall \boldsymbol{U}, \boldsymbol{V} \in l_s^2(\mathbb{Z}) : \quad \|\boldsymbol{U} \star \boldsymbol{V}\|_{l_s^2} \leqslant C \|\boldsymbol{U}\|_{l_s^2} \|\boldsymbol{V}\|_{l_s^2}, \qquad \forall s > \tfrac{1}{2}.$$

Since the norm (3.4) for vector space $X_{s,p}$ is separable on $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ and $\boldsymbol{D} \in l_s^2(\mathbb{Z})$ for all $s > \frac{1}{2}$, each term of the vector field $\boldsymbol{F}$ in (3.3) maps an element of $X_{s,p}$ to an element of $X_{s,p}$. □

Remark 3.2. The left-hand side of the lattice system (3.2) maps a vector space $X_{s+1,p}$ to $X_{s,p}$ for any $s \in \mathbb{R}$, $p \geqslant 0$.

If $\epsilon = 0$, the lattice system (3.2) is reduced to the difference equation (2.6) for special solutions in the form

$$V_{m,n} = \delta_{m,0}\phi_n, \qquad \forall (m,n) \in \mathbb{Z}^2, \tag{3.5}$$

where $\delta_{m,0}$ is the Kronecker symbol ($\delta_{m,0} = 0$ for any $m \neq 0$ and $\delta_{0,0} = 1$). Using lemma 3.1, one can immediately prove theorem 2.2. We add this proof for consistence of presentation, although it is merely a remake of the original proof in [MA94].

Proof of theorem 2.2. We first prove that all solutions of the difference equation (2.6) with $\phi_{n_0} \in \mathbb{R}\backslash\{0\}$ for any fixed $n_0 \in \mathbb{Z}$ and $\lim_{|n|\to\infty} \phi_n = 0$ are real-valued for $\delta \neq 0$. Indeed, if ϕ solves the difference equation (2.6) for $\delta \neq 0$, then $J_n = J_{n-1}$, $\forall n \in \mathbb{Z}$, where

$$J_n = \bar{\phi}_n \phi_{n+1} - \phi_n \bar{\phi}_{n+1}.$$

Because (2.6) is a symmetric second-order difference map for $\delta \neq 0$, if $\phi \neq \mathbf{0}$, there exists at most one consequent node, say $n_1 \in \mathbb{Z}$, with $\phi_{n_1} = 0$. In this case, $\phi_{n_1+1} = -\phi_{n_1-1} \neq 0$. If $\lim_{|n|\to\infty} \phi_n = 0$ or if there exists $n_1 \in \mathbb{Z}$ such that $\phi_{n_1} = 0$, then $J_n = 0$ for all $n \in \mathbb{Z}$. If $\phi_n, \phi_{n+1} \neq 0$ for any $n \in \mathbb{Z}$, then $2\arg(\phi_{n+1}) = 2\arg(\phi_n) \bmod (2\pi)$. On the other hand, if there exists n_1 such that $\phi_{n_1} = 0$, then $2\arg(\phi_{n_1+1}) = 2\arg(\phi_{n_1-1}) \bmod (2\pi)$. In both cases, if $\phi_{n_0} \in \mathbb{R}\backslash\{0\}$ for at least one $n_0 \in \mathbb{Z}$, then $\phi_n \in \mathbb{R}$ for any $n \in \mathbb{Z}$.

By lemma 3.1, the vector field of the difference equation (2.6) maps $l_p^2(\mathbb{Z})$ to $l_p^2(\mathbb{Z})$ for any $p \geqslant 0$. We consider now the linear matrix operator L_+ defined by

$$(L_+\boldsymbol{u})_n = \left(\lambda - 3\phi_n^2\right) u_n - \delta \Delta u_n, \qquad \forall n \in \mathbb{Z}, \tag{3.6}$$

which is the Jacobian of the difference equation (2.6) at the real-valued solution ϕ with the perturbation $\boldsymbol{u}$. It is clear that (i) if $\phi \in l_p^2(\mathbb{Z})$ for any $p \geqslant 0$, then $L_+ : l_p^2(\mathbb{Z}) \to l_p^2(\mathbb{Z})$, (ii) L_+ is analytic with respect to δ, and (iii) if $\phi = \phi_0$ is a compact solution (2.7) for $\delta = 0$, then

$\mathrm{Ker}(L_+) = \varnothing$ for $\delta = 0$. Therefore, L_+ is continuously invertible near $\phi = \phi_0$ and $\delta = 0$ in $l^2_p(\mathbb{Z}) \times \mathbb{R}$ for any $p \geqslant 0$. By the implicit function theorem, there exists a unique solution ϕ_δ in $l^2_p(\mathbb{Z})$ for sufficiently small δ, such that $\phi_\delta \to \phi_0$ as $\delta \to 0$. Moreover, the dependence ϕ_δ is real analytic in δ.

In the discrete space, convergence in $l^2_p(\mathbb{Z})$ for any $p \geqslant 0$ implies convergence in $l^\infty(\mathbb{Z})$. Therefore, the distance $|(\phi_\delta)_{n_0} - (\phi_0)_{n_0}|$ is small and, if $n_0 \notin U_0$, then $(\phi_\delta)_{n_0} \neq 0$ for sufficiently small δ. □

Remark 3.3. Consider another linear matrix operator L_- defined by

$$(L_- \boldsymbol{w})_n = \left(\lambda - \phi_n^2\right) w_n - \delta \Delta w_n, \qquad \forall n \in \mathbb{Z}, \tag{3.7}$$

which corresponds to the linearization of the difference equation (2.6) at the real-valued solution ϕ with respect to the purely imaginary perturbation i$\boldsymbol{w}$. If $\phi = \phi_0$ is a compact solution (2.7) for $\delta = 0$, then L_- has a k-dimensional kernel at $\delta = 0$, where $k = \dim(U_+) + \dim(U_-)$. The kernel of L_- introduces a technical obstacle on a continuation of the real-valued solution ϕ_0 to a complex-valued solution ϕ_δ. However, all localized solutions ϕ_δ are real-valued if $(\phi_\delta)_{n_0} \in \mathbb{R}\backslash\{0\}$ for a $n_0 \in \mathbb{Z}$, such that operator L_- need not be considered in the proof of theorem 2.2.

