Minimal Jacobi geodesics in the Newtonian n-center problem

Panayotis Panayotaros*

Department of Applied Mathematics

University of Colorado at Boulder

Boulder, CO 80309-0526

Abstract.

We show that minimal geodesics of the Jacobi metric for the Newtonian planar $n-$center problem avoid collisions and therefore yield classical trajectories of prescribed energy connecting any two configurations of the system. The proof involves a minimization argument for the Jacobi arc-length, and an analysis of the approach to collision.

* *email: panos@colorado.edu*

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1. Introduction

We consider a particle of mass m moving on the plane under the influence of a finite number of fixed masses m_i located at points σ_i, $i = 1, \dots, n$ (i.e. the planar $n-$center problem). The equations of motion for the position $x(t)$ of the particle are $mx'' = -\nabla U(x)$, with the potential U being the finite sum of central force Newtonian potentials, i.e. $U(x) = -\sum_{i=1}^{n} m_i |x - \sigma_i|^{-1}$. We show that for any $E \geq 0$, any two points a, b outside the singularities $\sigma_1, \dots, \sigma_n$ can be connected by a trajectory of Newton's equations that avoids the singularities of the potential, has total energy E and minimizes the Jacobi action. The precise statement is given in section 2.

The present work is inspired by recent variational constructions in the $n-$body problem. While the main focus has been on the existence of periodic orbits that avoid collisions, the problem involves the question of finding open collision-free trajectories with prescribed endpoints. This question is much simpler for potentials with $\frac{1}{r^\alpha}$, $\alpha \geq 2$ singularities ("strong force" potentials), where the action of any test path passing through a collision diverges (see e.g. [P], [G], [CGMS]). This fact leads to several existence results for periodic collision-free orbits via the direct method of the calculus of variations (see [Mon1], [CGMS]). There is also a significant body of literature on more general critical points of a modified Jacobi action (see e.g. [ACZ]) where the "strong force" assumption is also used to avoid collisions.

For the Newtonian potential, we should not in general expect to exclude collisions by minimizing the action over classes of loops (see [G], [V], [Mon2], [C1]). On the other hand, it was recently shown in [M1] (see also [C2]) that minimizers of the action over paths with prescribed endpoints in the planar and spatial Newtonian $n-$body problems avoid collisions. This is a generalization of earlier partial results on collision-free action minimizers in the Newtonian n-body problem (see [D]), and we note that some special results of this type were also used in works on the "figure-8" periodic orbit of the equal mass planar 3-body problem (see [CM], [Ch1]), a generalization to the equal mass planar 4-body problem (see [Ch2]), and a new periodic orbit in the spatial equal mass 3-body problem (see [M2]). These works suggest a general strategy for combining variational and symmetry arguments in problems with singular potentials that may lead to further dynamically interesting constructions (see [C2] for an overview and some earlier related works).

In this work we consider the existence of collision-free minimizers for the Jacobi action (arc-length) where instead of prescribing the time interval we prescribe the total energy of the path connecting two given points. We are not aware of any analogous results for Jacobi minimizers in the $n-$body problem, and we here consider the simpler planar $n-$center problem. In particular we start with the Jacobi action $\int_{\tau_1}^{\tau_2} [2(E-U)]^{\frac{1}{2}} |x'| d\tau$ over paths with given endpoints $a = x(\tau_1)$ and $b = x(\tau_2)$ and show the existence of minimizers using arguments of the direct method of the calculus of variations (see Section 3). We use some basic notions from metric geometry and look for minimizers of an induced length defined for continuous curves connecting two given points a, b. Compactness follows from the existence of parameterization by arc-length, the fact that we include paths that may pass through the singularities, and the divergence of the length for paths wandering too far from the endpoints. In the $E = 0$ case, the last property is a consequence of the slow decay at infinity of the Newtonian potential.

To show that minimizers of the Jacobi length avoid singularities, we assume that a minimizer passes through a singularity, and produce continuous curves with the same endpoints that avoid the singularity and have smaller Jacobi length. To construct shorter deformed curves we study the geometry of collision and ejection trajectories using Sundman's regularization, and we see that the approach to the singularities is very nearly "radial" (in a sense made precise in Section 4). This allows us to define small deformations off the singularity in a rather explicit way, estimate the first variation of the Jacobi arc-length, and see that moving off the singularities decreases the length. The strategy becomes very simple in the test problem where we have only one singularity. There the approach to the singularity is radial and the variation of the length as we deform off the singularity is estimated easily (e.g. deforming in a direction normal to the collision trajectories). By controlling the geometry of the approach to the singularities, we see that similar estimates are also valid for the case of n singularities (see section 4).

After completing this work we have found that the existence of Jacobi geodesics connecting two points in the planar $n-$center problem can be shown by a different method (see [B], [KK], and [BT], [K] for a generalization to the spatial case). We believe that the approach in these works also leads to the existence of minimal collision-free geodesics. Also, the idea of considering local deformations off the singularities was used by [D] for action minimizers, it is interesting however that the deformations used there do not allow us to conclude that Jacobi minimizers avoid collisions (see remarks in [D]).

The main result is stated in Section 2, where we also set the notation. In Section 3 we extend the Jacobi length to a larger class of admissible paths and prove the existence of minimizers connecting two given points. In Section 4 we analyze the approach to the singularities, and prove the main theorem. We conclude by discussing some possible extensions in Section 5.

2. The Jacobi action

We consider the configuration space $M = \mathbf{R}^2$ with Cartesian coordinates $x = (x_1, x_2)$, and Lagrangians L of the form

$$L = K(x') - U(x), \quad \text{with} \quad K = \tfrac{1}{2}m|x'|^2, \quad U = -\sum_{i=1}^{n} \frac{m_i}{|x - \sigma_i|}. \tag{2.1}$$

The set of the n distinct points $\sigma_1, \dots, \sigma_n$ will be denoted by Σ. The masses m, $m_1, \dots, m_n$ are positive.

The action $A(x)$ of a sufficiently smooth path $x : [t_1, t_2] \to M \setminus \Sigma$ is

$$A(x) = \int_{t_1}^{t_2} L(x(t), x'(t))dt, \tag{2.2}$$

and a general strategy for finding trajectories of the system (2.1) joining two points a, $b \in M \setminus \Sigma$ is to look for extrema of the action $A(x)$ over a suitable class of paths $x : [t_1, t_2] \to M \setminus \Sigma$ with $x(t_1) = a$, $x(t_2) = b$. By a classical argument, an extremum of $A(x)$ that is C^2 in $[t_1, t_2]$ will be a solution of the Euler-Lagrange equations for the system of (2.1).

A related approach is based on the Maupertuis principle, where we extremize the action over paths with fixed total energy $E = K + U$. In particular, fix $E \in \mathbf{R}$ and let the Hill's region $\mathcal{H}_E$ be the set of points in $M \setminus \Sigma$ where $E - U(x) > 0$. Also, let $x : [\tau_1, \tau_2] \to \mathcal{H}_E$ be a sufficiently smooth parameterized curve with $x(\tau_1) = a$, $x(\tau_2) = b$, and define the Jacobi arc-length (or Jacobi action) $l(x)$ of the path x by

$$l(x) = \int_{\tau_1}^{\tau_2} [2(E - U(x(\tau))]^{1/2} |x'(\tau)| d\tau. \tag{2.3}$$

Then, a C^2 curve extremizing the action $A(x)$ over paths $x : [t_1, t_2] \to \mathcal{H}_E$ with $x(t_1) = a$, $x(t_2) = b$ and satisfying $K(x') + U(x) = E$ is, after a suitable reparameterization, a geodesic of the Jacobi metric

$$g_E(x) = \sum_{i,j=1}^{2} 2(E - U(x))\delta_{ij} dx_i dx_j, \quad x \in \mathcal{H} \tag{2.4}$$

(see e.g. [A]). Conversely, we can obtain trajectories of the Euler-Lagrange equations from geodesics of the Jacobi metric.

Proposition 2.1 Let $x(\tau) : [\tau_1, \tau_2] \to \mathcal{H}_E$ be a C^2 geodesic of the Jacobi metric above, and let $t(\tau)$ satisfy

$$\frac{dt}{d\tau} = \left|\frac{dx(\tau)}{d\tau}\right| [2(E - U(x(\tau))]^{-\frac{1}{2}}. \tag{2.5}$$

Then, $x(t) = x(t(\tau))$ satisfies the Euler-Lagrange equations for the Lagrangian $L = K - U$ of (2.1), and has energy $K + U = E$.

Proof: The geodesic $x(\tau)$ must satisfy the geodesic equations for g_E, which reduce to

$$\frac{d}{d\tau}\left[2(E-U)\frac{dx_i}{d\tau}\right] = \left[\frac{\partial}{\partial x_i}(E-U)\right]\left|\frac{dx}{d\tau}\right|^2, \quad i = 1, 2. \tag{2.6}$$

In addition, $x(\tau)$ must be parameterized by a multiple of the Jacobi arc-length, thus satisfying

$$[2(E - U(x(\tau))]^{\frac{1}{2}}\left|\frac{dx(\tau)}{d\tau}\right| = c, \quad \text{with} \quad c > 0. \tag{2.7}$$

Combining (2.5) and (2.7), the geodesic equations (2.6) become

$$c[2(E-U)]^{-\frac{1}{2}}\frac{d^2x_i}{dt^2} = -\frac{\partial U}{\partial x_i}\left|\frac{dx}{d\tau}\right|, \quad i = 1, 2. \tag{2.8}$$

By (2.5), equations (2.8) are the Euler-Lagrange equations for the Lagrangian of (2.1). Also, using (2.5) to express $|\frac{dx}{dt}|$ in terms of $|\frac{dx}{d\tau}|$ at $t = t(\tau)$, the energy $K(x'(t)) + U(x(t))$ at $t = t(\tau_1)$ is E, and remains constant by the Euler-Lagrange equations.

