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Invariant tori in the discrete NLS with small amplitude diffraction management

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Abstract

We consider the question of persistence of breather solutions of the discrete NLS equation under time-periodic perturbations corresponding to small amplitude diffraction management. The question is formulated as a problem of continuation of tori in an infinite-dimensional Hamiltonian system with symmetries and we show that one-peak breathers of the discrete NLS with zero residual diffraction can be continued to periodic or quasiperiodic solutions of the discrete NLS with small residual diffraction and small amplitude diffraction management, provided that a nonresonance condition is satisfied. We also present numerical evidence that a similar continuation should be possible for certain single-, and multi-peak breathers of the discrete NLS with small diffraction.

Keywords: Nonlinear Schrödinger equations; Discrete breathers; Hamiltonian systems

1. Introduction

In this paper we show the existence of periodic or quasiperiodic orbits in the discrete cubic NLS equation with diffraction management, a lattice NLS system with periodic parametric forcing proposed by Ablowitz and Musslimani [1] to describe an array of coupled waveguides with the zig–zag "diffraction management" geometry introduced and studied experimentally in [5].

Our results concern the special case of small amplitude parametric forcing. In particular, we consider the question of persistence of breather solutions of the discrete NLS equation under periodic perturbations that correspond to small amplitude diffraction management. The persistence question is formulated as a question of the continuation of invariant tori in an infinite-dimensional Hamiltonian system with symmetries. Using this general framework we show that one-peak breathers of the discrete NLS with vanishing residual diffraction can be continued to periodic or quasiperiodic solutions of the perturbed discrete NLS system of [1], provided that the ratio between the breather and forcing frequencies is a noninteger. We also see numerical evidence that a similar continuation should be possible for single peak, and a class of multi-peak breathers of the discrete NLS with small diffraction. A similar persistence question also arises in the continuous analogue of the system we study, the cubic NLS with small dispersion management on the line. There is theoretical and numerical evidence (see [12]) that the (localized) breather soliton of the cubic NLS decays in the perturbed system. One interpretation of the [12] results is that the breather solution cannot be continued to a solution of the perturbed system, although it is also possible that the breather can be continued to an unstable solution.

The Hamiltonian formulation of the persistence problem allows us to use the ideas of Nekhoroshev [9] and Bambusi and Gaeta [3] on the continuation of invariant tori in Hamiltonian systems with additional conserved quantities (see also [6]). The theory is based on an equivariant version of the Poincare map and in the present problem we use an infinite-dimensional generalization (see also [4] for another application). The idea is to write the full NLS system as an autonomous Hamiltonian system in an extended phase space and then decompose it to

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an "unperturbed part" with an invariant 2-torus corresponding to the breather, plus a small perturbation. In order to continue the 2-torus we must verify a nonresonance condition on the Floquet spectrum of an appropriate linear combination of the Hamiltonian vector fields of the unperturbed Hamiltonian and an additional conserved quantity (the orbits of this linear combination are periodic on the 2-torus). In the case of one-peak breathers of the anticontinuous limit system the Floquet spectrum is understood easily and the above strategy leads to a proof of existence of invariant tori in the perturbed system. In the multi-peak case we present some numerical computations of the Floquet spectrum that suggest that a similar continuation should also be possible for breather of NLS systems that are near the anticontinuous limit.

The paper includes a proof of an infinite-dimensional version of the theorem of Nekhoroshev [9] and Bambusi and Gaeta [3]. From the proof we see that the existence of invariant 2-tori of the periodically forced NLS implies the existence of invariant circles of the time-T map of the parametrically forced system (T is the period of the forcing). These invariant circles are near the orbits of the breather solutions.

The discrete NLS system of [1] can be also studied in the regime where the frequency of the parametric forcing is large. In that regime the system can be approximated by an averaged equation that is autonomous and has localized breather solutions (see [7,10,11]). Preliminary results suggest that certain one-peak breathers of the averaged equation can also be continued to solutions of the full system, but the argument is more involved and will be presented elsewhere. (Some asymptotic and numerical results on the continuous problem on the line are in [14].)

The paper is organized as follows. In Section 2 we describe the [1] model, set up the problem of continuation of breathers and state a general theorem on the continuation of tori in equivariant Hamiltonian systems. We also use the theorem to continue single-peak anticontinuous limit breathers. In Section 3 we present some numerical results on the continuation of single-, and multi-peak breathers. In Section 4 we prove the theorem on the continuation of tori in equivariant Hamiltonian systems. We point out its geometrical interpretation in the particular problem, and the extra information it provides on the geometry of the invariant tori.

2. Continuation of breathers under small amplitude forcing

We consider the parametrically forced discrete cubic nonlinear Schrödinger equation

$$\partial_t u = i D(t) \Delta u - 2\mathrm{i}\gamma g(u), \quad \text{with} \tag{2.1}$$

$$(\Delta u)_j = u_{j+1} - 2u_j + u_{j-1}, \qquad g_j(u) = |u_j|^2 u_j \tag{2.2}$$

and u a complex valued function on the integers $\mathbf{Z}$. (f_j denotes the value of $f : \mathbf{Z} \to \mathbf{C}$ at the site j.) Also, γ is a real constant and D is a T-periodic real valued function (for some $T > 0$). We also write

$$D(t) = \overline{D} + \tilde{D}(t), \quad \text{where } \overline{D} = \frac{1}{T}\int_0^T D(\tau)\mathrm{d}\tau \tag{2.3}$$

as the average over the period (also referred to as residual diffraction).

Eqs. (2.1) and (2.2) are a nonautonomous Hamiltonian system in $X = l_2(\mathbf{Z}^d, \mathbf{C})$, the set of square-summable complex valued functions on $\mathbf{Z}$ with the norm $\|.\| = \|.\|_{l_2}$. The Hamiltonian structure is specified below. Physically, t in (2.1) is the distance along the waveguides, and u_j is the complex amplitude of (any) one of the components of the electric field at the waveguide j (see [1,5]). The initial condition $u(t_0)$ for (2.1) is the emitted light.

A general problem for (2.1) is the existence of localized periodic or quasiperiodic solutions. The solutions we will consider will be defined indirectly, specifically, we look for solutions of (2.1) that are close to known localized periodic solutions of simpler equations whose solutions approximate the solutions of (2.1) in certain ranges of the parameters.