Lemma 3.4. *Let m be the number of sign-differences in the sets $U_+ \cup U_-$. There exists a $\delta_1 > 0$ in $0 < \delta_1 < \delta_0$, such that for any $0 < \delta < \delta_1$ the operator L_- has m small negative eigenvalues, $p = k - 1 - m$ small positive eigenvalues, and a simple zero eigenvalue, all of which belong to an interval $(-b, b)$, where $b > 0$ for $\delta \neq 0$ and $b \to 0$ as $\delta \to 0$.*

Proof. If $\phi \in l^2_p(\mathbb{Z})$ for any $p \geqslant 0$, then $L_- : l^2_p(\mathbb{Z}) \to l^2_p(\mathbb{Z})$. In addition, L_- is analytic with respect to δ. The simple zero eigenvalue corresponds to the eigenvector ϕ_δ in the kernel of L_-. Since $L_- \boldsymbol{w} = \boldsymbol{0}$ is a linear second-order difference equation for $\delta \neq 0$, it has two fundamental solutions. Since the solution $\boldsymbol{w} = \phi_\delta$ is decaying as $|n| \to \infty$, the linearly independent solution $\boldsymbol{w}$ is growing as $|n| \to \infty$. Therefore, the zero eigenvalue is simple for any $\delta \neq 0$. By the discrete Sturm–Liouville theorem [LL92], the number m of small negative eigenvalues of L_- equals the number of times the solution ϕ_δ changes sign on $n \in \mathbb{Z}$. It follows by diffusivity smoothing in lemma 2.3 of [PKF05a] that the number of sign changes of ϕ_δ is continuous in δ as $\delta \to 0$, such that it is equal to the number of sign-differences in the sets $U_+ \cup U_-$. The other $p = k - 1 - m$ small eigenvalues of L_- are positive. Existence of an interval $[-b, b]$ with $b > 0$ for $\delta \neq 0$ and $b \to 0$ as $\delta \to 0$ follows by theorem 4.10 in [K76] for semi-simple isolated eigenvalues of self-adjoint operators. □

At $\epsilon = 0$, consider the spectrum of the linearization operator associated with the lattice equations (3.2) at the real-valued stationary solution (3.5). Assuming a decomposition

$$V_{m,n} = \delta_{m,0}\phi_n + U_{m,n} + \mathrm{i} W_{m,n}, \qquad \forall (m, n) \in \mathbb{Z}^2, \tag{3.8}$$

where elements $U_{n,m}$ and $W_{n,m}$ are real-valued, we derive the linearized operator $(\mathcal{L}_+, \mathcal{L}_-)$ acting on $(\boldsymbol{U}, \boldsymbol{W})$:

$$(\mathcal{L}_+ \boldsymbol{U})_{m,n} = \left(\lambda - m\Omega - 2\phi_n^2\right) U_{m,n} - \phi_n^2 U_{-m,n} - \delta \Delta U_{m,n}, \qquad \forall (m, n) \in \mathbb{Z}^2,$$

$$(\mathcal{L}_- \boldsymbol{W})_{m,n} = \left(\lambda - m\Omega - 2\phi_n^2\right) W_{m,n} + \phi_n^2 W_{-m,n} - \delta \Delta W_{m,n}, \qquad \forall (m, n) \in \mathbb{Z}^2.$$

It is clear that $\mathcal{L}_\pm|_{m=0} = L_\pm$, where $\mathrm{Ker}(L_+) = \varnothing$ for sufficiently small $0 \leqslant \delta < \delta_0$ (see proof of theorem 2.2) and $\mathrm{Ker}(L_-) = \{\phi_\delta\}$ for $0 < \delta < \delta_1$ (see proof of lemma 3.4). The mth component of the vectors $\boldsymbol{U}$ and $\boldsymbol{W}$ is coupled with the $-m$th component, such that one can introduce matrix operators $L_{\pm}^{(m)}$ for any $m \in \mathbb{N}$ with the elements

$$\left[L_{\pm}^{(m)}\begin{pmatrix}\boldsymbol{u}_m\\ \boldsymbol{u}_{-m}\end{pmatrix}\right]_n = \begin{pmatrix}(\lambda - m\Omega - 2\phi_n^2)u_{m,n} - \delta\Delta u_{m,n} & \mp\phi_n^2 u_{-m,n}\\ \mp\phi_n^2 u_{m,n} & (\lambda + m\Omega - 2\phi_n^2)u_{-m,n} - \delta\Delta u_{-m,n}\end{pmatrix},$$

where $n \in \mathbb{Z}$.

Lemma 3.5. *Let $m\Omega \neq \lambda$ for all $m \in \mathbb{N}$. There exists a $\delta_2 > 0$ in $0 < \delta_2 < \delta_0$, such that for any $0 < \delta < \delta_2$, the direct product of operators $L_{\pm}^{(m)}$ on $m \in \mathbb{N}$ has no eigenvalues in an interval $(-b_0, b_0)$ for some $b_0 > 0$.*

Proof. If $\phi = \phi_0$ is a compact solution (2.7) for $\delta = 0$, then $L_{\pm}^{(m)}$ for any fixed $m \in \mathbb{N}$ has eigenvalues $-\lambda + \sqrt{\lambda^2 + m^2\Omega^2}$ and $-\lambda - \sqrt{\lambda^2 + m^2\Omega^2}$ of multiplicity k and eigenvalues $\lambda - m\Omega$ and $\lambda + m\Omega$ of infinite multiplicity. All these eigenvalues are bounded away from zero if $m\Omega \neq \lambda$ for all $m \in \mathbb{N}$. If $\phi \in l_p^2(\mathbb{Z})$ for any $p \geqslant 0$, then $L_{\pm}^{(m)} : l_p^2(\mathbb{Z}, \mathbb{C}^2) \to l_p^2(\mathbb{Z}, \mathbb{C}^2)$. Moreover, $L_{\pm}^{(m)}$ is analytic with respect to δ. The assertion of the lemma follows by theorem 4.10 in [K76], where eigenvalues of infinite multiplicities for $\delta = 0$ transform for $\delta \neq 0$ into spectral bands of small width and a set of isolated eigenvalues. □

Incorporating lemmas 3.1, 3.4 and 3.5, we prove theorem 2.4.