■

Remark 2.1 By (2.5), the function $t(\tau)$ is strictly increasing, and the trajectory $x(t)$ obtained from the geodesic $x(\tau)$ joins the points $x(\tau_1)$, $x(\tau_2)$ in a time interval $t(\tau_2) - t(\tau_1)$.

Therefore, to connect any two points a, $b \in M \setminus \Sigma$ by a trajectory of the Lagrangian system of (2.1) that avoids Σ it is sufficient to fix some $E \geq 0$ and find a C^2 Jacobi geodesic in $M \setminus \Sigma$ that connects the two points. Our main result is that this is possible.

Theorem 2.1 Let $E \geq 0$ and a, $b \in M \setminus \Sigma$. Then there exists a C^2 curve $x : [\tau_1, \tau_2] \to M \setminus \Sigma$ that is a minimal geodesic of the Jacobi metric g_E of (2.4) and has endpoints $x_1(\tau_1) = a$, $x(\tau_2) = b$.

To show the theorem we will minimize a functional related to the Jacobi arc-length over a suitable class of curves connecting the two points a, b (Section 3), and show that the minimizers are C^2 Jacobi geodesics that avoid the singularity set (Section 4). By Proposition 2.1, we will then have a family of trajectories joining a, b, at least one trajectory for each value of $E \geq 0$.

Remark 2.2 It will be seen that the theorem also holds in the case where the endpoints a, b are in Σ, i.e. we then show the existence of a Jacobi length minimizing trajectory that avoids Σ at all intermediate times. In the case where $E < 0$, the Hill's region $\mathcal{H}_E$ is a bounded subset of M, and the metric vanishes at $\partial\mathcal{H}_E$. We consider that in this case, the strategy of seeking trajectories joining pairs of points in $\mathcal{H}_E$ by minimizing the Jacobi arc-length is only appropriate for points that are sufficiently far from $\partial\mathcal{H}_E$.

3. Minimizers of the Jacobi arc-length

In this section we show the existence of a continuous curve that joins any two points a , $b \in M \setminus \Sigma$ and minimizes a length functional induced by the Jacobi arc-length.

We start with an outline of the argument. We first define a class $P(a, b)$ of continuous, piecewise C^2 parameterized curves joining points a and $b \in M \setminus \Sigma$ for which the the Jacobi arc-length l of (2.3) is well defined. The Jacobi arc-length over such curves is used to define a metric ρ_l in M, and from this metric we obtain an induced length $\hat{l}$, defined for any continuous curve joining a and b. The induced length we use is a well-known construction in metric spaces, and is lower semi-continuous in the C^0 topology. The existence of a minimizer will follow from the fact that the induced length of curves that join a and b and cross a circle of (a sufficiently large) radius R are bounded below by a function $f(R)$ that diverges as $R \to \infty$. This is shown in Lemma 3.2, and implies that minimizing sequences of curves will have their image in a compact subset of M. A standard application of the Arzela-Ascoli theorem for curves parameterized by arc-length will then show that the minimum of $\hat{l}$ is attained (Theorem 3.1). The regularity of the minimizers of the Jacobi length will be considered in the next section.

Remark 3.1 Note that the induced length $\hat{l}$ can be also defined for continuous paths that may pass through Σ. In the case $E = 0$ where the Jacobi metric vanishes at infinity, Lemma 3.2 relies on the slow decay of the Newtonian potential.

We consider M with the Euclidean metric $\rho_{\mathcal{E}}$. A continuous parameterized curve

$\gamma : [\tau_1, \tau_2] \to M$ is *never-locally-constant* if there exists no interval $[\tau, \tau'] \subset [\tau_1, \tau_2]$, $\tau \neq \tau'$ for which $\gamma|_{[\tau,\tau']}$ is constant.

Definition 3.1 Let a, $b \in M$, and $P(\tau_1, \tau_2; a, b)$ be the set of C^0, piecewise C^2, never-locally-constant curves $\gamma : [\tau_1, \tau_2] \to M$, $\gamma(\tau_1) = a$, $\gamma(\tau_2) = b$ whose image intersects Σ for a discrete set of points $\tau \in [\tau_1, \tau_2]$. Also, let $P(a, b) = \cup_{[\tau_1,\tau_2]\subset \mathbf{R}} P(\tau_1, \tau_2; a, b)$, and $P = \cup_{(a,b)\in M\times M} P(a, b)$

The above sets are non-empty, moreover $P(a, b)$ is closed under C^0, piecewise C^2 reparameterizations (C^0, piecewise C^2 homeomorphisms between intervals). Also, that fact that the image of every $\gamma \in P(a, b)$ intersects Σ for a discrete set of points of its domain allows us to define the Jacobi length of these paths in a natural way.

Definition 3.2 Let $\gamma : [\tau_1, \tau_2] \to M$ be in $P(a, b)$ for some pair a, $b \in M$, and fix $E \geq 0$. Then the (Jacobi) length $l : P(a, b) \to \mathbf{R}_+ \cup \{\infty\}$ is defined by

$$l(\gamma) = \int_{\tau_1}^{\tau_2} [2(E - U(\gamma(\tau))]^{1/2} |\gamma'(\tau)| d\tau, \tag{3.1}$$

with the notation of section 2.

Notation: ¿From now on we suppress from the notation the dependence of l and the corresponding Jacobi metric on the energy E. What follows will hold for any fixed $E \geq 0$. Also, we set all masses m, $m_1, \dots, m_n$ to unity; all arguments below will be valid for arbitrary positive masses.

We check that l is additive, depends continuously on restrictions to subintervals, and is bounded away from zero for admissible paths connecting any $a \in M$ with points outside any ball radius $\epsilon > 0$ around a. Moreover, we can easily see that l is invariant under C^0, piecewise C^2 reparameterizations of paths in $P(a, b)$, i.e. as in the case where the metric is smooth, and regardless of whether l is finite or infinite.

Proposition 3.1 Let a, $b \in M$. Then $\rho_l(a, b) = \inf_{\gamma \in P(a,b)} l(\gamma)$ is a metric on M.

Proof: The symmetry of ρ_l and the triangle inequality follow immediately from the definition of ρ_l, and we also have $\rho_l(a, b) > 0$ for any $a \neq b \in M$, and $\rho_l(a, a) = 0$, $\forall a \in M \setminus \Sigma$. For $a \in \Sigma$, let the curve $\gamma_{\tilde{a}}$ be the concatenation of two line segments, from $a \in \Sigma$ to $\tilde{a} \notin \Sigma$ and back respectively, both traversed with constant (Euclidean) speed. The singularity in $l(\gamma_{\tilde{a}})$ is integrable, and we easily estimate that $l(\gamma_{\tilde{a}})$ is of $O(|a - \tilde{a}|^{\frac{1}{2}})$, and therefore vanishes as $\tilde{a}$ tends to a.

$\blacksquare$

We can use the metric ρ_l to define a length functional $\hat{l}$ on continuous parameterized curves. Unless otherwise specified, the topology on M in the definition of continuity of curves will be the one induced by the Euclidean metric $\rho_{\mathcal{E}}$. We then let $C^0(I, M)$ denote the set of continuous parameterized curves $\gamma : I = [\tau_1, \tau_2] \to M$.

Definition 3.3 Let $\gamma \in C^0(I, M)$, and let Y denote the set of all finite partitions $y = \{y_0 = \tau_1 \leq y_1 \leq \ldots \leq y_{N-1} = \tau_2\}$ of the interval $I = [\tau_1, \tau_2]$. Then the induced Jacobi arc-length $\hat{l}(\gamma)$ is defined by

$$\hat{l}(\gamma) = \sup_{y \in Y} \sum_{j=0}^{N-2} \rho_l(\gamma(y_j), \gamma(y_{j+1})). \tag{3.2}$$

Theorem 3.1 Consider the set $C^0(I, M)$ of continuous parameterized curves with endpoints a, $b \in M$. Then, for every pair a, $b \in M$ there exists a continuous curve $\gamma : I \to M$ joining a and b whose length $\hat{l}(\gamma)$ attains the infimum of $\hat{l}$ over the set.

To show the theorem we collect a number of intermediate statements. The proof follows the general outline of the direct method. First, we consider a minimizing sequence of curves $\{\gamma_i\} \in C^0(I, M)$ connecting a and b, and seek a subsequence that converges uniformly. We need to show that the curves of the minimizing sequence have their image in a bounded subset of M. We observe that a radial curve that reaches a point on a circle of a sufficiently large radius R has length l of least $O(R)$ if $E > 0$, and $O(R^{\frac{1}{2}})$ if $E = 0$. In Lemma 3.2 we show a similar statement for arbitrary continuous curves. We will use the following result on "short geodesics":

Lemma 3.1 Let $x \in M \setminus \Sigma$, and consider an open set $U_x \subset M \setminus \Sigma$ and an open (Euclidean) disc $B_x(\delta) \subset U_x$ of radius $\delta > 0$ around x. Then for δ sufficiently small and any points x_1, $x_2 \in B_x(\delta)$ there exists a unique C^2 Jacobi geodesic $\phi : [0, 1] \to U_x$ minimizing the arc length l over all continuous, piecewise C^2 paths with endpoints x_1, x_2 and image in U_x.

The lemma follows from the general local theory of the exponential map for smooth metrics (see e.g. [M, par. 10]).