The simplest construction of this type concerns the case where $D(t) = \epsilon d(\Omega t)$, with $|\epsilon|$ small, d a 2π-periodic function, and $\Omega = \frac{2\pi}{T}$, i.e. $D(t)$ is T-periodic in t. The solutions we seek will be continuations from solutions of (2.1) with $D \equiv 0$ (the "anticontinuous limit"). First, Eqs. (2.1) and (2.2) with the initial condition $u(t_0) = v \in X$ are equivalent to the autonomous system

$$\partial_t u = \mathrm{i}\epsilon d(\phi)\Delta u - 2\mathrm{i}\gamma g(u), \qquad \dot{\phi} = \Omega, \tag{2.4}$$

$\phi \in S^1$, with the initial condition $u(0) = v$, $\phi(0) = \phi_0$. We view the linear part of (2.1) as a small perturbation of the cubic term.

Eq. (2.4) can be also written as a Hamiltonian system by adding an extra variable $J \in \mathbf{R}$. The phase space will be $X \times S^1 \times \mathbf{R}$. The Hamiltonian H_ϵ is

$$H_\epsilon = -\Omega J + \sum_{j\in\mathbf{Z}} \left(\epsilon d(\phi)|u_{j+1} - u_j|^2 + \gamma |u_j|^4\right), \tag{2.5}$$

and we formally obtain (2.4) by the first two of Hamilton's equations

$$\partial_t u = -\mathrm{i}\frac{\partial H_\epsilon}{\partial u^*}, \qquad \dot{\phi} = -\frac{\partial H_\epsilon}{\partial J}, \qquad \dot{J} = \frac{\partial H_\epsilon}{\partial \phi}. \tag{2.6}$$

We consider the case $\epsilon = 0$. Then for any $n_0 \in \mathbf{Z}$, $A \in \mathbf{C} \setminus \{0\}$, and $\phi_0 \in S^1$ we have the "one-peak" breather solution

$$\begin{aligned} &u_{n_0}(t) = \mathrm{e}^{-\mathrm{i}\lambda t} A \quad \text{with } \lambda = 2\gamma|A|^2; \\ &u_n(t) = 0, \quad \forall t \in \mathbf{R} \quad \text{if } n \neq n_0; \end{aligned} \tag{2.7}$$

$$\phi(t) = \Omega t + \phi_0; \quad J(t) = 0, \quad \forall t \in \mathbf{R}. \tag{2.8}$$

The choice $J = 0$ is arbitrary here. Let $C(n_0, A, \phi_0)$ be the set of points of $X \times S^1 \times \mathbf{R}$ in the orbit defined by (2.7) and (2.8). Also let $\Lambda_0(n_0, A) = \cup_{\phi_0 \in S^1} C(n_0, A, \phi_0)$. The set $\Lambda_0(n_0, A)$ is an invariant 2-torus of the $\epsilon = 0$ system, e.g. foliated by periodic orbits if $\frac{\lambda}{\Omega}$ is rational. We now consider (2.6) with $|\epsilon|$ being small. We show that the invariant tori $\Lambda_0(n_0, A)$ can persist, provided that λ and Ω satisfy a nonresonance condition.

Proposition 2.1. *Consider $n_0 \in \mathbf{Z}$, $A \in \mathbf{C} \setminus \{0\}$, and the set $\Lambda_0(n_0, A)$ as above. Let r be an integer, $r \geq 2$, and suppose that* *the 2π-periodic function d above is C^r in* $\mathbf{R}$. *Assume that* $\frac{\lambda}{\Omega} \notin \mathbf{Z}$, *where* $\lambda = 2|\gamma||A|^2$ *(and* $\Omega \neq 0$*). Then there exist* ϵ_0, $\beta_0 > 0$ *such that for any* ϵ *with* $|\epsilon| < \epsilon_0$ *the corresponding system* (2.4) *has a* C^r 2*-parameter family of invariant* 2*-tori* $\Lambda_{\epsilon,\beta}$, *with* $\beta \in (-\beta_0, \beta_0)^2$, *and* $\Lambda_{0,0} = \Lambda_0(n_0, A)$. *The motion on each torus is periodic or quasiperiodic (with two quasiperiods). Also, for any* $\epsilon \in (-\epsilon_0, \epsilon_0)$ *there is a two-parameter family of* C^r *functions* $f_{\epsilon,\beta} : \Lambda_0(n_0, A) \to X \times S^1$, $\beta \in (-\beta_0, \beta_0)^2$ *with* $\Lambda_{\epsilon,\beta} = f_{\epsilon,\beta}(\Lambda_0(n_0, A))$, *and* $f_{0,0}(\Lambda_0(n_0, A)) = \Lambda_0(n_0, A)$.

The proof of the proposition uses Theorem 2.2 which generalizes results of [9] and [3] on the continuation of invariant tori in Hamiltonian systems with symmetry.

Note that periodic and quasiperiodic solutions of (2.6) in $X \times S^1 \times \mathbf{R}$ project to periodic and quasiperiodic solutions of (2.4) in $X \times S^1$ (with the same periods and quasiperiods respectively). This follows from the fact that the respective right-hand sides of the first two equations of (2.6), i.e. Eqs. (2.4), do not depend on J.

Remark 2.1.1. The conclusions of Proposition 2.1 are also valid for less regular 2π-periodic functions d. The proof uses the C^r regularity of the flow defined by (2.6), but since system (2.6) comes from a nonautonomous system, the C^r regularity of the flow can be also shown for d that is less regular, e.g. it would be sufficient to require that d be piecewise C^r.