Proof of theorem 2.4. The vector field $\boldsymbol{F}$ of the lattice equations (3.2) is closed in $\boldsymbol{V} \in X_{s,p}$ for any $s > \frac{1}{2}$ and $p \geqslant 0$ by lemma 3.1 and it is analytic in $\epsilon \in \mathbb{R}$. The Jacobian operator J of the lattice equations (3.2) at the solution (3.5) with $\phi \in l_p^2(\mathbb{Z})$ for any $p \in \mathbb{R}$ maps $X_{s+1,p} \subset X_{s,p}$ to $X_{s,p}$ and has a one-dimensional kernel for $\epsilon = 0$ and $0 < \delta < \min(\delta_0, \delta_1, \delta_2)$ by lemmas 3.4 and 3.5. Due to the symmetry of the lattice equations (3.2) with respect to the gauge transformation $\boldsymbol{V} \mapsto e^{i\theta}\boldsymbol{V}$, $\forall\theta \in \mathbb{R}$, the Jacobian operator J for any $\epsilon > 0$ inherits a one-dimensional kernel. Due to the same reason, the nonlinear operator $\boldsymbol{F}(\boldsymbol{V})$ is orthogonal to the kernel of J. To define uniquely a projection to the one-dimensional kernel of J, we add a condition $(\boldsymbol{v})_{n_0}(0) \in \mathbb{R}\backslash\{0\}$ for a fixed $n_0 \in \mathbb{Z}$ on the solution of the differential lattice equation (2.5), which is equivalent to the following constraint on the solution $\boldsymbol{V}$ of the lattice equations (3.2):

$$G(\boldsymbol{V}) = \mathrm{Im}\sum_{m\in\mathbb{Z}} V_{m,n_0} = 0. \tag{3.9}$$

The constraint (3.9) is satisfied at the solution (3.5) for any $\delta \in \mathbb{R}$. Let us apply the Lyapunov–Schmidt decomposition of $\boldsymbol{V} \in X_{s,p}$ with respect to the one-dimensional kernel of J:

$$V_{m,n} = \delta_{m,0}\phi_n + i\theta\delta_{m,0}\phi_n + U_{m,n} + iW_{m,n}, \qquad \forall(m,n) \in \mathbb{Z}^2, \tag{3.10}$$

where $(\boldsymbol{U}, \boldsymbol{W})$ lies in the orthogonal complement of the kernel of the Jacobian operator J. The value of θ is defined uniquely from the vector $\boldsymbol{W}$ by the constraint (3.9) resulting in the relation

$$\theta\phi_{n_0} = -\sum_{m\in\mathbb{Z}} W_{m,n_0}, \tag{3.11}$$

where $\phi_{n_0} \neq 0$ for a $n_0 \in U_+ \cup U_-$ by theorem 2.2 for sufficiently small $\delta > 0$. Let P be the projection to the orthogonal complement of the kernel of J. By lemma 3.5, the projected Jacobian operator PJP is continuously invertible near $(\boldsymbol{U}, \boldsymbol{W}) = (\boldsymbol{0}, \boldsymbol{0})$ and $\epsilon = 0$ in $X_{s,p} \times X_{s,p} \times \mathbb{R}$ for $0 < \delta < \min(\delta_0, \delta_1, \delta_2)$ and any $s > \frac{1}{2}$ and $p \geqslant 0$. By the implicit function theorem, there exists a unique solution $(\boldsymbol{U}_\epsilon, \boldsymbol{W}_\epsilon)$ in vector space $X_{s,p} \times X_{s,p}$ such that $(\boldsymbol{U}_\epsilon, \boldsymbol{W}_\epsilon) \to (\boldsymbol{0}, \boldsymbol{0})$ as $\epsilon \to 0$. Moreover, the dependence $(\boldsymbol{U}_\epsilon, \boldsymbol{W}_\epsilon)$ is real analytic in ϵ. Therefore, θ_ϵ defined by (3.11) is real analytic in ϵ and $\theta \to 0$ as $\epsilon \to 0$. □

Remark 3.6.

(i) When $k \geqslant 2$, $\dim\mathrm{Ker}(L_-) = k > 1$ at $\delta = 0$ such that $b_0 \to 0$ as $\delta \to 0$ in the last assertion of lemma 3.4. Therefore, the Jacobian operator of the lattice system (3.2) is not invertible for $\delta = 0$ and $k > 1$.

(ii) When the resonance condition $m\Omega = \lambda$ is satisfied for a particular $m = m_0 \in \mathbb{Z}_+$, the spectrum of $L_\pm^{(m)}$ is not bounded away from zero. Again, the Jacobian operator of the lattice system (3.2) is not invertible if the resonance occurs at any $m = m_0$.

In both cases, theorem 2.4 may be invalid and a separate study is needed.

Using results on the linear operators L_+ and L_-, we finally prove theorem 2.6.

Proof of theorem 2.6. Let $v_\epsilon(t)$ be a periodic solution of theorem 2.4 for sufficiently small $\epsilon > 0$ and $\delta > 0$. In order to investigate spectral stability of periodic solutions in the differential lattice equation (2.5), one has to study the linearized differential lattice equation

$$\mathrm{i}\dot{w}_n + \left(\lambda - 2|v_n(t)|^2\right) w_n - v_n^2(t)\bar{w}_n = \delta\Delta w_n + \epsilon D(t)\Delta w_n. \tag{3.12}$$

We recall that $D(t)$ and $\boldsymbol{v}(t)$ are periodic with the same period T. By the Floquet theorem, all solutions of the linearized equation (3.12) are given by the vectors $\boldsymbol{w}(t)$ such that $\boldsymbol{w}(t+T) = \mu\boldsymbol{w}(t)$ and $\boldsymbol{w} \in L^2([0,T], l^2(\mathbb{Z}))$, where $\mu \in \mathbb{C}$ is the Floquet multiplier. At $\epsilon = 0$, we have $\boldsymbol{v}(t) = \boldsymbol{\phi}$ and $\boldsymbol{w}(t) = \boldsymbol{\psi}\mathrm{e}^{\gamma t}$ with time-independent vectors $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ such that $\mu = \mathrm{e}^{\gamma T}$. If we further decompose $\boldsymbol{\psi} = \boldsymbol{u} + \mathrm{i}\boldsymbol{w}$ with $(\boldsymbol{u}, \boldsymbol{w}) \in l^2(\mathbb{Z}, \mathbb{C}^2)$, then the linear system for $\boldsymbol{\psi}$ reduces to the eigenvalue problem

$$L_+\boldsymbol{u} = \gamma\boldsymbol{w}, \qquad L_-\boldsymbol{w} = -\gamma\boldsymbol{u}, \tag{3.13}$$

where operators L_+ and L_- are given by (3.6) and (3.7). Let $\delta \neq 0$ be sufficiently small. By theorem 3.6 in [PKF05a], the spectral problem (3.13) has a double zero eigenvalue γ, a finite number of semi-simple purely imaginary or real eigenvalues γ near the points $\gamma = 0$ and $\gamma = \pm\mathrm{i}\lambda$ and two bands of the continuous spectrum on the imaginary axis near points $\gamma = \pm\mathrm{i}\lambda$. Therefore, the linear problem (3.12) for $\epsilon = 0$ has a double unit multiplier μ, a finite number of semi-simple multipliers μ on the unit circle or on the real positive axis near the points $\mu = 1$ and $\mu = \mathrm{e}^{\pm\mathrm{i}\lambda T}$, and two bands of the continuous spectrum on the unit circle near points $\mu = \mathrm{e}^{\pm\mathrm{i}\lambda T}$. If $m\Omega \neq 2\lambda$ for all $m \in \mathbb{N}$, it follows that $\lambda T \neq \pi m$ and all spectral data of the linear problem (3.12) are bounded away of the bifurcation points $\mu = \pm 1$, except for the double unit multiplier. However, the double unit multiplier is structurally stable due to the gauge invariance of the linear equation (3.12). In addition, semi-simple eigenvalues on the unit circle are structurally stable in the linearized symplectic equations such as (3.12), while the continuous spectrum is preserved under deformations of $\boldsymbol{v}(t) \in H^s([0,T], l_p^2(\mathbb{Z}))$ for $s > \frac{1}{2}$ and $p \geqslant 0$ by Weyl's theorem. □