Lemma 3.2 Let $\gamma : I = [\tau_1, \tau_2] \to M$ be a C^0 parameterized curve with $\gamma(\tau_1) = a$, $\gamma(\tau_2) = b$ and finite length $\hat{l}(\gamma)$. Fix $R_0 > 2$ such that the open disk of radius $\frac{R_0}{2}$ centered at the origin contains a, b, and the singularity set Σ, and assume that there exist $\tau \in I$ and $R > R_0$ for which $|\gamma(\tau)| \geq R$. Then, $\hat{l}(\gamma) > h(R)$, with h an increasing function satisfying $h(R) \to +\infty$ as $R \to +\infty$.

Proof: Fix R_0 as above, and choose $\tau_{R_0} < \tau_R$ satisfying $|\gamma(\tau_{R_0})| = R_0$ and $\gamma(\tau_R) = R$ respectively, with $R > R_0$. Also, let I_A be the intersection of $[\tau_{R_0}, \tau_R]$ with the preimage under γ of the closed annulus between R_0 and R. Denoting the restriction of γ to I_A by γ_A, we have $\hat{l}(\gamma_A) < \hat{l}(\gamma)$.

To estimate $\hat{l}(\gamma_A)$ we cover its image by a finite collection $\mathcal{C}$ of (Euclidean) open discs of radius ϵ. Since γ_A is continuous we can cover its image by a finite subcollection $\mathcal{C}$, for any $\epsilon > 0$. We also partition I_A as follows: let $y_0 = \tau_{R_0}$, and $B^0 \in \mathcal{C}$ a disc containing $\gamma_A(y_0)$. Also, let y_1 be the largest $\tau \in I_A$ for which $\gamma_A(\tau) \in \overline{B}^0$. Let B^1 be a disc in $\mathcal{C}$ containing $\gamma_A(y_1)$. By the assumption that $\mathcal{C}$ is an open cover such a disk exists, moreover, $B^1 \neq B^0$ by the definition of y_1. We inductively define y_{j+1} to be the largest $\tau \in I_A$ for which $\gamma_A(\tau) \in \overline{B}^j$, and letting B^{j+1} be a disk in $\mathcal{C}$ containing $\gamma_A(y_{j+1})$. When $y_{j+1} = \tau_R$ the process terminates. Note that for each j we have $B^{j+1} \neq B^0, \ldots, B^j$. We therefore have a finite partition $y = \{y_0 = \tau_{R_0} \leq y_1 \leq \ldots \leq y_{N-1} = \tau_R\}$, $N \leq \operatorname{card}(\mathcal{C})$, where every pair of points $\gamma_A(y_j)$, $\gamma_A(y_{j+1})$ belongs to the same closed disk $\overline{B}^j$.

We can choose ϵ sufficiently small so that, by Lemma 3.1 all pairs $\gamma_A(y_j)$, $\gamma_A(y_{j+1})$ can be connected by a short geodesic of the Jacobi metric. The short geodesics are the unique C^2 curves whose Jacobi length l attains $\rho_l(\gamma_A(y_j), \gamma_A(y_{j+1}))$. Reparameterizing each short geodesic that connects $\gamma_A(y_j)$ to $\gamma_A(y_{j+1})$ by an appropriate multiple of the Jacobi arc-length so that its domain is $[y_j, y_{j+1}]$, and concatenating the short geodesics, we then obtain a parameterized curve $z : I_A \to M \setminus \Sigma$. By construction, z has the same endpoints as γ_A, and is C^0 and piecewise C^2. By the definition of $\hat{l}$ we have that

$$\hat{l}(\gamma_A) \geq \sum_{j=0}^{N-2} \rho_l(\gamma_A(y_j), \gamma_A(y_{j+1})) = l(z), \tag{3.3}$$

and we can now bound $l(z)$ below by direct calculation: first, assuming without loss of generality that one of the singularities is at the origin, we have $2(E - U(x)) \geq -2U(x) \geq |x|^{-1}$. Also, for ϵ sufficiently small $|z(\tau)| > \frac{R_0}{2} > 1$, $\forall \tau \in I_A$, hence

$$l(z) \geq \sqrt{2} \int_{I_A} \frac{|z'|}{|z|^{\frac{1}{2}}} d\tau. \tag{3.4}$$

Using polar coordinates r, θ we can write $z' = r'\hat{e}_r + r\theta'\hat{e}_\theta$ ($\hat{e}_r$ points outward). Since z is piecewise C^2 and $R > R_0$, the set $I_A^+ \subset I_A$ where $r' \geq 0$ is a non-empty countable union of closed intervals with non-empty interior. Since $|z'| \geq |r'| = r'$ in I_A^+, (3.4) becomes

$$\int_{I_A} \frac{|z'|}{|z|^{\frac{1}{2}}} d\tau \geq \int_{I_A^+} \frac{r'}{r^{\frac{1}{2}}} d\tau \geq \tfrac{1}{2}(R^{\frac{1}{2}} - R_0^{\frac{1}{2}}), \tag{3.5}$$

since at the maximum and minimum of I_A^+, we have $r \geq R$ and $r \leq R_0$ respectively. ¿From (3.3), we therefore have $\hat{l}(\gamma) > l(\gamma_A) \geq CR^{\frac{1}{2}}$, with C depending on R_0, i.e. on the points a, b and the singularity set Σ.

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Therefore, if $\{\gamma_i\}$ is a minimizing sequence, then there exists $R > 0$ and a closed disc $\overline{B}_0(R) = \{x \in M : |x| \leq R\}$ such that $\gamma_i \in C^0(I, \overline{B}_0(R))$, for all i after some N. Note that by the definition of $\hat{l}$, if γ is continuous, piecewise C^2 and avoids Σ then $\hat{l}(\gamma) \leq l(\gamma)$ (the two lengths are in fact equal for such curves, but we do not need this here). The infimum of $\hat{l}$ over the curves of interest is therefore finite, and the curves γ_i of the minimizing sequence must be uniformly equicontinuous, for i after some N'. To see this, first note that the curves γ_i are also continuous as functions from I to M with the topology of ρ_l. Then, since $\hat{l}(\gamma_i) < +\infty$ for all i, the curves γ_i admit a *constant velocity parameterization* (see [BBI], ch.2), i.e. parameterizations by arc-length. Reparameterizing the γ_i by arc-length we then have $\rho_l(\gamma_i(\tau), \gamma_i(\tau')) \leq \hat{l}(\gamma_i|_{[\tau,\tau']}) \leq c|\tau' - \tau|$, for all τ, $\tau' \in I$, and all i. Also, letting c_R^{-1} be the maximum of $[2(E-U)]^{-\frac{1}{2}}$ in $\overline{B}_0(R)$, we see that $\rho_{\mathcal{E}}(x, x') \leq c_R \rho_l(x, x')$, for all x, x' in the image of the γ_i. Therefore, the minimizing sequence $\{\gamma_i\}$, with the γ_i parameterized by arc-length, satisfies the assumptions of the following version of the Arzela-Ascoli theorem:

Lemma 3.3 Consider a sequence $\{\gamma_i\}$ of parameterized curves $\gamma_i \in C^0(I, M)$ with endpoints a, $b \in M$. Also, suppose that there exist constants R, C such that for all i

$$|\gamma_i(\tau)| \leq R, \quad \forall \tau \in I, \quad \text{and} \quad |\gamma_i(\tau) - \gamma_i(\tau')| \leq C|\tau - \tau'|, \quad \forall \tau, \tau' \in I. \tag{3.6}$$

Then the sequence $\{\gamma_i\}$ has a subsequence that converges uniformly to a curve $\gamma \in C^(I, M)$ with endpoints a and b.

To show that the limit of the convergent subsequence of the minimizing sequence $\{\gamma_i\}$ attains the infimum of $\hat{l}$ we show below that $\hat{l}$ is lower-semicontinuous in C^0. The argument is well-known (see e.g. [BBI], ch.2), a small difference here is that we are considering continuous curves for M with the topology of the Euclidean metric $\rho_{\mathcal{E}}$.

Lemma 3.4 The induced Jacobi length $\hat{l}$ is lower semi-continuous on $C^0(I, M)$ with the C^0 (uniform convergence) topology.

Proof: Consider a sequence $\{\gamma_i\} \in C^0(I, M)$ converging to γ in C^0. Fix a partition $y = \{y_0 = \tau_1 \leq y_1 \leq \ldots \leq y_{N-1} = \tau_2\}$ of I for which $\hat{l}(\gamma) - \sum_{j=0}^{N-2} \rho_l(\gamma(y_j), \gamma(y_{j+1})) < \epsilon$. Each $\gamma_i(y_j)$ converges to $\gamma(y_j)$ in $\rho_{\mathcal{E}}$, and by the definition of ρ_l and the fact that we have a finite set of points we can choose i large enough so that $\rho_l(\gamma(y_j), \gamma_i(y_j)) < \epsilon$, $\forall j = 0, \ldots, N-1$. Then, the triangle inequality yields $\rho_l(\gamma(y_j), \gamma(y_{j+1})) \leq \rho_l(\gamma_i(y_j), \gamma_i(y_{j+1})) + 2\epsilon$, and we therefore have

$$\hat{l}(\gamma) < \sum_{j=0}^{N-2} \rho_l(\gamma(y_j), \gamma(y_{j+1})) + \epsilon \leq 2N\epsilon + \sum_{j=0}^{N-2} \rho_l(\gamma_i(y_j), \gamma_i(y_{j+1})) + \epsilon \leq \hat{l}(\gamma_i) + (2N+1)\epsilon,$$

for all $\epsilon > 0$.

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Proof of Theorem 3.1: By the lower-semicontinuity of $\hat{l}$, the limit γ of an appropriate subsequence of a minimizing sequence $\{\gamma_i\}$ has length $\hat{l}(\gamma) = \liminf_{i\to\infty} \hat{l}(\gamma_i)$.