To set up Theorem 2.2, let r be an integer, $r \geq 2$, and consider a real C^r Hilbert manifold M modeled on a real (separable) Hilbert space E with inner product $\langle , \rangle$. Assume that M also has a weak symplectic structure $\boldsymbol{\omega}$ with corresponding Poisson bracket $\{\ ,\}$. Consider s real functions $H_1^\epsilon, \ldots, H_s^\epsilon$ on M that have the form $H_j^\epsilon = H_j^0 + \epsilon \tilde{H}_j$, $j = 1, \ldots, s$. (The parameter ϵ is real and the H_j^0, $\tilde{H}_j$ are independent of ϵ.) The Hamiltonian vector fields of H_j^ϵ, H_j^0, $\tilde{H}_j$ are respectively denoted by X_j^ϵ, X_j^0, $\tilde{X}_j$, $j = 1, \ldots, s$. We are assuming that s is finite, and in the case of $\dim(M) = 2n$ also $1 \leq s \leq n$. We further assume that there exists $\tilde{\epsilon} > 0$ such that for any $\epsilon \in (-\tilde{\epsilon}, \tilde{\epsilon})$ the following hold:

- AI. The Hamiltonian vector fields X_j^ϵ of the H_j^ϵ, $j = 1, \ldots, s$ are C^r, and their time-t maps exist and are C^r in M, $\forall t \in \mathbf{R}$.
- AII. There exists an s-dimensional torus Λ that is invariant under the Hamiltonian flows of the H_j^0, $j = 1, \ldots, s$. Moreover, Λ is a C^r submanifold of M and has a C^r tubular neighborhood in M.
- AIII. The H_j^ϵ, $j = 1, \ldots, s$ mutually Poisson commute and are functionally independent in a neighborhood of Λ in M.

We are interested in the flow of H_1^ϵ, and in particular on whether the invariant torus Λ of X_1^0 can be continued to an invariant torus of the perturbed system X_1^ϵ for $|\epsilon|$ sufficiently small. First note that given any $\alpha \in \pi_1(\Lambda) \simeq \mathbf{Z}^s$ there exists a $c = [c_1, \ldots, c_s] \in \mathbf{R}^s$ such that the integral curves of the restriction of the vector field $K_0(\alpha) = \sum_{j=1}^s c_j X_j^0$ to Λ are 1-periodic orbits that belong to the homotopy class α. Denote the time-1 map of $K_0(\alpha)$ by g_0^c. The Fréchet derivative $Dg_0^c(x)$ of g_0^c at any $x \in \Lambda$ is a bounded linear operator in $E \simeq T_x M$. It is easily seen that the derivatives at two different points of Λ are related by a similarity transformation. The spectrum of $Dg_0^c(x)$ in E is therefore independent of the point $x \in \Lambda$ and will be denoted by $\sigma(Dg_0^c)$.

Theorem 2.2. *Consider the functions* H_j^ϵ, $j = 1, \ldots, s$ *as above and assume that there exist* $\alpha \in \pi_1(\Lambda)$ *and a corresponding vector* $c = c(\alpha) \in \mathbf{R}^s$ *with the property that* $\sigma(Dg_0^c)$ *has exactly s eigenvalues that are unity and that* $\sigma(Dg_0^c) \setminus \{1\}$ *lies outside an open disc around* 1. *Then there exist* ϵ_0, $\beta_0 > 0$ *such that for any* ϵ *with* $|\epsilon| < \epsilon_0$ *there exists an s-parameter family of s-tori* $\Lambda_{\epsilon,\beta}$, $\beta \in (-\beta_0, \beta_0)^s$, *that are invariant under the flow of each of the* X_j^ϵ, $j = 1, \ldots, s$. *The motion on each* $\Lambda_{\epsilon,\beta}$ *is periodic or quasiperiodic (with at most s quasiperiods). The family* $\Lambda_{\epsilon,\beta}$ *is also* C^r *in* β *and there exists* $\beta_* \in (-\beta_0, \beta_0)^s$ *for which* $\Lambda_{0,\beta_*} = \Lambda$.

A proof of Theorem 2.2 is given in the Section 4. The geometrical ideas are similar to the ones used by Bambusi and Gaeta [3] to prove the original version of Nekhoroshev [9].

Proof of Proposition 2.1. Let $M = X \times S^1 \times \mathbf{R}$, and $H_1^\epsilon = H_\epsilon$, with H_ϵ as in (2.5). The corresponding Hamiltonian vector field is denoted by X_1^ϵ. Also, let $\Lambda = \Lambda_0(n_0, A)$, with $\Lambda_0(n_0, A)$ as defined by (2.7). Thus Λ is a 2-torus that is invariant under the vector field X_1^0. We also let

$$P = \sum_{j \in \mathbf{Z}} |u_j|^2 \tag{2.9}$$

and define a second family of functions H_2^ϵ by $H_2^\epsilon = P$, for any $\epsilon \in \mathbf{R}$. The corresponding Hamiltonian vector field is denoted by X_2^ϵ. We observe Λ is also invariant under X_2^0 and it is easy to check that the two functions H_1^ϵ, H_2^ϵ satisfy the conditions AI–AIII of Theorem 2.2. In particular, they Poisson commute and are independent (provided that $\Omega \neq 0$). We now verify the nonresonance condition for the Floquet map of an appropriate linear combination of X_1^0, X_2^0. To parametrize $\Lambda \in M$ define the function $\overline{\mathcal{A}} : \mathbf{Z} \to \mathbf{C}$ by $\overline{\mathcal{A}}_{n_0} = A$, $\overline{\mathcal{A}}_n = 0$ for $n \neq n_0$. Then

$$\Lambda = \{[\mathrm{e}^{\mathrm{i}\theta}\overline{\mathcal{A}}, \phi, 0] \in X \times S^1 \times \mathbf{R} : \theta \in \mathbf{R}, \phi \in S^1\}. \tag{2.10}$$

Also, given any $F : M \to \mathbf{R}$, let g_F^t be the time-t maps of the flows of the Hamiltonian vector field of F. On Λ we then have

$$g^t_{H_1^0}([\mathrm{e}^{\mathrm{i}\theta}\overline{\mathcal{A}}, \phi, 0]) = [\mathrm{e}^{\mathrm{i}(\theta - \lambda t)}\overline{\mathcal{A}}, (\phi + \Omega t) \bmod 2\pi, 0], \tag{2.11}$$

$$g^t_{H_2^0}([\mathrm{e}^{\mathrm{i}\theta}\overline{\mathcal{A}}, \phi, 0]) = [\mathrm{e}^{\mathrm{i}(\theta - t)}\overline{\mathcal{A}}, \phi, 0]. \tag{2.12}$$