Remark 3.7. Theorem 3.6 of [PKF05a] states that the localized mode $\boldsymbol{\phi}$ is spectrally stable in the linear problem (3.13) for sufficiently small $\delta \neq 0$ if and only if $m = k - 1$, where m is defined by lemma 3.4 and k is defined by remark 2.3. If $m < k - 1$, the stationary solution is unstable with exactly $k - 1 - m$ unstable (real or complex) eigenvalues γ in the linear problem (3.13). By theorem 2.6, the same conclusion extends to the periodic solutions of the linear equation (3.12) for sufficiently small ϵ if $m\Omega \neq 2\lambda$ for all $m \in \mathbb{N}$ but the role of eigenvalues γ is taken by Floquet multipliers $\mu = e^{\gamma T}$.

4. Extensions to higher dimensional lattices

We shall reformulate theorems 2.2 and 2.4 for time-periodic localized solutions of the discrete NLS equation in higher spatial dimensions. The main difference is that localized solutions in two and three dimensions can be complex valued and persistence of these solutions from the anti-continuum limit depends on the non-degeneracy assumption on the kernel of a finite-dimensional reduction. The method of Lyapunov–Schmidt reductions near the anti-continuum limit was developed for localized solutions in [PKF05b] and [LPK08] in two and three dimensions, respectively.

The existence of single-peak localized modes in higher dimensions has been shown by variational methods in [W99, M05, P06]. Although these methods work equally well near and far from the anti-continuum limit, they provide less information on the solution than the methods based on the implicit function theorem. On the other hand, the interpretation of the discrete NLS equation as a symplectic map in [BBJ00] does not seem applicable in the space of higher dimensions.

We shall consider the main equation in the form

$$\mathrm{i}\dot{v}_n + \left(\lambda - |v_n|^2\right) v_n = \delta \Delta v_n + \epsilon D(t) \Delta v_n, \qquad \forall n \in \mathbb{Z}^d, \tag{4.1}$$

where $v_n(t) : \mathbb{R} \mapsto \mathbb{C}$, $\Delta u_n = \sum_{j=1}^{d} (u_{n+e_j} + u_{n-e_j}) - 2d u_n$ for unit vectors $(e_1, e_2, \ldots, e_d) \in \mathbb{R}^d$, and we assume that $\lambda > 0$. Stationary solutions satisfy the system of algebraic equations

$$\epsilon = 0: \quad \left(\lambda - |\phi_n|^2\right) \phi_n = \delta \Delta \phi_n, \qquad n \in \mathbb{Z}^d. \tag{4.2}$$

To characterize solutions of the stationary problem (4.2) for small values of δ, we shall repeat arguments of the method of Lyapunov–Schmidt reductions in space $X = l_p^2(\mathbb{Z}^d)$ for $p \geqslant 0$ [LPK08]. Let S be a finite subset of $\mathbb{Z}^d$ with $N = \dim(S)$ and denote $S^\perp = \mathbb{Z}^d \setminus S$. The stationary problem (4.2) with $\delta = 0$ has solutions ϕ_0 with the components

$$(\phi_0)_n = 0, \quad \forall n \in S^\perp; \qquad (\phi_0)_n = \sqrt{\lambda} \mathrm{e}^{\mathrm{i}\psi_n}, \quad \forall n \in S, \tag{4.3}$$

where $\psi_n \in \mathbb{T}([0, 2\pi])$. Fix $\lambda > 0$ and define the function $\boldsymbol{F} : X \times \mathbb{R} \to X$ by

$$F_n(\boldsymbol{u}, \delta) = (\lambda - |u_n|^2) u_n - \delta (\Delta u)_n, \qquad \forall \boldsymbol{u} \in X, \quad \forall \delta \in \mathbb{R}. \tag{4.4}$$

The function $\boldsymbol{F}$ is a polynomial in $\boldsymbol{u}$, δ, and thus real analytic in $X \times \mathbb{R}$. Let $D_{\boldsymbol{u}} \boldsymbol{F}(\boldsymbol{u}, \delta)$ denote the Fréchet derivative of $\boldsymbol{F}(\boldsymbol{u}, \delta)$ with respect to $\boldsymbol{u}$. If we identify $\mathbb{C}$ with $\mathbb{R}^2$ by taking real and imaginary parts such that $u_n = u_n^r + \mathrm{i} u_n^i$, we can then define the vector field in the form $F_n = F_n^r + \mathrm{i} F_n^i$.

Proposition 4.1. *The spectrum of $D_{\boldsymbol{u}} \boldsymbol{F}(\phi_0, 0)$, where ϕ_0 is given by (4.3), consists of eigenvalues $0, -2\lambda$, both of multiplicity N, and an eigenvalue λ, of (doubly) infinite multiplicity. The eigenvectors solve the linear systems*

$$D_{\boldsymbol{u}} \boldsymbol{F}(\phi_0, 0) \boldsymbol{e}^r(k) = \lambda \boldsymbol{e}^r(k), \quad D_{\boldsymbol{u}} \boldsymbol{F}(\phi_0, 0) \boldsymbol{e}^i(k) = \lambda \boldsymbol{e}^i(k), \quad \forall k \in S^\perp \tag{4.5}$$

and

$$D_{\boldsymbol{u}} \boldsymbol{F}(\phi_0, 0) \boldsymbol{r}(k, \psi_k) = -2\lambda \boldsymbol{r}(k, \psi_k), \quad D_{\boldsymbol{u}} \boldsymbol{F}(\phi_0, 0) \boldsymbol{\theta}(k, \psi_k) = 0, \qquad \forall k \in S, \tag{4.6}$$

with

$$e_n^r(k) = \delta_{n,k} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_n^i(k) = \delta_{n,k} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$r_n(k, \psi_k) = \delta_{n,k} \begin{bmatrix} \cos \psi_k \\ \sin \psi_k \end{bmatrix}, \qquad \theta_n(k, \psi_k) = \delta_{n,k} \begin{bmatrix} -\sin \psi_k \\ \cos \psi_k \end{bmatrix}.$$