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4. Collisionless Jacobi geodesics

We now show that the minimizers of the induced Jacobi arc-length $\hat{l}$ are smooth, and that they can not pass though the singularity set Σ.

As a preliminary step, we first show that minimizers of $\hat{l}$ are C^2 geodesics of the Jacobi metric as long as their image is outside the singularity set Σ. By Proposition 2.1 this implies that the minimizers are solutions of the Euler-Lagrange equations outside Σ. We also see that the Jacobi length l and induced Jacobi length $\hat{l}$ of the minimizers coincide.

To see that minimizers of $\hat{l}$ avoid Σ, we will argue that allowing a minimizer to pass through Σ leads to a contradiction. In particular, assuming that a minimizer γ of $\hat{l}$ passes through Σ we will produce paths with smaller length $\hat{l}$ that avoid the singularity set. The argument uses the fact that γ can be reparameterized to satisfy the Euler-Lagrange equations outside Σ. To study the approach of γ to the singularity set Σ, we use a variant of Sundman's result on the "branch regularization" of binary collisions (Lemma 4.2). We see that the approach to Σ is almost "radial", in the sense that the angular velocity decreases rapidly as we approach the singularity (see Lemmas 4.3, 4.4). This quite simple geometry of the minimizers near the singularities allows us to define one-parameter families of small deformations of γ normal to γ that avoid Σ. The deformations alter γ only near the singularity set, and using the information on the geometry of the approach to Σ, we estimate the first variation of the arc-length with respect to the deformation parameter in Lemma 4.5. We show that deforming off the singularities decreases the Jacobi arc-length. The deformations used in Lemma 4.5 generally produce discontinuous curves. In Lemma 4.6 we show that by choosing the deformations appropriately, the shorter curves off the singularities have self-intersections, and can therefore be "glued" to yield continuous curves connecting the given endpoints.

The see that Jacobi minimizers are regular off the singularity set Σ we will use the properties of short geodesics stated in Lemma 3.1.

Lemma 4.1 Let $\gamma \in C^0(I, M)$ with endpoints a, $b \in M \setminus \Sigma$ be a minimizer of the induced Jacobi arc-length $\hat{l}$ over all continuous parameterized curves with the same endpoints. Also, assume that $\hat{l}(\gamma) < +\infty$. Then, γ is a C^2 Jacobi geodesic for all $\tau \in I$ such that $\gamma(\tau) \in M \setminus \Sigma$. Moreover, $\hat{l}(\gamma) = l(\gamma)$.

Proof: Consider $\tilde{\tau} \in I$ such that $\tilde{x} = \gamma(\tilde{\tau}) \in M \setminus \Sigma$. Also, let $D_{\tilde{x}}(\delta) \subset M \setminus \Sigma$ be an open (Euclidean) disc of radius δ around $\tilde{x}$, and pick $\tau_1 < \tilde{\tau} < \tau_2$ such that $\gamma|_{[\tau,\tau']} \subset D_{\tilde{x}}(\delta)$. By Lemma 3.1, we can choose $\delta > 0$ sufficiently small so that the points $x_1 = \gamma(\tau_1)$ and $x_2 = \gamma(\tau_2)$ can be connected by a short geodesic of the Jacobi metric. Denoting the short geodesic by ϕ, we can be reparameterize ϕ so that $\phi(\tau_1) = x_1$, $\phi(\tau_2) = x_2$. We want to show that $\gamma|_{[\tau_1,\tau_2]}$ and ϕ coincide. By the definitions of ϕ and $\hat{l}$ we have that $\hat{l}(\phi) = l(\phi)$, and also $\hat{l}(\gamma|_{[\tau_1,\tau_2]}) \geq l(\phi) = \rho_l(x_1, x_2)$. On the other hand, $\hat{l}(\gamma|_{[\tau_1,\tau_2]}) \leq \hat{l}(\phi)$ since γ is a minimizer. Therefore $\hat{l}(\gamma|_{[\tau_1,\tau_2]}) = \hat{l}(\phi)$. Suppose now that there exists y in the image of $\gamma|_{[\tau_1,\tau_2]}$ that is not in the image of ϕ. Consider the curve $\tilde{\phi}$ made by concatenating two short geodesics connecting x_1 to y and y to x_2 respectively, and reparameterize $\tilde{\phi}$ so that $\tilde{\phi}(\tau_1) = x_1$, $\tilde{\phi}(\tau_2) = x_2$. By the definition of $\hat{l}$ we have $\hat{l}(\tilde{\phi}) = l(\tilde{\phi})$, and $\hat{l}(\gamma|_{[\tau_1,\tau_2]}) \geq l(\tilde{\phi})$. Moreover, since the parameterized curves ϕ and $\tilde{\phi}$ are C^2 they differ in some open subset of $[\tau_1, \tau_2]$. By the uniqueness of ϕ we then have $l(\tilde{\phi}) > l(\phi)$, hence $\hat{l}(\gamma|_{[\tau_1,\tau_2]}) > \hat{l}(\phi)$, a contradiction. Thus $\gamma|_{[\tau_1,\tau_2]}$ and ϕ are the same curve. The argument applies to any point $\gamma(\tau) \notin \Sigma$, and the first statement of the lemma follows.

To see that $\hat{l}(\gamma) = l(\gamma)$, we first observe that the set of times $\tau_j \in I$ for which

$\gamma(\tau_j) \in \Sigma$ is finite. Consider a union I'_ϵ of neighborhoods of size ϵ around each such τ_j. Using the first statement of the lemma, we have $\hat{l}(\gamma|_{I\setminus I'_\epsilon}) = l(\gamma|_{I\setminus I'_\epsilon})$. Then, $\hat{l}(\gamma) < +\infty$ implies that $l(\gamma) < +\infty$, and hence $l(\gamma|_{I'_\epsilon}) \to 0$ as $\epsilon \to 0$. Similarly, $\hat{l}(\gamma|_{I'_\epsilon}) \to 0$ as $\epsilon \to 0$, by the continuity of $\hat{l}$ with respect to endpoints. Letting $\epsilon \to 0$ we can thus make $|\hat{l}(\gamma) - l(\gamma)| = |\hat{l}(\gamma|_{I'_\epsilon}) - l(\gamma|_{I'_\epsilon})|$ arbitrarily small.

■

Therefore, a C^0 parameterized curve $\gamma : I \to M$ joining a, $b \in M \setminus \Sigma$ that minimizes $\hat{l}$ over all continuous curves with the same endpoints is a C^2 geodesic of the Jacobi metric for all $\tau \in I$ for which $\gamma(\tau) \notin \Sigma$, and can be reparameterized to solve the Euler-Lagrange equations for (2.1).

We are assuming that there are $\tau \in I$ for which $\gamma(\tau) \in \Sigma$. The set T_* of all such τ is discrete: since γ is a minimizer and the length between points of Σ is bounded away from zero, any accumulation point in T_* would correspond to loops around a point of Σ. Thus each $\tau \in T_*$ has an open neighborhood where the curve γ does not pass through Σ, and we want to study some of the geometric properties of the minimizer γ as it approaches a singularity time $\tau \in T_*$. The goal is eventually to show that there exists a curve $\tilde{\gamma}$ which does not pass through the singularity set Σ and satisfies $\hat{l}(\tilde{\gamma}) < \hat{l}(\gamma)$. To produce the curve $\tilde{\gamma}$ we study γ in neighborhoods of the points $\tau \in T_*$. This is a local analysis of $\hat{l}$ near such points, and it is sufficient to describe it for any given $\tau \in T_*$. We will also assume without loss of generality that one of the singularities of the potential U is located $x = 0$, and that there exists $\tau_* \in T_*$ for which $\gamma(\tau_*) = 0$. Otherwise, given a, $b \in M \setminus \Sigma$, and a minimizer γ with $\gamma(\tau_*) = \sigma_i$ for some $\sigma_i \in \Sigma$, $\tau_* \in I$, we can translate our system of coordinates so that σ_i goes to the origin, produce the shorter curve $\tilde{\gamma}$ that avoids the origin (and other singularities), and translate back to the original coordinate system. The procedure can be performed for any singularity encountered by γ. The changes of coordinates are justified by the fact that the Jacobi length is invariant under translations.

We can also consider the minimizer γ parameterized by t so that $\gamma(t)$ satisfies the Euler-Lagrange equations for all t with $\gamma(t) \in M \setminus \Sigma$. Assuming that $\tau \in I$ above defines a parameterization of γ satisfying (2.6), and setting $t = 0$ when $\tau = 0$, we easily see that $l(\gamma) < +\infty$ and (2.5), (2.6) imply that $t_* < +\infty$. By translating the variable t appropriately, we may simplify the notation by also assuming that $t_* = 0$.

Lemma 4.2 Suppose that $\gamma(t) \to 0 \in \Sigma$ as $t \to 0$, where $\gamma(t)$ satisfies the Euler-

Lagrange equations for (2.1) for $t \neq 0$. Then, for $|t|$ sufficiently small, $\gamma(t)$ can be expressed as

$$\gamma(t) = t^{\frac{2}{3}}\phi(\tilde{\sigma}), \quad \tilde{\sigma} = t^{\frac{1}{3}}, \tag{4.1}$$

where $\phi(\tilde{\sigma})$ is analytic in a sufficiently small neighborhood of $\tilde{\sigma} = 0$, and satisfies $|\phi(0)| \neq 0$.

Note that the function ϕ depends on the initial condition, i.e. $\gamma(t_0)$ and $\gamma'(t_0)$ for some $t_0 \neq 0$. The lemma is a slight modification of Sundman's branch regularization of binary collisions in the Newtonian 3-body problem, and we shall only sketch the proof, referring to [SM, I.6-I.8] for more details.