Also, $g^t_{cF} = g^{ct}_F$, $\forall c \in \mathbf{R}$, therefore the time-t map of the Hamiltonian vector field of $c_1 H_0 + c_2 P$ is $g^{c_1 t}_{H_0} g^{c_2 t}_P$. Using (2.11) and (2.12) we therefore see that the condition for the orbits of the Hamiltonian vector field of $c_1 H_0 + c_2 P$ to be 1-periodic on Λ and to belong to the homotopy class $[n_1, n_2] \in \mathbf{Z}^2$ is that

$$-c_1 \lambda - c_2 = 2\pi n_1, \qquad c_1 \Omega = 2\pi n_2, \tag{2.13}$$

hence

$$c_1 = n_2 \frac{2\pi}{\Omega}, \qquad c_2 = -2\pi n_1 - n_2 \frac{2\pi\lambda}{\Omega}. \tag{2.14}$$

To calculate the Floquet map around any such 1-periodic orbit, i.e. Dg_0^c, let

$$u = \mathrm{e}^{-\mathrm{i}(c_1\lambda + c_2)t}\mathrm{e}^{\mathrm{i}\theta}\overline{\mathcal{A}} + w, \qquad \phi = c_1\Omega t + \phi_0 + \psi, \qquad J = I, \tag{2.15}$$

with $\overline{\mathcal{A}}$ as above. Using (2.1), (2.5) and (2.6) and keeping only linear terms we obtain the variational equation for $\dot{w}$, $\dot{\psi}$, $\dot{I}$. The variational equation can be made autonomous by the change of variables $v = \mathrm{e}^{\mathrm{i}(c_1\lambda + c_2)t} w$. We obtain

$$\dot{v}_n = \mathrm{i}c_1\lambda v_n, \quad n \in \mathbf{Z} \setminus \{n_0\} \tag{2.16}$$

$$\dot{v}_{n_0} = -\mathrm{i}c_1\lambda(v_{n_0} + v_{n_0}^*), \tag{2.17}$$

$$\dot{\psi} = 0, \qquad \dot{I} = 0. \tag{2.18}$$

Since $v = w$ at $t = 1$, the Floquet map around the 1-periodic orbit coincides with the time-1 map of the equations for $\dot{v}$, $\dot{\psi}$, $\dot{I}$. The spectrum of the time-1 map of the linear system (2.16)–(2.18) is calculated readily since the system is block diagonal with 2×2 blocks: (2.18) yields two unit eigenvalues. From (2.17), we obtain another pair of unit eigenvalues. Finally, Eqs. (2.16) yield the pair of eigenvalues $\mathrm{e}^{\pm 2\pi \mathrm{i} n_2 \frac{\lambda}{\Omega}}$ for each integer $n \neq n_0$. Choose $[n_1, n_2] = [-1, 1]$. Then, by our assumption that $\frac{\lambda}{\Omega} \notin \mathbf{Z}$ we have exactly 4 unit eigenvalues with the rest of the Floquet spectrum bounded away from unity. Therefore, the proposition follows from Theorem 2.2. ■

The proposition and the proof also generalize to higher-dimensional lattices. On the other hand, the proof would not work for a quasiperiodic diffraction management function D with at least two quasiperiods. Also note that expressions (2.14) for c_1, c_2 are independent of the specific form of the function $\mathcal{A}$.

The proof of Theorem 2.2 provides some additional information on the invariant tori $\Lambda_{\epsilon,\beta}$. Let $\overline{\Lambda}_{\epsilon,\beta}$ denote the projection of $\Lambda_{\epsilon,\beta}$ to $X \times S^1$, the phase space of (2.4).

Proposition 2.3. *Let $|\epsilon| < \epsilon_0$, $\beta \in (-\beta_0, \beta_0)^2$, with ϵ_0, β_0 as in Proposition 2.1. Then the intersection of $\overline{\Lambda}_{\epsilon,\beta}$ with any $X \times \{\phi_0\}$, $\phi_0 \in S^1$, is a circle $S^1(\epsilon, \beta, \phi_0)$ that belongs to a plane through the origin of X and whose center is at the origin of X. The circles $S^1(\epsilon, \beta, \phi_0)$, $\phi_0 \in S^1$, are invariant under the time-T map of the flow of (2.4), and $\overline{\Lambda}_{\epsilon,\beta} = \cup_{\phi_0 \in S^1} S^1(\epsilon, \beta, \phi_0)$.*

The proposition is shown in Section 4. By $\Lambda_{0,0} = \Lambda$, any circle $S^1(0, 0, \phi)$, $\phi_0 \in S^1$, coincides with the unperturbed breather orbit.

Also note that the strategy above can be used to examine the continuation of any breather of the discrete NLS. The main difficulty is that we do not have an adequate understanding of the Floquet spectra of the different classes of breathers known to exist (see e.g. [13,8,11]). A first step towards extending the continuation results to more breathers is to study Floquet numerically. This is done in the next section.

Remark 2.2.1. One case where the Floquet spectrum is calculated trivially is the one of the trivial "k-peak" breather solutions of the anticontinuous limit. These solutions have the form $u_n(t) = \mathrm{e}^{-\mathrm{i}\lambda t} A$, $\lambda = 2\gamma|A|^2$ for n in some $U \subset \mathbf{Z}$ that contains k sites (k finite), and $u_n(t) \equiv 0$ at all other sites (with ϕ, J as in (2.8)). The argument of Proposition 2.1 does not apply for $k > 1$ since each peak contributes to a block of the form (2.17), i.e. a pair or unit eigenvalues.

3. Numerical Floquet spectra

We now examine the continuation question numerically. We start by considering (2.1)–(2.3) with $D(t) = \overline{D} + \epsilon\tilde{d}(t)$ and $|\epsilon|$ small. The function $\tilde{d}$ is 2π-periodic, and $\Omega = \frac{2\pi}{T}$. Note that $|\overline{D}|$ is not assumed a priori small and the system we consider can be viewed as a small time-periodic perturbation of the discrete NLS. Eqs. (2.1) and (2.2) with the initial condition $u(t_0) = v \in X$ are then written as

$$\partial_t u = \mathrm{i}\overline{D}\Delta u - 2\mathrm{i}\gamma g(u) + \mathrm{i}\epsilon\tilde{d}(\phi)\Delta u, \qquad \dot{\phi} = \Omega, \tag{3.1}$$

$\phi \in S^1$, with the initial condition $u(0) = v$, $\phi(0) = \phi_0$. As before, $\phi \in S^1$ and the Hamiltonian structure is as in (2.5) and (2.6).