Proof. For any vectors $\boldsymbol{u}, \boldsymbol{v} \in X$, we compute

$$(D_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u},\delta)\boldsymbol{v})_n = \lambda v_n - 2|u_n|^2 v_n - u_n^2 \bar{v}_n - \delta(\Delta v)_n. \tag{4.7}$$

Therefore, we have

$$\begin{aligned}
&(D_{\boldsymbol{u}}\boldsymbol{F}(\phi_0,0)\boldsymbol{v})_n = \lambda v_n, \quad \forall n \in S^{\perp};\\
&(D_{\boldsymbol{u}}\boldsymbol{F}(\phi_0,0)\boldsymbol{v})_n = -\lambda(v_n + \mathrm{e}^{\mathrm{i}2\psi_n}\bar{v}_n), \quad \forall n \in S.
\end{aligned} \tag{4.8}$$

Equations (4.5)–(4.6) are now established by direct computations. □

We note that the set of vectors $\{\boldsymbol{e}^r(k), \boldsymbol{e}^i(k)\}_{k\in S^\perp}$ and $\{\boldsymbol{r}(k,\psi_k), \boldsymbol{\theta}(k,\psi_k)\}_{k\in S}$ provide an orthonormal basis in X. Let $Z_0(\psi) = \mathrm{Span}\,(\{\boldsymbol{\theta}(k,\psi_k)\}_{k\in S})$ be the kernel of $D_{\boldsymbol{u}}\boldsymbol{F}(\phi_0,0)$ and $Z(\psi) = \mathrm{Span}\left(\{\boldsymbol{e}^r(k), \boldsymbol{e}^i(k)\}_{k\in S^\perp}; \{\boldsymbol{r}(k,\psi_k)\}_{k\in S}\right)$ be the orthogonal complement of $Z_0(\psi)$. By proposition 4.1, $Z_0(\psi)$ is in the tangent space to the N-torus of the limiting solution (4.3), while $Z(\psi)$ is orthogonal to the N-torus. This geometry suggests that we use a mixed coordinate system in X, i.e. polar coordinates in each of the planes spanned by $\{\boldsymbol{r}(k,\psi_k), \boldsymbol{\theta}(k,\psi_k)\}_{k\in S}$ and rectangular coordinates in all other planes.

Let $u_n = u_n^r + i u_n^i$, $\forall n \in \mathbb{Z}^d$ and define new coordinates by

$$\begin{aligned}
&u_n^r = q_n, \quad u_n^i = p_n, \qquad \forall n \in S^{\perp};\\
&u_n^r = r_n\cos\theta_n, \quad u_n^i = r_n\sin\theta_n, \qquad \forall n \in S.
\end{aligned} \tag{4.9}$$

Denoting the old and new coordinates of a point by $\boldsymbol{x}, \boldsymbol{y}$ respectively, the correspondence (4.9) defines a map $\boldsymbol{x} = \boldsymbol{G}(\boldsymbol{y})$ from $\mathbb{R}_+^N \times \mathbb{T}^N \times P_0X \ni \boldsymbol{y}$ to $X \ni \boldsymbol{x}$, where P_0 is the projection onto the span of $\{\boldsymbol{e}^r(k), \boldsymbol{e}^i(k)\}_{k\in S^\perp}$. The map $\boldsymbol{G}$ is real analytic in $\mathbb{R}_+^N \times \mathbb{T}^N \times P_0X$.

Remark 4.2. Since we are interested in solving $\boldsymbol{F}(\boldsymbol{u},\delta) = 0$ for $\boldsymbol{u}$ near ϕ_0, we expect that r_n is close to $\sqrt{\lambda}$, i.e. away from the origin. We therefore expect solutions in the domain of G.

Proposition 4.3. *Equation $\boldsymbol{F}(\boldsymbol{u},\delta) = \boldsymbol{0}$, $\forall \boldsymbol{u} \in X$, is equivalent to*

$$F_n^r(\boldsymbol{u},\delta) = 0, \quad F_n^i(\boldsymbol{u},\delta) = 0, \qquad \forall n \in S^{\perp}, \tag{4.10}$$

$$\cos\theta_n F_n^r(\boldsymbol{u},\delta) + \sin\theta_n F_n^i(\boldsymbol{u},\delta) = 0, \qquad \forall n \in S, \tag{4.11}$$

$$-\sin\theta_n F_n^r(\boldsymbol{u},\delta) + \cos\theta_n F_n^i(\boldsymbol{u},\delta) = 0, \qquad \forall n \in S, \tag{4.12}$$

where $\boldsymbol{u}$ is expressed in the coordinates $\{q_n, p_n\}_{n\in S^\perp}$ and $\{r_n, \theta_n\}_{n\in S}$.

Proof. We view the function $\boldsymbol{F}(\cdot,\delta) : X \to X$ as a vector field in X by mapping the basis vectors $\{\boldsymbol{e}^r(n), \boldsymbol{e}^i(n)\}_{n\in\mathbb{Z}^d}$ to the respective basis vectors $\partial/\partial u_n^r$, $\partial/\partial u_n^i$ of the tangent space at each point in X. Then,

$$\boldsymbol{F} = \sum_{n\in\mathbb{Z}^d}\left(F_n^r\frac{\partial}{\partial u_n^r} + F_n^i\frac{\partial}{\partial u_n^i}\right), \tag{4.13}$$

and since

$$\begin{aligned}
&\frac{\partial}{\partial u_n^r} = \frac{\partial}{\partial q_n}, \quad \frac{\partial}{\partial u_n^i} = \frac{\partial}{\partial p_n}, \qquad \forall n \in S^{\perp},\\
&\frac{\partial}{\partial u_n^r} = \cos\theta_n\frac{\partial}{\partial r_n} - \frac{\sin\theta_n}{r_n}\frac{\partial}{\partial\theta_n}, \quad \frac{\partial}{\partial u_n^i} = \sin\theta_n\frac{\partial}{\partial r_n} + \frac{\cos\theta_n}{r_n}\frac{\partial}{\partial\theta_n}, \qquad \forall n \in S,
\end{aligned}$$

expression (4.13) becomes

$$\begin{aligned}
\boldsymbol{F} = &\sum_{n\in S^\perp}\left(F_n^r\frac{\partial}{\partial q_n} + F_n^i\frac{\partial}{\partial p_n}\right) + \sum_{n\in S}\left(\cos\theta_n F_n^r + \sin\theta_n F_n^i\right)\frac{\partial}{\partial r_n}\\
&+ \sum_{n\in S}\frac{1}{r_n}\left(-\sin\theta_n F_n^r + \cos\theta_n F_n^i\right)\frac{\partial}{\partial\theta_n}.
\end{aligned}$$