Proof of Lemma 4.2: The equations of motion for (2.1) can be written as Hamilton's equations for the position and momentum $q(t)$, $p(t)$ respectively. Denote the Hamiltonian by H, and define the new "time" σ by $\sigma(t) = \int_{t_0}^{t} |\gamma(\tilde{t})|^{-1} d\tilde{t}$. We can see that $q(\sigma)$, $p(\sigma)$ satisfy the Hamiltonian system

$$\frac{dq}{d\sigma} = \frac{dF}{dp}, \quad \frac{dp}{d\sigma} = -\frac{dF}{dq}, \quad \text{with} \quad F = |q|(H - E), \quad E \in \mathbf{R}, \tag{4.2}$$

if and only if $q(t)$, $p(t)$ are trajectories of Hamilton's equations for (2.1) with total energy E. Defining the new variables ξ, η by the symplectic transformation

$$\xi = q|p|^2 - 2p(q \cdot p), \quad \eta = p|p|^{-2}, \tag{4.3}$$

and writing $H = \frac{1}{2}|p|^2 - |q|^{-1} - f(q)$, with f smooth near $|q| = 0$, the Hamiltonian F becomes

$$F(\xi, \eta) = \frac{1}{2}|\xi| - 1 + |\xi||\eta|^2(f(\xi, \eta) - E)). \tag{4.4}$$

By conservation of energy, as σ, $|q(\sigma)| \to 0$, the norm of the momentum $|p|$ diverges. Therefore, by (4.2), as we approach the origin, we have $\eta \to [0, 0]$, and $|\xi| \to 1$. By the local existence theory for $\xi(\sigma)$, $\eta(\sigma)$, and the fact that $\xi(\sigma)$, $\eta(\sigma)$ remain bounded in a set where F is analytic as $\sigma \to 0$, $\xi(\sigma)$, $\eta(\sigma)$ are analytic for $|\sigma|$ small enough (i.e. at $\sigma = 0$ as well). Moreover, as $\sigma \to 0$, ξ has a limit ξ_0, with $|\xi_0| = 1$. Matching powers of σ in (4.4), and using (4.3), we also have $\eta(\sigma) = -\frac{1}{2}\xi_0\sigma + O(\sigma^2)$, $q(\sigma) = -\frac{1}{2}\xi_0\sigma^2 + O(\sigma^3)$, and therefore $|\phi(0)| \neq 0$. Also, by the definition of σ we have $t = \frac{1}{6}\sigma^3 + \ldots$, and inverting near $\sigma = 0$, we have that ϕ is analytic in $\tilde{\sigma} = t^{\frac{1}{3}}$.

■

Corollary 4.1 Suppose that $\gamma(t) \to 0 \in \Sigma$ as $t \to 0$, where $\gamma(t)$ satisfies the Euler-Lagrange equations for (2.1) for $t \neq 0$. Then, there exist $\tilde{t}_1 < 0 < \tilde{t}_2$ for which $r(t) = |\gamma(t)|$ is decreasing in $[\tilde{t}_1, 0]$ and increasing in $[0, \tilde{t}_2]$.

Proof: Let $\phi_0 = \phi(0)$ and $r(t) = |\gamma(t)|$. ¿From (4.1) we immediately have $\frac{d}{dt}(r^2) = t^{\frac{1}{3}}(\frac{4}{3}|\phi_0|^2 + O(t^{\frac{1}{3}}))$ for $|t|$ small, i.e. $\frac{d}{dt}(r^2)$ negative and positive respectively as t increases changing sign.

■

The fact that the radius $r = |\gamma|$ decreases and then increases monotonically as we pass the singularity allows us to reparameterize a segment of the curve near the origin by a multiple of r. Consider the times $\tilde{t}_1$, $\tilde{t}_2$ in Corollary 4.1, and pick $\alpha > 0$ satisfying $|\gamma(\tilde{t}_1)| > \alpha$ and $|\gamma(\tilde{t}_2)| > \alpha$. Then define $t_1 < t_2$ by $|\gamma(t_1)| = |\gamma(t_2)| = \alpha$, and parameterize the segment of $\gamma|_{[t_1,t_2]}$ by the scaled radius $s(t)$, defined by $s(t) = -\alpha^{-1}r(t)$, for $t \in [t_1, 0]$, and $s(t) = \alpha^{-1}r(t)$, for $t \in [0, t_2]$. The minimizer γ is now parameterized by $s \in [-1, 1]$, with $\gamma(0) = 0$, and $\gamma(s) \in M \setminus \Sigma$ if $s \neq 0$. Clearly, γ is smooth in $[-1, 0) \cup (0, 1]$. Also, the reparameterization does not change the length l of the segment.

Lemma 4.3 Let $\gamma(t) \to 0$ as $t \to 0$, where $\gamma(t)$ satisfies the Euler-Lagrange equations of (2.1) if $t \neq 0$. Also let $r(t)$, $\theta(t)$ denote the polar coordinates of $\gamma(t)$. Then, **(i)** $\mathcal{L} = \gamma \times \frac{d\gamma}{dt} \to 0$, $\frac{d}{dt}\mathcal{L} \to 0$, and **(ii)** $r^{\frac{1}{2}} \frac{d\theta}{dt} \to 0$.

Proof: Let prime denote the derivative with respect to t. For x near the origin we have $E - U(x) = \frac{1}{r} + f(x)$ with $f \geq 0$ and smooth. By the conservation of energy,

$$|\gamma'| = \left(\frac{2}{r} + 2f\right)^{\frac{1}{2}}, \tag{4.5}$$

and using Corollary 4.1, $|\mathcal{L}| \leq |\gamma||\gamma'| \leq Cr^{\frac{1}{2}}$, with C some constant as $t \to 0$. Thus $|\mathcal{L}|$ vanishes as we approach the singularity. Also, $\mathcal{L}' = \gamma \times \gamma'' = \gamma \times \nabla f$ by the Euler-Lagrange equations. By Corollary 4.1 we then have $|\mathcal{L}'| \leq C'|\gamma|$, with C' some constant as $t \to 0$. Hence $\mathcal{L}'$ also vanishes as we approach the origin. To see (ii) we use polar coordinates, where

$$\frac{d\theta}{dt} = \frac{\gamma \times \gamma'}{|\gamma|^2}. \tag{4.6}$$

Expanding $\gamma(t)$ as

$$\gamma(t) = \phi_0 t^{\frac{2}{3}} + \phi_1 t + \phi_2 t^{\frac{4}{3}} + \dots$$

(for appropriate ϕ_i), we calculate

$$\mathcal{L} = \gamma \times \gamma' = \tfrac{1}{3}(\phi_0 \times \phi_1)t^{\frac{2}{3}} + \tfrac{2}{3}(\phi_0 \times \phi_2)t + O(t^{\frac{4}{3}}),$$

$$\mathcal{L}' = \tfrac{2}{9}(\phi_0 \times \phi_1)t^{-\frac{1}{3}} + \tfrac{2}{3}(\phi_0 \times \phi_2) + O(t^{\frac{1}{3}}),$$

where using Lemma 4.2 all series are absolutely convergent for $|t|$ small. On the other hand, since $\mathcal{L}' \to 0$ as $t \to 0$ we must have $\phi_0 \times \phi_1 = \phi_0 \times \phi_2 = 0$. Therefore, $\mathcal{L}$ is $O(t^{\frac{4}{3}})$ as $t \to 0$. ¿From $|\phi_0| \neq 0$, we also have that $|\gamma|^2$ is $O(t^{\frac{4}{3}})$ as $t \to 0$, so that by (4.6) the angular velocity θ' is bounded as $t \to 0$.

■

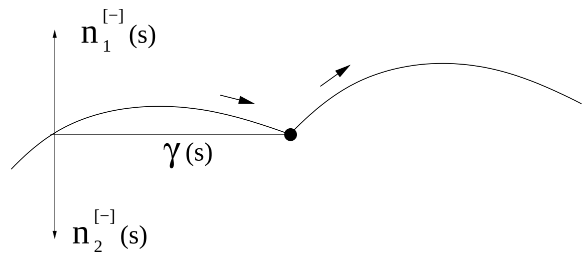

Figure 1

To construct families of curves near $\gamma|_{[-1,1]}$ that avoid the singularity, we now deform $\gamma|_{[-1,0]}$ and $\gamma|_{[0,1]}$ in the direction of unit vectors that are normal to the position vector γ. Specifically, by the smoothness of γ for $s \neq 0$ there exist (exactly) two C^1 one-parameter families of unit vectors $n_1^{[-]}(s)$, $n_2^{[-]}(s)$, $s \in [-1, 0)$ that are normal to $\gamma(s)$ for all $s \in [-1, 0)$ (see e.g. Fig. 1). Similarly, there exist (exactly) two C^1 one-parameter families $n_1^{[+]}(s)$, $n_2^{[+]}(s)$, $s \in (0, 1]$, of unit vectors normal to $\gamma(s)$ for all $s \in (0, 1]$. Notice that $n_1^{[\pm]} = -n_2^{[\pm]}$, i.e. the indices i are arbitrary at this point.

Lemma 4.4 Let $\gamma(s)$ be as above. Then, as $s \to 0$, we have **(i)** $|\frac{d\gamma}{ds}| \to \alpha$, **(ii)** $\frac{d}{ds}n[\pm]_i \to 0$, and **(iii)** $\frac{d\gamma}{ds} \cdot n_i^{[\pm]} \to 0$, $i = 1, 2$.