We are interested in continuing solutions of the discrete NLS that have the "breather" form

$$u(t) = \mathrm{e}^{-\mathrm{i}\lambda t}\mathcal{A}, \quad \text{with } \lambda \in \mathbf{R}, \tag{3.2}$$

and $\mathcal{A} : \mathbf{Z} \to \mathbf{C}$ a function that decays at infinity. The breathers correspond to solutions

$$u(t) = \mathrm{e}^{-\mathrm{i}\lambda t}\mathcal{A}; \qquad \phi(t) = \Omega t + \phi_0; \qquad J(t) = 0, \quad \forall t \in \mathbf{R} \tag{3.3}$$

of Eq. (3.1) with $\epsilon = 0$. The invariant 2-torus of the unperturbed system is $\Lambda_0(\mathcal{A}) = \cup_{\phi_0 \in S^1} C(\mathcal{A}, \phi_0)$, where $C(\mathcal{A}, \phi_0)$ is the set of points of $X \times S^1 \times \mathbf{R}$ in the orbit defined by (3.3).

Although there are several theoretical results on the existence of breather solutions (see e.g. [13,7,11]), we here compute breathers numerically. To do this we fix λ and solve the nonlinear system resulting from (3.2) and (3.1) using minpack routines. As in the proof of Proposition 2.1, given any homotopy class α we have a $c = [c_1, c_2] \in \mathbf{R}^2$ for which solutions of the form of (3.3) correspond to 1-periodic solutions of the modified system

$$u_t = c_1[i\overline{D}\Delta u - 2i\gamma g(u)] + c_2[-iu], \qquad \dot{\phi} = c_1\Omega, \qquad \dot{J} = 0. \tag{3.4}$$

The modified Floquet map is obtained by integrating numerically the variational equation around any such 1-periodic solution. Integration of the equations for $\dot{\phi}$, $\dot{J}$ yield a trivial Floquet 2×2 block and we refer to the remaining (infinite dimensional) block as the nontrivial (Floquet) block. The spectrum of the nontrivial block is obtained numerically using eispack routines. Note that since the first equation of (3.4) is Hamiltonian, the spectrum of the nontrivial block should include at least two unit eigenvalues, while the $\dot{\phi}$, $\dot{J}$ block contributes another pair of unit eigenvalues. Thus the criterion for continuation is that the spectrum of the nontrivial block has exactly two unit eigenvalues.

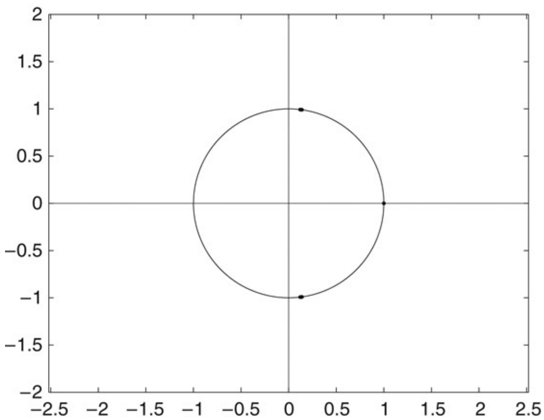

Fig. 1(a). Spectrum of (nontrivial modified) Floquet map for single-peak breather. The unit eigenvalue is double. $\overline{D} = -0.01$.

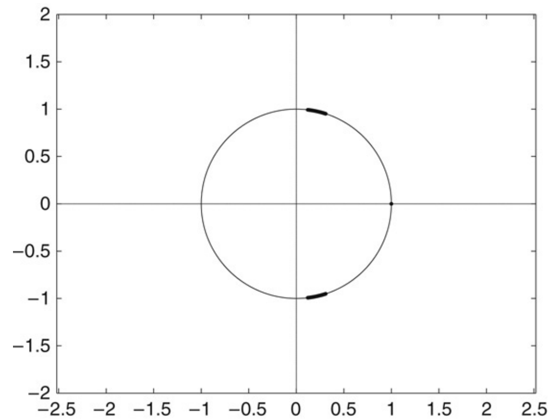

Fig. 2(a). Spectrum of (nontrivial modified) Floquet map for single-peak breather. The unit eigenvalue is double. $\overline{D} = -0.1$.

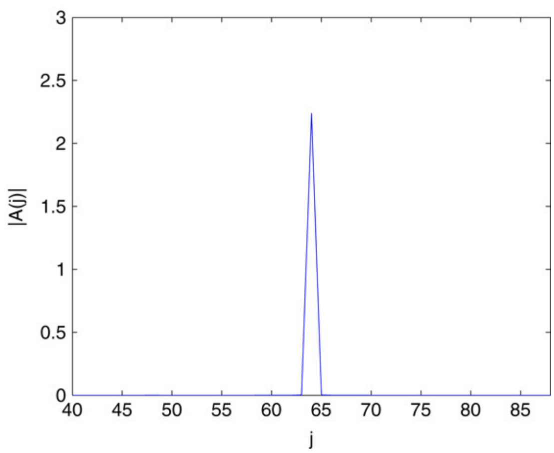

Fig. 1(b). Single-peak breather. $\overline{D} = -0.01$.

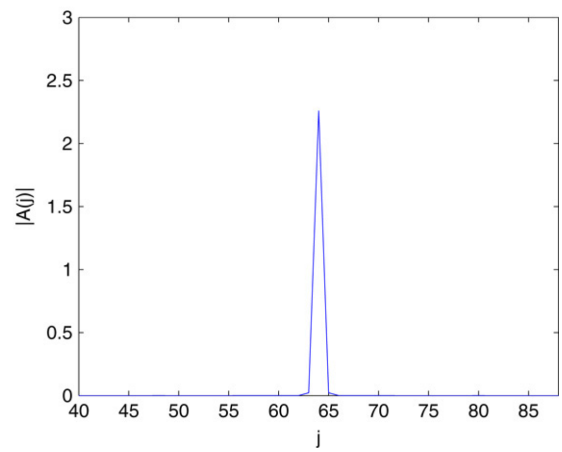

Fig. 2(b). Single-peak breather. $\overline{D} = -0.1$.