Therefore, the equation $\boldsymbol{F}(\boldsymbol{u}, \delta) = \boldsymbol{0}$, $\forall \boldsymbol{u} \in X$, is equivalent to the vanishing the coefficients of $\{\partial_{q_n}, \partial_{p_n}\}_{n \in S^\perp}$ and $\{\partial_{r_n}, \partial_{\theta_n}\}_{n \in S}$. □

To solve system (4.10)–(4.12), we group the variables $\{q_n, p_n\}_{n \in S^\perp}$ and $\{r_n\}_{n \in S}$ into a radial variable $z \in Z = P_0 X \times \mathbb{R}_+^N$ and the variables $\{\theta_n\}_{n \in S}$ into an angular variable $\boldsymbol{\theta} \in Z_) = \mathbb{T}^N$. Then, the left-hand side of system (4.10)–(4.11) defines a function $\boldsymbol{f}_r : Z \times \mathbb{T}^N \times \mathbb{R} \to Z$. Similarly, the left-hand side of system (4.12) defines a function $\boldsymbol{f}_\theta : Z \times \mathbb{T}^N \times \mathbb{R} \to \mathbb{R}^N$. Consider a point $z_0 \in Z$, such that $q_n(z_0) = p_n(z_0) = 0$, $\forall n \in S^\perp$ and $r_n(z_0) = \sqrt{\lambda}$, $\forall n \in S$. Since $\boldsymbol{F}(\boldsymbol{\phi}_0, 0) = 0$, we have

$$\boldsymbol{f}_r(z_0, \boldsymbol{\theta}, 0) = \boldsymbol{0}, \quad \boldsymbol{f}_\theta(z_0, \boldsymbol{\theta}, 0) = \boldsymbol{0}, \qquad \forall \boldsymbol{\theta} \in \mathbb{T}^N. \tag{4.14}$$

We seek a function $z = \boldsymbol{r}(\delta)$, $\boldsymbol{\theta} = \boldsymbol{\theta}(\delta)$ satisfying the system

$$\boldsymbol{f}_r(z, \boldsymbol{\theta}, \delta) = \boldsymbol{0}, \qquad \boldsymbol{f}_\theta(z, \boldsymbol{\theta}, \delta) = \boldsymbol{0}. \tag{4.15}$$

We show that for any $\boldsymbol{\theta} \in \mathbb{T}^N$, and sufficiently small δ, the first equation has a unique solution $z = z(\boldsymbol{\theta}, \delta)$. Then, the second equation is reduced to finding $\boldsymbol{\theta} \in \mathbb{T}^N$ satisfying

$$\boldsymbol{f}_\theta(z(\boldsymbol{\theta}, \delta), \boldsymbol{\theta}, \delta) = \boldsymbol{0}. \tag{4.16}$$

Lemma 4.4. *Fix $\boldsymbol{\theta} \in \mathbb{T}^N$. There exists a $\delta_0 > 0$, independent of $\boldsymbol{\theta}$, such that for every δ in $0 \leqslant \delta < \delta_0$, there exists a unique function $z = z(\boldsymbol{\theta}, \delta) \in Z$ satisfying $\boldsymbol{f}_r(z(\boldsymbol{\theta}, \delta), \boldsymbol{\theta}, \delta) = \boldsymbol{0}$. The dependence of $z(\boldsymbol{\theta}, \delta)$ on $\boldsymbol{\theta}$ and δ is real analytic and $z(\boldsymbol{\theta}, 0) = z_0$, $\forall \boldsymbol{\theta} \in \mathbb{T}^N$.*

Proof. Fix $\boldsymbol{\theta} \in \mathbb{T}^N$. Since $\boldsymbol{f}_r(z_0, \boldsymbol{\theta}, 0) = \boldsymbol{0}$, we want to apply the implicit function theorem to find solutions for small δ. Let $P : X \to Z$ be the orthogonal projection onto Z. We then find that

$$D_z \boldsymbol{f}_r(z_0, \boldsymbol{\theta}, 0) = P D_{\boldsymbol{u}} \boldsymbol{F}(\boldsymbol{\phi}_0, 0) P. \tag{4.17}$$

By proposition 4.1, the spectrum of $D_z \boldsymbol{f}_r(z_0, \boldsymbol{\theta}, 0)$ consists of an eigenvalue λ of infinite multiplicity and an eigenvalue -2λ of multiplicity N, which are independent of $\boldsymbol{\theta}$. Therefore, this operator has a bounded inverse. The function $\boldsymbol{f}_r(z, \boldsymbol{\theta}, \delta)$ is also real analytic in $Z \times \mathbb{R}$ and by the real analytic implicit function theorem [Z], there exists δ_0 that is independent of $\boldsymbol{\theta}$ such that, if $|\delta| < \delta_0$, then there exists a unique function $z(\boldsymbol{\theta}, \delta)$ that satisfies $\boldsymbol{f}_r(z(\boldsymbol{\theta}, \delta), \boldsymbol{\theta}, \delta) = \boldsymbol{0}$ with $z(\boldsymbol{\theta}, 0) = z_0$ and is real analytic in δ. The argument clearly works for any $\boldsymbol{\theta} \in \mathbb{T}^N$.

Instead of applying the implicit function theorem for each fixed $\boldsymbol{\theta}$, i.e. treating only δ as the parameter, we can also choose any $\boldsymbol{\theta}_0 \in \mathbb{T}^N$ and let the parameter space consist of $(\boldsymbol{\theta}, \delta)$ in a neighbourhood of $(\boldsymbol{\theta}_0, 0)$. The solutions for $\boldsymbol{f}_r(z, \boldsymbol{\theta}, \delta)$ are also real analytic in $\boldsymbol{\theta}$ near $\boldsymbol{\theta}_0$ and coincide with the ones obtained by the above argument by uniqueness. As a result, the existence of solutions is established for $0 \leqslant \delta < \delta_0$, where δ_0 is independent of $\boldsymbol{\theta}$ in the torus $\mathbb{T}^N$. □

Remark 4.5. By the continuity of $z(\boldsymbol{\theta}, \delta)$ on δ, and the property $z(\boldsymbol{\theta}, 0) = z_0$ we see that the components r_n of the vector $z(\boldsymbol{\theta}, \delta)$ are close to $\sqrt{\lambda}$ for sufficiently small δ. In particular, these components remain positive. By the continuity of $z(\boldsymbol{\theta}, \delta)$ in $\boldsymbol{\theta}$, and compactness, we can find $\delta_0 > 0$ for which $z(\boldsymbol{\theta}, \delta) \in P_0 X \times \mathbb{R}_+^N$, i.e. the domain of the change of coordinates is $\mathbb{T}^N$ for $0 \leqslant \delta < \delta_0$.