Proof: Let prime and dot denote the derivative with respect to t and s respectively. Using polar coordinates, $|\gamma'|^2 = (r')^2 + r^2(\theta')^2$, and by Lemma 4.3 (ii), the second term vanishes as $s \to 0$. Then, $|\dot{\gamma}(s)| = |\gamma'||s'|^{-1} = \alpha|\gamma'||r'|^{-1} \to \alpha$ as $s \to 0$. To see (ii), we start with $|\dot{n}_i^{[\pm]}| = |(n_i^{[\pm]})'||s'|^{-1}$, $i = 1, 2$. Conservation of energy implies $|s'|^{-1} \leq Cr^{-\frac{1}{2}}$ for some constant C. Also note that $|(n_i^{[\pm]})'| \leq |\theta'|$, $i = 1, 2$. Combining with Lemma 4.3

(ii), we therefore have $|\dot{n}_i^{[\pm]}| \leq Cr^{\frac{1}{2}}|\theta'| \to 0$, $i = 1, 2$. To see (iii), we use $\gamma \cdot n_i^{[\pm]} = 0$, $\forall s$. Then $|\dot{\gamma} \cdot n_i^{[\pm]}| \leq |\gamma||\dot{n}_i^{[\pm]}|$, and the statement follows by (ii).

■

The families $n_i^{[-]}(s)$, $n_i^{[+]}(s)$, $i = 1, 2$ are therefore C^1 in $[-1, 0]$, $[0, 1]$ respectively. By Lemma 4.4 (iii), we can define two one-parameter families of closed half-discs $H_\beta^{[-]}$, $H_\beta^{[+]}$, $\beta > 0$, with radius β and center at the origin that in addition satisfy: (i) the diameters of $H_\beta^{[-]}$, $H_\beta^{[+]}$ are along $n_1^{[-]}(0)$, $n_1^{[+]}(0)$ respectively, (ii) $H_{\beta_1}^{[\pm]} \subset H_{\beta_2}^{[\pm]}$ for $0 < \beta_1 < \beta_2 \leq \alpha$, and (iii) $H_\beta^{[-]}$, $H_\beta^{[+]}$ intersect $\gamma|_{[-\beta,0]}$, $\gamma|_{[0,\beta]}$ respectively for all $\beta > 0$.

Letting $H^{[\pm]} = H_1^{[\pm]}$, notice that if $n_1^{[-]}(0)$, $n_1^{[+]}(0)$ are collinear, then $H^{[-]} \cap H^{[+]}$ is either $H^{[-]} = H^{[+]}$ or a line segment along $n_1^{[-]}(0)$. If $n_1^{[-]}(0)$, $n_1^{[+]}(0)$ are not collinear, the boundary of $H^{[-]} \cap H_1^{[+]}$ contains one of the $n_i^{[-]}(0)$ and one of the $n_j^{[+]}(0)$ (see e.g. Fig. 2).

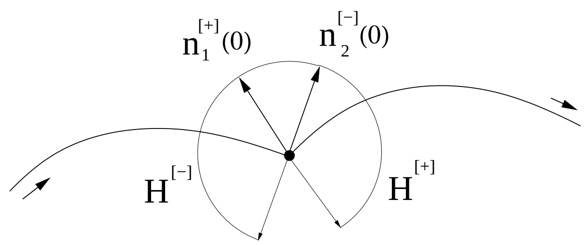

Figure 2

With this notation we define the one parameter families of unit vectors $n^{[-]} : [-1, 0] \to S^1$ and $n^{[+]} : [-1, 0] \to S^1$ as follows: in the case where $n_1^{[-]}(0)$ and $n_1^{[+]}(0)$ are collinear we let

$$n^{[-]} = n_1^{[-]}, \quad n^{[+]} = n_i^{[+]}, \quad \text{with } n_i^{[+]} \text{ such that } \quad n_i^{[+]}(0) = n_1^{[-]}(0), \tag{4.7}$$

i.e. $n^{[-]}$ is chosen arbitrarily, and then determines $n^{[+]}$ uniquely. In the case where $n_1^{[-]}(0)$ and $n_1^{[+]}(0)$ are not collinear we let

$$n^{[-]} = \begin{cases} n_1^{[-]}, & \text{if } n_1^{[-]}(0) \in H^{[-]} \cap H^{[+]}; \\ n_2^{[-]}, & \text{otherwise,} \end{cases} \tag{4.8}$$

$$n^{[+]} = \begin{cases} n_1^{[+]}, & \text{if } n_1^{[+]}(0) \in H^{[-]} \cap H^{[+]}; \\ n_2^{[+]}, & \text{otherwise.} \end{cases} \tag{4.9}$$

For example, in Fig. 2 we would set $n^{[-]} = n_2^{[-]}$ and $n^{[+]} = n_1^{[+]}$.

Then, for $\epsilon \geq 0$ we define $\gamma_\epsilon^{[-]} : [-1, 0] \to M$, $\gamma_\epsilon^{[+]} : [0, 1] \to M$ by

$$\gamma_\epsilon^{[-]}(s) = \gamma(s) + \epsilon\alpha(1 + s)n^{[-]}(s), \quad s \in [-1, 0], \tag{4.10}$$

$$\gamma_\epsilon^{[+]}(s) = \gamma(s) + \epsilon\alpha(1 - s)n^{[+]}(s), \quad s \in [0, -1] \tag{4.11}$$

respectively. Note that $s \neq 0$ implies that $|\gamma_\epsilon^{[-]}(s)| = [\alpha^2 s^2 + \epsilon^2\alpha^2(1 + s)^2]^{\frac{1}{2}} \neq 0$, and $|\gamma_\epsilon^{[+]}(s)| \neq 0$. Also, we can chose ϵ_0, α_0 so that if $\epsilon \leq \epsilon_0$, $\alpha \leq \alpha_0$, the curves $\gamma_\epsilon^{[-]}$, $\gamma_\epsilon^{[+]}$ do not encounter any singularities other than the origin.

The curves $\gamma_\epsilon^{[-]}$, $\gamma_\epsilon^{[+]}$ with $\epsilon > 0$ (dotted lines in Fig. 3), are the desired "deflections" of the segments $\gamma|_{[-1,0]} = \gamma_0^{[-]}$, $\gamma|_{[0,1]} = \gamma_0^{[+]}$ respectively (solid lines in Fig. 3). In Lemma 4.5 below we show that if ϵ is positive and sufficiently small, then $l(\gamma_\epsilon^{[-]}) < l(\gamma_0^{[-]})$ and $l(\gamma_\epsilon^{[+]}) < l(\gamma_0^{[+]})$. The definition of the vectors $n^{[-]}$, $n^{[+]}$ from the respective $n_i^{[-]}$, $n_i^{[+]}$, $i = 1, 2$ above is meant to simplify the last part of the argument, where we must also show that the shorter deflected segments can be chosen to intersect.

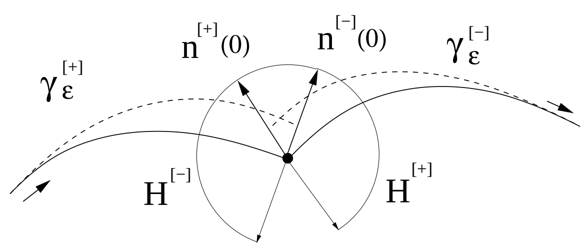

Figure 3

Lemma 4.5 There exists $\tilde{\epsilon} > 0$ for which $l(\gamma_\epsilon^{[-]})$ and $l(\gamma_\epsilon^{[+]})$ decrease as ϵ increases in $[0, \tilde{\epsilon}]$.

Proof: We will show the statement for the family $\gamma_\epsilon^{[-]}$, since the proof for the $\gamma_\epsilon^{[+]}$ is similar. To simplify the notation, let $\gamma_\epsilon = \gamma_\epsilon^{[-]}$, $\gamma = \gamma_0^{[-]} = \gamma|_{[-1,0]}$, and $n = n^{[-]}$. With this notation and (4.10), we then have the family of curves

$$\gamma_\epsilon(s) = \gamma(s) + \epsilon\alpha(1 + s)n(s), \quad s \in [-1, 0], \quad \epsilon \geq 0, \tag{4.12}$$

with $\alpha > 0$ fixed.

Let $l(\epsilon) = l(\gamma_\epsilon)$. For $\epsilon > 0$, the curves γ_ϵ avoid the singularity, and the integrand of $l(\epsilon)$ is smooth. Thus $l(\epsilon)$ is differentiable in ϵ for $\epsilon > 0$, and we can take the derivative inside the integral. We first estimate $l' = \frac{dl}{d\epsilon}$ and show that it is negative as $\epsilon \to 0^+$. ¿From (4.10) we have

$$l(\epsilon) = \int_{-1}^{0} [\alpha^2 s^2 + \epsilon^2\alpha^2(1+s)^2]^{-\frac{1}{4}}[1 + |\gamma_\epsilon| f(\gamma_\epsilon)]^{\frac{1}{2}}|\dot\gamma_\epsilon| ds, \quad \epsilon \geq 0. \tag{4.13}$$