We first consider single-peak breathers. Typical examples of spectra of the corresponding nontrivial Floquet blocks are shown in Figs. 1(a), 2(a) and 3(a). The corresponding breathers are shown in Figs. 1(b), 2(b) and 3(b). Here we are using the homotopy class $[n_1, n_2] = [-1, 1]$, with c as in (2.14). Also, $\gamma = 1$, $\lambda = 10$, $\Omega = 13$. In all the figures we see an isolated double unit eigenvalue, and this suggests that the corresponding breathers can be continued. For $\overline{D} = 0$ we obtain two more eigenvalues $e^{\pm 2\pi i \frac{\lambda}{\Omega}}$, in agreement with the computation in the proof of Proposition 2.1. As we increase $|\overline{D}|$ with $\overline{D} < 0$, we see the appearance of two "arcs" of nearby Floquet eigenvalues on the unit circle, extending from $e^{2\pi i \frac{\lambda}{\Omega}}$, and $e^{-2\pi i \frac{\lambda}{\Omega}}$ respectively towards 1. The arcs are interpreted as a continuous Floquet spectrum and become wider as we increase $|\overline{D}|$, e.g. as in Figs. 1(a), 2(a) and 3(a). For $\overline{D} > 0$ we see a similar widening of the arcs on the unit circle, this time from $e^{2\pi i \frac{\lambda}{\Omega}}$ and $e^{-2\pi i \frac{\lambda}{\Omega}}$ respectively towards -1. As we increase $|\overline{D}|$ further, the convergence to the breather solution slows down and the existence of single-peak breathers becomes less certain (the existence of these breathers is shown for small $|\overline{D}|$, see [11]).

For $|\overline{D}|$ sufficiently small we also find numerically multi-peak breather solutions. The existence of multi-peak breathers

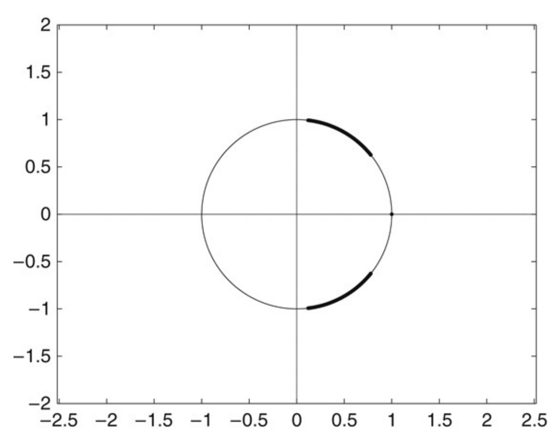

Fig. 3(a). Spectrum of (nontrivial modified) Floquet map for single-peak breather. The unit eigenvalue is double. $\overline{D} = -0.4$.

can be shown by an implicit function argument (see [8,11]), i.e. by continuation from the multi-peak breathers of the anticontinuous limit system. However, by Remark 2.2.1, the modified Floquet spectrum of k-peak breathers of the anticontinuous limit system has $2k$ unit eigenvalues and this does not allow us to continue to solutions of the full system by the argument of Proposition 2.1. On the other

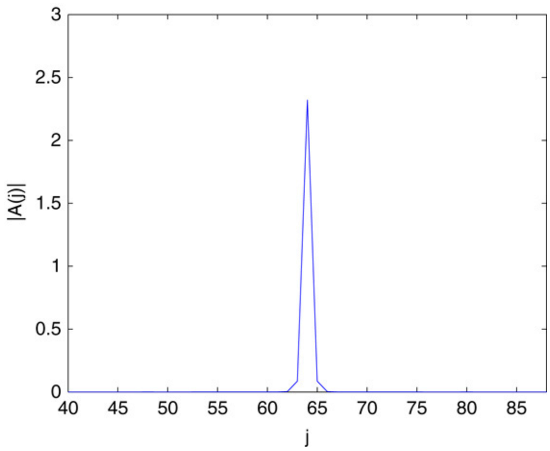

Fig. 3(b). Single-peak breather. $\overline{D} = -0.4$.

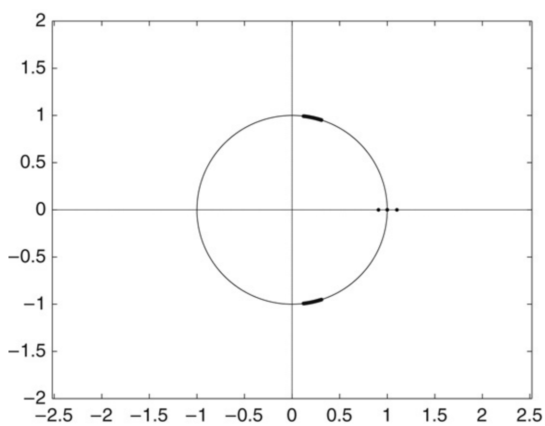

Fig. 4(a). Spectrum of (nontrivial modified) Floquet map for 2-peak breather. The unit eigenvalue is double. $\overline{D} = -0.1$.

hand, in the numerical experiments we considered "in-phase" k-peak breathers (i.e. k-peak breathers with U_{-} empty, see Remark 2.2.1) and found that as we increase $|\overline{D}|$ away from the origin, the nontrivial Floquet block has $2k - 2$ eigenvalues that move away from 1 along the real axis. This suggests that the k-peak in-phase breathers obtained for nontrivial $|\overline{D}|$ can be continued, provided that $\frac{\lambda}{\Omega}$ is noninteger.

The first example is shown in Fig. 4(a), where we see the nontrivial block of the modified Floquet map around the 2-peak breather of Fig. 4(b). The eigenvalue 1 is double and we have two other eigenvalues that have moved off 1. In Fig. 5(a) we see the nontrivial block of the modified Floquet map around the 3-peak breather shown in Fig. 5(b). The eigenvalue 1 is again double and we have four other eigenvalues that have moved off 1. We also observe the appearance of arcs of continuous spectrum on the unit circle. These are similar to the arcs observed for the single-peak breathers, i.e. one of their endpoints is at $e^{\pm 2\pi i \frac{\lambda}{\Omega}}$.