With δ_0 and $z(\boldsymbol{\theta}, \delta)$, as in lemma 4.4, define the function $\boldsymbol{g}(\boldsymbol{\theta}, \delta)$ by

$$\boldsymbol{g}(\boldsymbol{\theta}, \delta) = \boldsymbol{f}_\theta(z(\boldsymbol{\theta}, \delta), \boldsymbol{\theta}, \delta), \qquad \forall \boldsymbol{\theta} \in \mathbb{T}^N, \quad |\delta| < \delta_0. \tag{4.18}$$

The function $\boldsymbol{g}(\boldsymbol{\theta}, \delta)$ can be extended by periodicity in each direction to $\boldsymbol{\theta} \in \mathbb{R}^N$. Also, let $\boldsymbol{d}_N = \sum_{j=1}^N \boldsymbol{e}_j \in \mathbb{R}^N$. If $\boldsymbol{\theta}_0 \in \mathbb{R}^N$ is a root of $\boldsymbol{g}(\boldsymbol{\theta}, \delta)$, then $\boldsymbol{\theta}_0 + s\boldsymbol{d}_N$ is also a root of $\boldsymbol{g}(\boldsymbol{\theta}, \delta)$,

$\forall s \in \mathbb{R}$. This follows from the fact that $\boldsymbol{F}(\boldsymbol{u}_0, \delta) = \boldsymbol{0}$ implies $\boldsymbol{F}(\mathrm{e}^{\mathrm{i}\chi}\boldsymbol{u}_0, \delta) = \boldsymbol{0}$, $\forall \chi \in \mathbb{R}$. Thus, any root $\boldsymbol{\theta}_0 \in \mathbb{T}^N$ of $\boldsymbol{g}(\boldsymbol{\theta}, \delta)$ belongs to a circle $S^1(\boldsymbol{\theta}_0) \subset \mathbb{T}^N$ of roots of $\boldsymbol{g}(\boldsymbol{\theta}, \delta)$. Consequently, if $\boldsymbol{\theta}_0$ is a solution of $\boldsymbol{g}(\boldsymbol{\theta}, \delta) = 0$, then the derivative $D_{\boldsymbol{\theta}}\boldsymbol{g}(\boldsymbol{\theta}_0, \delta)$ has an eigenvalue $\lambda_1 = 0$ with corresponding eigenvector $\boldsymbol{d}_N$.

Definition 4.6. *A solution $\boldsymbol{\theta}_0 \in \mathbb{T}^N$ of $\boldsymbol{g}(\boldsymbol{\theta}, \delta) = \boldsymbol{0}$ is non-degenerate if the null eigenvalue of $D_{\boldsymbol{\theta}}\boldsymbol{g}(\boldsymbol{\theta}_0, \delta)$ has geometric multiplicity one.*

By lemma 4.4, there exists $\delta_0 > 0$, for which we can use the Taylor series expansions

$$z(\boldsymbol{\theta}, \delta) = \sum_{k=0}^{\infty} \delta^k z_k(\boldsymbol{\theta}), \qquad \forall |\delta| < \delta_0. \tag{4.19}$$

Recall that all components r_n of the vector $z_0(\theta)$ are equal to $\sqrt{\lambda}$. Also, by the real analyticity of $\boldsymbol{f}_\theta(z, \boldsymbol{\theta}, \delta)$ in z and δ, we can write

$$\boldsymbol{g}(\boldsymbol{\theta}, \delta) = \sum_{k=1}^{\infty} \delta^k \boldsymbol{g}_k(\boldsymbol{\theta}), \qquad \forall |\delta| < \delta_0. \tag{4.20}$$

If we let

$$z_m(\boldsymbol{\theta}, \delta) = \sum_{k=0}^{m} \delta^k z_k(\boldsymbol{\theta}), \qquad \boldsymbol{g}_m(\boldsymbol{\theta}, \delta) = \sum_{k=1}^{m} \delta^k \boldsymbol{g}_k(\boldsymbol{\theta}), \tag{4.21}$$

then the finite sums $z_m(\boldsymbol{\theta}, \delta)$ and $\boldsymbol{g}_m(\boldsymbol{\theta}, \delta)$ are real analytic on $\mathbb{T}^N \times \mathbb{R}$. The remarks above about the equivariance of $\boldsymbol{g}(\boldsymbol{\theta}, \delta)$ under the translation $\boldsymbol{\theta}_0 \mapsto \boldsymbol{\theta}_0 + s\boldsymbol{d}_N$, $\forall s \in \mathbb{R}$ and $\boldsymbol{\theta}_0 \in \mathbb{T}^d$ also apply to the derivatives of $\boldsymbol{g}(\boldsymbol{\theta}, \delta)$ in δ, and therefore to the function $\boldsymbol{g}_k(\boldsymbol{\theta})$. We thus solve the equation $\boldsymbol{f}_r(z, \boldsymbol{\theta}, \delta) = \boldsymbol{0}$ by $z_m(\boldsymbol{\theta}, \delta)$ up to some order $m \geqslant 1$ in δ, compute $\boldsymbol{g}_m(\boldsymbol{\theta}, \delta)$, and solve the equation $\boldsymbol{g}_m(\boldsymbol{\theta}, \delta) = \boldsymbol{0}$.

Definition 4.7. *A solution $\boldsymbol{\theta}_0 \in \mathbb{T}^N$ of $\boldsymbol{g}_m(\boldsymbol{\theta}, \delta) = \boldsymbol{0}$ is m-non-degenerate if $D_{\boldsymbol{\theta}}\boldsymbol{g}_m(\boldsymbol{\theta}_0, \delta)$ has one eigenvalue $\lambda_1 = 0$ of geometric multiplicity one and all remaining eigenvalues $\lambda_2, \ldots, \lambda_N$ satisfy $|\lambda_j| > C_m|\delta|^m$ for some $C_m > 0$.*

Remark 4.8. By the fact that $\boldsymbol{g}(\boldsymbol{\theta}, 0) = \boldsymbol{0}$, $\forall \boldsymbol{\theta} \in \mathbb{T}^N$, we have that any eigenvalue λ_j of $D_{\boldsymbol{\theta}}\boldsymbol{g}_m(\boldsymbol{\theta}, \delta)$, evaluated at a solution $\boldsymbol{\theta}$ of $\boldsymbol{g}_m(\boldsymbol{\theta}, \delta) = \boldsymbol{0}$, $m \geqslant 1$, must satisfy $|\lambda_j| < C|\delta|$ for some $C > 0$. This follows from the fact that $D_{\boldsymbol{\theta}}\boldsymbol{g}_m(\boldsymbol{\theta}, \delta)$ depends continuously on δ and hence its eigenvalues also depend continuously on δ [K76].