Letting $u = s^2 + \epsilon^2(1+s)^2$, we have $l'(\epsilon) = I_1 + I_2$, where

$$I_1 = -\tfrac{1}{2}a^{-\frac{1}{2}}\epsilon \int_{-1}^{0} u^{-\frac{5}{4}}(1+s)^2[1 + \alpha u^{\frac{1}{2}} f(\gamma_\epsilon)]^{\frac{1}{2}}|\dot\gamma_\epsilon| ds, \tag{4.14}$$

$$I_2 = \int_{-1}^{0} u^{-\frac{1}{4}} \frac{d}{d\epsilon}\left\{[1 + \alpha u^{\frac{1}{2}} f(\gamma_\epsilon)]^{\frac{1}{2}}|\dot\gamma_\epsilon|\right\} ds, \tag{4.15}$$

with $\epsilon > 0$. We will show that as $\epsilon \to 0^+$ the first integral I_1 is negative and of $O(\epsilon^{-\frac{1}{2}})$, while the second integral I_2 is bounded uniformly in ϵ. Letting $\beta = \alpha u^{\frac{1}{2}} f$, and $w^{\frac{1}{2}} = \alpha^{-1}|\dot\gamma_\epsilon|$, the integral I_2 is further decomposed as $I_2 = I_{2A} + I_{2B} + I_{2C}$, with

$$I_{2A} = \alpha^{\frac{3}{2}}\epsilon \int_{-1}^{0} u^{-\frac{3}{4}}(1+s)^2[1+\beta]^{-\frac{1}{2}} f w^{\frac{1}{2}} ds, \tag{4.16}$$

$$I_{2B} = \tfrac{1}{2}\alpha^{\frac{3}{2}} \int_{-1}^{0} u^{\frac{1}{2}}[1+\beta]^{-\frac{1}{2}} f'_\epsilon w^{\frac{1}{2}} ds, \quad I_{2C} = \alpha^{\frac{1}{2}}\epsilon \int_{-1}^{0} u^{-\frac{1}{4}}[1+\beta]^{\frac{1}{2}} w^{-\frac{1}{2}} \frac{dw}{d\epsilon} ds,$$

where $f = f(\gamma_\epsilon(s))$ and $f'_\epsilon = \frac{d}{d\epsilon} f(\gamma_\epsilon(s))$.

To estimate the integrals we assume that $\epsilon \in (0, \epsilon_0]$, $\alpha \in [0, \alpha_0]$ with ϵ_0 and α_0 such that the only singularity of the potential U encountered by the segments γ_ϵ is the one at the origin. It follows from this assumption that there exist M, $M' > 0$ satisfying

$$|f| \leq M, \quad |f'_\epsilon| \leq M' \tag{4.17}$$

in (4.14)-(4.16) above. Also, by Lemma 4.4 there exist N, $m > 0$ satisfying

$$|\dot n| \leq N, \quad |s\dot\theta| < m. \tag{4.18}$$

We further restrict α_0 and ϵ_0 to satisfy

$$\text{(i)} \quad \alpha_0(\tfrac{1}{4} + \epsilon_0^2)M \leq \tfrac{1}{2}, \quad \text{(ii)} \quad \epsilon_0 N \leq \tfrac{1}{4}, \quad \text{(iii)} \quad \epsilon_0^2 N \leq \tfrac{1}{8}, \tag{4.19}$$

Fixing $\alpha \in (0, \alpha_0]$, we now see that the essential term in each of the four integrals is the one involving u (to some power). All other terms can be bounded uniformly in $s \in [-1, 0]$ and $\epsilon \in [0, \epsilon_0]$. In particular, using (4.10) and $\gamma(s) \cdot n(s) = 0$, $\forall s \in [-1, 0)$, we calculate

$$w^{\frac{1}{2}} = [1 + s^2\dot{\theta}^2 + \epsilon^2 + \epsilon^2(1+s)^2|\dot{n}|^2 + 2a^{-1}\epsilon s(\gamma \cdot \dot{n}) + 2\epsilon^2(1+s)^2(n \cdot \dot{n})]^{\frac{1}{2}}. \tag{4.20}$$

Then, from $s \in [-1, 0]$, $|\gamma| = |s|\alpha$, and conditions (4.19) (ii), (iii) we have

$$|2a^{-1}\epsilon s(\gamma \cdot \dot{n})| \leq \tfrac{1}{4}, \quad |2\epsilon^2(1+s)^2(n \cdot \dot{n})| \leq \tfrac{1}{4}$$

respectively, hence

$$w^{\frac{1}{2}} \geq 2^{-\frac{1}{2}}. \tag{4.21}$$

Similarly, $s \in [-1, 0]$, the definition of m, and conditions (4.19) (ii), (iii) imply

$$w^{\frac{1}{2}} \leq (2 + m^2 + \epsilon_0^2)^{\frac{1}{2}}. \tag{4.22}$$

In addition, the definition of β and condition (4.19) (i) imply

$$\beta \geq \tfrac{1}{2}. \tag{4.23}$$

Then, from (4.14), and (4.21), (4.23) we have

$$|I_1| \geq \tfrac{\alpha^{\frac{1}{2}}}{2}\epsilon \int_{-\frac{1}{2}}^{0} u^{-\frac{5}{4}}(1+s)^2[1 + \alpha u^{\frac{1}{2}} f(\gamma_\epsilon)]^{\frac{1}{2}} |\dot{\gamma}_\epsilon| ds \geq \tfrac{\alpha^{\frac{1}{2}}}{24} \int_{-\frac{1}{2}}^{0} \epsilon[s^2 + \epsilon^2]^{-\frac{5}{4}} ds,$$

so that setting $v = \tan^{-1} \frac{s}{\epsilon}$ in the last integral we obtain

$$|I_1| \geq \epsilon^{-\frac{1}{2}} \tfrac{\alpha^{\frac{1}{2}}}{24} \int_{-\tan^{-1}\frac{1}{2\epsilon}}^{0} \cos^{\frac{1}{2}} v dv \geq C_1 \epsilon^{-\frac{1}{2}}, \tag{4.24}$$

with $C_1 > 0$ a constant uniform in ϵ as $\epsilon \to 0^+$. For the integrals of (4.16) we similarly use (4.21)-(4.23), obtaining

$$\begin{aligned} I_{2A} \leq & M(2 + m^2 + \epsilon_0^2)^{\frac{1}{2}} \alpha^{\frac{3}{2}} \int_{-1}^{0} \epsilon u^{-\frac{3}{4}} ds \leq \\ \leq & 2M(2 + m^2 + \epsilon_0^2)^{\frac{1}{2}} \alpha^{\frac{3}{2}} \left[\epsilon^{\frac{1}{2}} \int_{-\tan^{-1}\frac{1}{\epsilon}}^{0} \cos^{-\frac{1}{2}} v dv + \epsilon \int_{-1}^{-\frac{1}{2}} u^{-\frac{3}{4}} ds \right], \end{aligned} \tag{4.25}$$

$$(4.26)\quad I_{2B} \leq \tfrac{M'}{2}(2+m^2+\epsilon_0^2)^{\frac{1}{2}}\alpha^{\frac{3}{2}} \int_{-1}^{0} u^{\frac{1}{2}} ds, \quad I_{2C} \leq 4(1+m+N+\epsilon_0)\alpha^{\frac{1}{2}} \int_{-1}^{0} u^{-\frac{1}{4}} ds.$$

We check from (4.25)-(4.26) that I_{2A}, I_{2B}, I_{2C} are bounded uniformly in $\epsilon \in [0, \epsilon_0]$. Collecting the bounds on the four integrals we therefore have that as $\epsilon \to 0^+$

$$l'(\epsilon) \leq -C_1 \epsilon^{-\frac{1}{2}} + C_2, \tag{4.27}$$

with C_1, C_2 positive constants.

Choosing ϵ_1 sufficiently small, $l(\epsilon)$ is therefore decreasing for $\epsilon \in (0, \epsilon_1]$. For the continuity of $l(\epsilon)$ at $\epsilon = 0$, we let $l(\epsilon) = \int_{-1}^{0} F_\epsilon(s) ds$, with F_ϵ the integrand of (4.12). Using (4.17)-(4.22), we see that

$$F_\epsilon(s) \leq a^{\frac{1}{2}} C |s|^{-\frac{1}{2}}, \quad \forall s \in [-1, 0], \quad \forall \epsilon \in [0, \epsilon_1],$$

with some $C > 0$ independent of ϵ. Thus the functions F_ϵ are uniformly bounded by an integrable function, and by the dominated convergence theorem we have

$$l(\epsilon) \to \int_{-1}^{0} F_0(s) ds = l(0) \quad \text{as} \quad \epsilon \to 0^+.$$

In a similar way, we can find $\epsilon_2 >$ for which $l(\gamma_\epsilon^{[+]})$ decreases for $\epsilon \in [0, \epsilon_2]$. Then the statement follows with $\tilde{\epsilon} = \min\{\epsilon_1, \epsilon_2\}$.

■

Remark 4.1 In the test case where Σ consists of only one singularity located at the origin, conservation of the angular momentum $\gamma \times \gamma'$ implies that all collision and ejection orbits are radial. The image of the curve γ is then the union of two linear segments, from a to the origin and then to b respectively, while the vectors $n^{[\pm]}(s)$ above are constant and normal to γ for all s. The integrals in (4.14)-(4.16) simplify and we again obtain (4.27). The choice of the $n^{[\pm]}$ ensures that the deflected curves intersect.

Remark 4.2 The above test case also suggests a simpler way of deforming γ off Σ that seems however to require a similar analysis. Specifically, consider two intermediate points a', b' with $|a'| = |b'| = \epsilon$ on each of the two radial segments of γ, and let θ be the angle between the segments. The Jacobi lengths of the circular arc connecting a' to b' and of the two line segments from a' to the origin and then to b' are $\theta\epsilon^{\frac{1}{2}}$ and $4\epsilon^{\frac{1}{2}}$ respectively (up to a common constant), i.e. too close for θ near π. In the n-center case we would therefore need a rather precise control of the geometry of approach to the singularities.

Remark 4.3 Replacing any of the two families $n^{[-]}$, $n^{[+]}$ by $-n^{[-]}$, $-n^{[+]}$ respectively does not affect the proof of Lemma 4.5. Such a change is also equivalent to reversing the sign of ϵ in the segments $\gamma_\epsilon^{[-]}$, $\gamma_\epsilon^{[+]}$. We have therefore also shown that the lengths of $\gamma_\epsilon^{[-]}$ and $\gamma_\epsilon^{[+]}$ decrease as ϵ decreases in $[-\tilde{\epsilon}, 0]$.