The distance of the real eigenvalues that move off 1 increases with $|\overline{D}|$ and also decreases rapidly as we increase the spacing between the peaks. For instance, for a 3-peak soliton obtained for $|\overline{D}| = 0.4$ with a spacing of 4 sites between the peaks, the distance of the 4 eigenvalues that are supposed to have

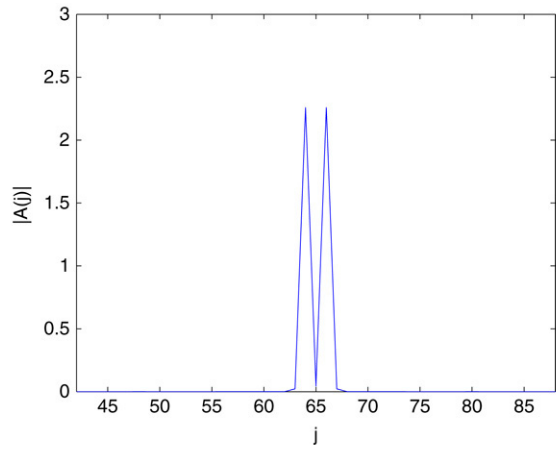

Fig. 4(b). 2-peak breather. $\overline{D} = -0.1$.

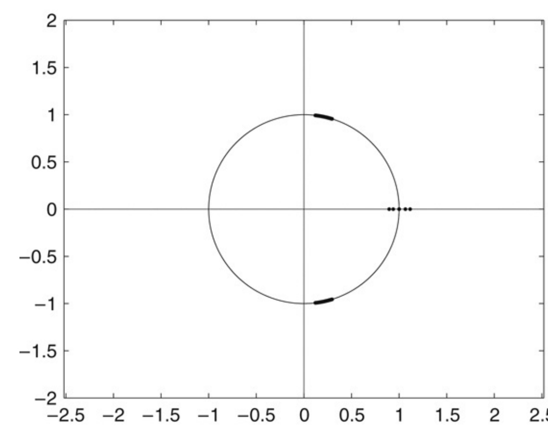

Fig. 5(a). Spectrum of (nontrivial modified) Floquet map for 3-peak breather. The unit eigenvalue is double. $\overline{D} = -0.093$.

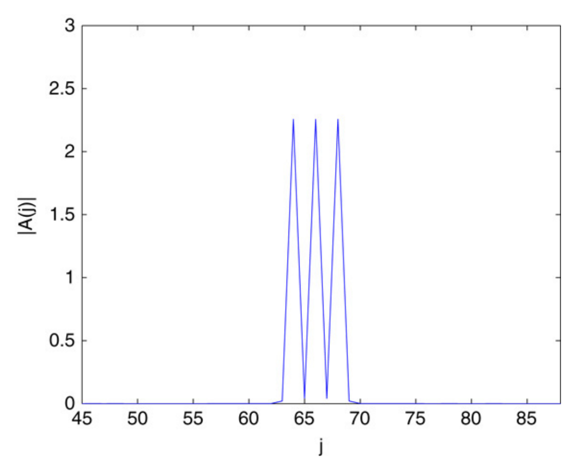

Fig. 5(b). 3-peak breather. $\overline{D} = -0.093$.

moved off 1 cannot be distinguished from numerical error. It appears therefore that as the (smallest) distance between consecutive peaks increases, continuation can only be possible for a decreasing range of the parameter ϵ. (Note that the numerical computation of multi-peak breathers is more reliable for small $|\overline{D}|$ and for small separation between the peaks.)

Remark 3.0.1. The above calculations also suggest that as we increase $|\overline{D}|$ away from the origin, the corresponding in-phase multi-peak breathers become linearly unstable since the modified Floquet map gives us relative stability information. This linear instability is more pronounced when the peaks are near each other. The stability of the "out-of-phase" multi-peaks (i.e. with both U_+, U_- nonempty, see Remark 2.2.1) is not considered here. There is evidence that some out-of-phase multi-peak breathers are linearly stable.

The continuous spectrum in the modified Floquet map appears to arise from exponentiating the discrete Laplacian, which is a bounded operator in X. We can then expect that in the continuation problem for the breather soliton of the NLS on the line analogous arcs of continuous spectrum will "wrap around" the unit circle an infinite number of times and cannot be "separated" from the unit eigenvalue by varying the homotopy class. Thus the continuation strategy we used for the discrete case does not seem directly applicable to the continuous problem and it is also possible that the continuation question does not have a positive answer for the problem on the line. This possibility is consistent with the results of [14,12].

4. Continuation of invariant tori

To prove Theorem 2.2 we follow the notation used in setting up the theorem in Section 2. Let r be a fixed integer, with $r \geq 2$. Also let $g^t_{i,\epsilon}$ denote the time-t map of the vector field X^ϵ_i. Assumption AI implies that the maps $g^t_{i,\epsilon}$ are C^r, $\forall t \in \mathbf{R}$. Letting $c = [c_1, \ldots, c_s] \in \mathbf{R}^s$, $t \in \mathbf{R}$, we also use the notation $g^{ct}_\epsilon = g^{c_1 t}_{1,\epsilon} \cdots g^{c_s t}_{s,\epsilon}$. By assumption AIII the maps $g^t_{i,\epsilon}$, $g^{t'}_{j,\epsilon}$ mutually commute, for all $i, j \in \{1, \ldots, s\}$, and $t, t' \in \mathbf{R}$. Also, $B(Z, Y)$ denotes the bounded linear operators from a Banach space Z to Banach space Y. The operator norm in $B(Z, Y)$ is denoted by $\|.\|_0$ (the spaces Z, Y will be clear from the context in each case). The ball of radius r around the point x is denoted by $\mathcal{B}_r(x)$, and $\mathcal{B}_r$ if x is the origin (the spaces and norms will be clear from the context in each case).

The plan is to use the maps g^{ct}_ϵ to construct a version of the Poincare map from a neighborhood of (each) m in Σ_m to Σ_m. The (family of) invariant tori we seek for $\epsilon \neq 0$ are identified with (a family) of fixed points of this map. The first step will be to define a suitable coordinate system in a neighborhood of m. This is done in Lemmas 4.1 and 4.2. Lemma 4.3 states the salient properties of the coordinate system and Lemma 4.4 completes the Poincare map construction. The existence of fixed points is shown in Lemma 4.5. Lemmas 4.6 and 4.7 show that the fixed points of the Poincare are the invariant tori we seek. Lemmas 4.1, 4.2 and 4.5 use the implicit function theorem, in Lemma 4.9. Also, Remarks 4.0.1, 4.1.1 and 4.4.1 describe the geometry of the Poincare map in the particular case of Proposition 2.1 and are used in the proof of Proposition 2.3.