Using definitions 4.6 and 4.7, we formulate analogue of theorems 2.2 and 2.4.

Theorem 4.9. *Let $\boldsymbol{\theta} = \boldsymbol{\theta}_0$ be an m-non-degenerate solution of $\boldsymbol{g}_m(\boldsymbol{\theta}, \delta) = \boldsymbol{0}$ for some $m \geqslant 1$. Then, there exists $\delta_0 > 0$ such that the difference equation (4.2) has a unique, modulo to the gauge translation, solution $\phi_\delta \in l_p^2(\mathbb{Z}^d)$ for any fixed $p \geqslant 0$, such that*

$$\forall 0 \leqslant \delta < \delta_0 : \quad \|\phi_\delta - \phi_0\|_{l_p^2} \leqslant C\delta, \tag{4.22}$$

for some $C > 0$. The solution ϕ_δ is real analytic in δ.

Proof. It follows from the analyticity of $\boldsymbol{g}(\boldsymbol{\theta}, \delta)$ and the implicit function theorem along the directions that are orthogonal to $\boldsymbol{d}_N$ that, if $\boldsymbol{\theta}_0$ is a root of $\boldsymbol{g}_m(\boldsymbol{\theta}, \delta)$, then there exists a $\tilde{\boldsymbol{\theta}}_0$ satisfying $\boldsymbol{g}(\tilde{\boldsymbol{\theta}}_0, \delta) = \boldsymbol{0}$, such that $\|\tilde{\boldsymbol{\theta}}_0 - \boldsymbol{\theta}_0\| < C|\delta|^{m+1}$. Also, $D_{\boldsymbol{\theta}}\boldsymbol{g}(\tilde{\boldsymbol{\theta}}_0, \delta)$ has one eigenvalue $\lambda_1 = 0$, of geometric multiplicity one, and the remaining eigenvalues $\lambda_2, \ldots, \lambda_N$ satisfying $|\lambda_j| > C_m|\delta|^m$ for some $C_m > 0$. Let $\tilde{z}_0 = z(\tilde{\boldsymbol{\theta}}_0, \delta)$. Then $\tilde{\boldsymbol{u}}_0 = (\tilde{z}_0, \tilde{\boldsymbol{\theta}}_0)$ satisfies $\boldsymbol{F}(\tilde{\boldsymbol{u}}_0, \delta) = \boldsymbol{0}$. Moreover, $D_{\boldsymbol{u}}\boldsymbol{F}(\tilde{\boldsymbol{u}}_0, \delta)$ has one eigenvalue $\lambda_1 = 0$, $N - 1$ eigenvalues $\lambda_2, \ldots, \lambda_N$ that satisfy

$C_-|\delta|^m < |\lambda_j| < C_+|\delta|$, for some $C_-, C_+ > 0$, and N eigenvalues $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_N$ that belong to the interval $(-2\lambda - C_1\delta, -2\lambda + C_1\delta)$ for some $C_1 > 0$. The rest of the spectrum of $D_{\boldsymbol{u}}\boldsymbol{F}(\tilde{\boldsymbol{u}}_0, \delta)$ belongs to the interval $(\lambda - C_2\delta, \lambda + C_2\delta)$ for some $C_2 > 0$. The statement of the theorem follows by the implicit function theorem. □

Theorem 4.10. *Let ϕ_δ be defined by theorem 4.9 and $0 < \delta < \delta_0$. Assume that $D(t)$ is a real-valued T-periodic function in $H^s([0, T])$ for any $s > \frac{1}{2}$. Fix $\Omega = \frac{2\pi}{T}$ such that $\lambda \neq m\Omega$ for all $m \in \mathbb{N}$. Then, there exist $\epsilon_0 > 0$ which depends on δ and $\min_{m\in\mathbb{N}} |\lambda - m\Omega|$ such that the differential lattice equation (4.1) has a unique, modulo to the gauge translation, solution $\boldsymbol{v}_\epsilon(t)$ in $X_{s,p} = H^s([0, T], l^2_p(\mathbb{Z}^d))$ for any fixed $s > \frac{1}{2}$ and $p \geqslant 0$, such that $\boldsymbol{v}_\epsilon(t+T) = \boldsymbol{v}_\epsilon(t)$ on $t \in \mathbb{R}$ and*

$$\forall 0 \leqslant \epsilon < \epsilon_0 : \quad \|\boldsymbol{v}_\epsilon(t) - \phi_\delta\|_{X_{s,p}} \leqslant C\epsilon, \tag{4.23}$$

for some $C > 0$. The solution $\boldsymbol{v}_\epsilon(t)$ is real analytic in ϵ.

Proof. The proof is similar to the proof of theorem 2.4 under the non-degeneracy assumption on $\boldsymbol{\theta}_0$ and is left for reader's exercise. □

Example 4.11. Let S be a closed discrete contour in $\mathbb{Z}^d$ that consists of nodes, each is connected to two nearest neighbours. Let the index $j = 1, \ldots, N$ enumerate nodes n_j along the contour S. Using the lowest-order solution $z_0(\boldsymbol{\theta})$, where all components r_n are equal to $\sqrt{\lambda}$, we compute

$$(\boldsymbol{g}_1(\boldsymbol{\theta}))_{n_j} = \sqrt{\lambda}\left(\sin(\theta_{n_j} - \theta_{n_{j+1}}) + \sin(\theta_{n_j} - \theta_{n_{j-1}})\right), \qquad \forall n_j \in S, \tag{4.24}$$

where the periodic boundary conditions $\theta_{n_{N+1}} = \theta_1$ and $\theta_0 = \theta_N$ are used. Particular examples of vortex solutions and contours S can be found in [PKF05b] for $d = 2$ and [LPK08] for $d = 3$, where the non-degeneracy assumptions are shown to be satisfied for every vortex configuration.

Remark 4.12. With exactly the same technique as in the proof of theorem 2.6, spectral stability of time-periodic solutions $\boldsymbol{v}_\epsilon(t)$ for sufficiently small $\epsilon > 0$ is inherited from spectral stability of stationary solutions ϕ_δ for $\epsilon = 0$, under the additional condition $m\Omega \neq 2\lambda$ for all $m \in \mathbb{N}$. Examples in [PKF05b] show, however, that no simple correlation exists between small positive and negative eigenvalues of the linear operator $D_{\boldsymbol{u}}\boldsymbol{F}(\boldsymbol{u}, \delta)$ and unstable eigenvalues of the spectral stability problem (3.13) for $d \geqslant 2$.

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