By Lemma 4.5, Remark 4.3, and noting that $\gamma_\epsilon^{[-]}(0) = \gamma_0^{[-]}(0)$ and $\gamma_\epsilon^{[+]}(1) = \gamma_0^{[+]}(1)$, $\forall \epsilon \in \mathbf{R}$, we can complete the argument by finding $\tilde{\epsilon}_1$, $\tilde{\epsilon}_2 \in [-\tilde{\epsilon}, \tilde{\epsilon}]$ for which the images of the corresponding segments $\gamma_{\tilde{\epsilon}_1}^{[-]}$ and $\gamma_{\tilde{\epsilon}_2}^{[+]}$ intersect at a point other than the origin.

Lemma 4.6 Let C_β be the set of all convex linear combinations of the vectors $bn^{[-]}(0)$, $bn^{[+]}(0)$, with $b \in [0, \beta]$. We can find $\beta > 0$ sufficiently small so that for every $x \in C_\beta \setminus \{0\}$ there exist unique pairs $[\tilde{\epsilon}_1, s_1] \neq [0, 0]$, $[\tilde{\epsilon}_2, s_2] \neq [0, 0]$ with $s_1 \in [-1, 0]$, $s_2 \in [0, 1]$ satisfying

$$x = \gamma_{\tilde{\epsilon}_1}^{[-]}(s_1) = \gamma_{\tilde{\epsilon}_2}^{[+]}(s_2).$$

Moreover, as $|x| \to 0$ we have $\tilde{\epsilon}_1$, $\tilde{\epsilon}_2 \to 0$ and $s_1 \to 0^-$, $s_2 \to 0^+$.

Proof: In the case where $n^{[-]}(0) = n^{[+]}(0)$, the conical region C_β degenerates to the set of all points $bn^{[-]}(0)$ with $b \in [0, \beta]$. The definitions of $\gamma_\epsilon^{[-]}$ and $\gamma_\epsilon^{[+]}$ immediately yield $\gamma_\epsilon^{[-]}(0) = \gamma_\epsilon^{[+]}(0) = \epsilon n^{[-]}(0)$, $\forall \epsilon \in \mathbf{R}$, and the statement follows.

For the case where $n^{[-]}(0) \neq n^{[+]}(0)$, we will show the statement for $\gamma_\epsilon^{[-]}$. The proof for $\gamma_\epsilon^{[+]}$ is similar. In essence, we want to solve the equation $\gamma_\epsilon(s) = x$ for any given $x \in C_\beta \setminus \{0\}$, and we will use the fact that $\gamma_0(0) = 0$ and the implicit function theorem. To simplify the notation we let $\gamma_\epsilon = \gamma_\epsilon^{[-]}$ and $\gamma = \gamma_0$. Also, let $N(s) = n^{[-]}(s)$, $N = N(0)$, and $M = n^{[+]}(0)$. We know that γ_0 and N are continuously differentiable at $s = 0$, and that $s \to 0^-$ implies $\dot{\gamma}_\epsilon(s) \cdot N(s) \to 0$ and $|\dot{\gamma}| \to \alpha$. We can then apply an orthogonal transformation to our coordinate system so that $\dot{\gamma}(0) = \alpha \hat{e}_1 = \alpha[1, 0]$, $N = [0, 1]$, and $M = [M_1, M_2]$ with $M_1 < 0$, $M_2 > 0$ and $|M| = 1$. Note that the signs of M_1, M_2 follow from the definition of $n^{[-]}$ and $n^{[+]}$.

We first consider a simpler version of $\gamma_\epsilon(s) = x$. Let

$$\tilde{\gamma}_\epsilon(s) = \alpha s \hat{e}_1 + \epsilon(1 + s)N, \quad s \in [-1, 0], \quad \epsilon \geq 0.$$

Then, writing $x \in C_\beta$ as $x = b\lambda N + b(1 - \lambda)M$, with $b \in [0, \beta]$, $\lambda \in [0, 1]$, the equation

$\tilde{\gamma}_\epsilon(s) = x$ has the unique solution

$$s = \frac{b}{\alpha}(1-\lambda)M_1, \quad \epsilon = \frac{b}{\alpha(1+s)}M_2, \quad \forall x \in C_\beta.$$

By the signs of M_1, M_2 above we therefore have that for any $\beta < \alpha$ there exists a neighborhood Q_β of $[0,0]$ in the $s \le 0$, $\epsilon \ge 0$ quadrant whose image under $\tilde{\gamma}_\epsilon(s)$ includes C_β. Moreover, $[s,\epsilon] \to [0,0]$ as $\beta \to 0$.

To solve $\gamma_\epsilon(s) = x$ for $x \in C_\beta$ it is enough to show that if $[\tau,\delta] \in Q_\beta$, and β is sufficiently small, the equation

$$F(s,\epsilon;\tau,\delta) = \gamma_\epsilon(s) - \tilde{\gamma}_\delta(\tau) = 0 \tag{4.28}$$

has a solution $s(\tau,\delta) \in [-1,0]$, $\epsilon(\tau,\delta)$ that depends continuously on τ and δ. Since $F(0,0;0,0) = 0$, showing that the solution is also unique will imply that if $x \ne 0$ (and hence $[\tau,\delta] \ne [0,0]$), then $[s,\epsilon] \ne [0,0]$. To solve (4.28) we note that the derivative D_1F of F with respect to the first two arguments at $[0,0;0,0]$ is

$$D_1F(0,0;0,0) = \begin{pmatrix} \dot{\gamma}_1(0) & \dot{\gamma}_2(0) \\ N_1 & N_2 \end{pmatrix},$$

with $\gamma = [\gamma_1,\gamma_2]$. By $\dot{\gamma}\cdot N = 0$ and $|\dot{\gamma}(0)| = \alpha$, we see that $\det(D_1F(0,0;0,0)) = \dot{\gamma}(0)\times N = \alpha^2 \ne 0$. Also, F can be extended to a neighborhood of $[0,0;0,0]$: we can let ϵ, τ, $\delta \in \mathbf{R}$ above, and $\gamma_\epsilon(s) = as\hat{e}_1$ for $s > 0$. The extended function is C^1 near $[0,0;0,0]$, and by the implicit function theorem we have unique C^1 functions $s(\tau,\delta)$, $\epsilon(\tau,\delta)$, with τ, δ near the origin. By continuity and our conclusions on the simplified equation $\tilde{\gamma}_\delta(\tau) = x$, we have that s, $\epsilon \to 0$ as x approaches the origin. Also, by the definition of the extension of F, for small $\tau > 0$ we have the unique solution $s(\tau,\delta) = \tau > 0$. On the other hand, solving $\tilde{\gamma}_\delta(\tau) = x$, $x \in C_\beta$, we had $\tau \le 0$. Therefore, $\gamma_\epsilon(s) = x$, $x \in C_\beta$, can only be satisfied with $s \in [-1,0]$.

■

Proof of Theorem 2.1: Assuming that a minimizer γ of $\hat{l}$ passes through some $\sigma \in \Sigma$, we can apply Lemma 4.5 (and Remark 4.3) to obtain "deflected" segments $\gamma_{\epsilon_1}^{[-]}$, $\gamma_{\epsilon_2}^{[+]}$ that avoid σ and have smaller length l. By Lemma 4.6 we can choose ϵ_1 and ϵ_2 arbitrarily small so that the images of the two deflected segments intersect. Concatenating appropriate restrictions of $\gamma_{\epsilon_1}^{[-]}$ and $\gamma_{\epsilon_2}^{[+]}$ we then obtain a continuous curve $\tilde{\gamma}$ joining a and b with length $\hat{l}(\tilde{\gamma}) \leq l(\tilde{\gamma}) < \hat{l}(\gamma)$, a contradiction. The argument applies to any singularity $\sigma \in \Sigma$. Minimizers of $\hat{l}$ therefore avoid the singularity set Σ, and by Lemma 4.1 and Proposition 2.1, are trajectories of the Euler-Lagrange equation joining a and b.

■

Remark 4.4 Note that the above argument does not give us any information on the homotopy class of the deflected shorter curve $\tilde{\gamma}$.

5. Discussion

Possible extensions will be considered in future work. Some directions are periodic lattices, higher dimensions, other types of point singularities, and singularity sets of higher dimension. For instance, we can consider Jacobi geodesics in a planar periodic lattice of potentials with $\frac{1}{r}$ singularities and fast decay at infinity. Arranging the potential energy U to be finite off the singularities, negative, and such that the perturbation to the approach to a singularity by other sites is analytic, the arguments will be the same.

Regarding higher dimensions, it is clear that the arguments leading to the existence of minimal geodesics do not involve the dimension of M. The same is also true for the analysis of the approach to the singularities in Lemmas 4.1 and 4.2. However, the construction of deflections off the singularities is special to the two dimensional case, and must be generalized.

Also, the definition of the induced length $\hat{l}$ in Section 3 is meaningful for $\frac{1}{r^\alpha}$, $\alpha < 2$ singularities, while Lemma 3.2 applies to all potentials that decay as $\frac{1}{r^\alpha}$ with $\alpha \leq 2$. Thus the existence of minimizers of the Jacobi length (with $E \geq 0$) can be extended to potentials with $\frac{1}{r^\alpha}$, $\alpha < 2$ singularities. The arguments of Section 4 must be modified however.

Also of interest are minimal Jacobi geodesics in configuration spaces with singularity sets of higher dimension, e.g. the shape space of the planar 3-body problem with total angular momentum $\omega = 0$.

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