We first define a system of coordinates around the set Λ. Note that for any $m \in \Lambda$ there exists a C^r Hilbert submanifold Σ_m of M that has codimension s and is transverse to Λ at m. Moreover, by assumption AII on the existence of a C^r tubular neighborhood around Λ, we can choose a family $\{\Sigma_m\}_{m \in \Lambda}$ of such submanifolds that constitutes a C^r foliation of a neighborhood U of Λ.

Letting $m \in \Lambda$ we can assume that a neighborhood of Σ_m in M has been identified with a neighborhood of the origin of $T_m\Sigma_m \simeq E$ by a C^r chart (that we do not make explicit). Also, let $h^0_m = H^0|_{\Sigma_m}$. Then, in a neighborhood of m in Σ_m we can use coordinates (y^0, z^0), where $y^0 \in Y^0$, the nullspace of Dh^0_m, and $z^0 \in Z^0$, the orthogonal complement of Y^0 in $T_m\Sigma_m$. Note that Y^0 splits in $T_m\Sigma_m$, and that m has coordinates $(0, 0)$. Also, in a neighborhood of $m \in \Lambda$ we can use C^r coordinates $(y^0, z^0, w) \in Y^0 \times Z^0 \times W$, $W \simeq \mathbf{R}^s$, where W is the orthogonal complement of Σ_m in $T_m\Sigma_m$. Points with coordinates $(y^0, z^0, 0)$ belong to Σ_m. The dependence of the coordinates of a given point on m is not made explicit in this notation.

Remark 4.0.1. In Proposition 2.1 the Σ_m can be chosen as follows: let $M = X \times S^1 \times \mathbf{R}$, let $m = (\mathcal{A}_0, \phi_0, 0) \in \Lambda$, $\mathcal{A}_0 = \mathrm{e}^{\mathrm{i}\theta_0}\mathcal{A}$ for some $\theta_0 \in \mathbf{R}$, and let $\mathcal{P}_{\mathcal{A}_0}$ be a hyperplane in X that is normal to the breather orbit. We let Σ_m be the set of points $(\varpi(\mathcal{A}_0), \phi_0, J)$ with $\varpi(\mathcal{A}_0)$ points of $\mathcal{P}_{\mathcal{A}_0}$ that satisfy $\|\varpi(\mathcal{A}_0) - m\| < \|\mathcal{A}\|$, and $J \in \mathbf{R}$. The transverse direction to Σ_m are along the angle ϕ (i.e. second) component, and along the breather orbit.

Consider $H^\epsilon = H^0 + \epsilon\tilde{H}$, where $H^\epsilon = [H^\epsilon_1, \ldots, H^\epsilon_s]$ (similarly for H^0, $\tilde{H}$), and let $\tilde{h}_m = \tilde{H}|_{\Sigma_m}$. By AI, $D_2h^0_m(y^0, z^0)$, $D_2\tilde{h}_m(y^0, z^0)$ are elements of $B(Z^0, \mathbf{R}^s)$, i.e. $s \times s$ matrices, for any $(y^0, z^0) \in \Sigma_m$. By the independence of the components of H^0, and the definition of Σ_m, $D_2h^0_m(y^0, z^0)$ is invertible, for any $(y^0, z^0) \in \Sigma_m$. Then by the continuity of $D_2h^0_m(y^0, z^0)$ in $m \in \Lambda$, there exist M_1, K_1 satisfying

$$\sup_{m \in \Lambda} \|[D_2h^0_m(0, 0)]^{-1}\|_0 < M_1 < \frac{1}{2}K_1. \tag{4.1}$$

Also, by the regularity assumption AI, there exists K_2 satisfying

$$\sup_{m \in \Lambda} \left(\sup_{(y^0, z^0) \in \Sigma_m} \|D_2\tilde{h}_m(y^0, z^0)\|_0 \right) < K_2. \tag{4.2}$$

Lemma 4.1. *There exist an $\epsilon_1 > 0$ and nonempty $\Sigma^1_m \subset \Sigma_m$ such that for $|\epsilon| < \epsilon_1$, and $(y^0, z^0) \in \Sigma^1_m$, the map $(y^0, z^0) \mapsto (y^0, \beta^\epsilon)$ defined by $\beta^\epsilon = h^\epsilon_m(y^0, z^0)$ is a C^r diffeomorphism in Σ^1_m, i.e. defines a new C^r coordinate system in Σ^1_m.*

Proof. Define $\phi^\epsilon_m : \Sigma_m \to Y^0 \times \mathbf{R}^s$ by $\phi^\epsilon_m(y^0, z^0) = (y^0, \beta(y^0, z^0))$. The map ϕ^ϵ_m is C^r in Σ_m and we need a subset of Σ_m where the derivative of ϕ^ϵ_m is an isomorphism. We have

$$D\phi^\epsilon_m = \begin{pmatrix} I & 0 \\ D_1h^\epsilon_m & D_2h^\epsilon_m. \end{pmatrix}$$

For $D\phi^\epsilon_m$ to be an isomorphism it is sufficient that the $s \times s$ matrix $D_2h^\epsilon_m$ is invertible. We have

$$D_2h^\epsilon_m = D_2h^0_m + \epsilon D_2\tilde{h}_m. \tag{4.3}$$

Let

$$D_2 h_m^0 = M + N, \quad \text{with } M = D_2 h_m^0(0,0),$$
$$N = D_2 h_m^0(y^0, z^0) - D_2 h_m^\epsilon(0,0). \tag{4.4}$$

By (4.1), M is invertible. By the continuity of $D_2 h_m^0$ we can choose Σ_m^1 so that

$$\|M^{-1}N\|_0 < \frac{1}{2}, \quad \forall (y^0, z^0) \in \Sigma_m^1.$$

Therefore $D_2 h_m^0$ is invertible in Σ_m^1 and

$$\|[D_2 h_m^0(y^0, z^0)]^{-1}\|_0 < K_1, \quad \forall (y^0, z^0) \in \Sigma_m^1. \tag{4.5}$$

Using (4.1), and $\epsilon_1 < (K_1 K_2)^{-1}$, we have that if $|\epsilon| < \epsilon_1$ then

$$|\epsilon| \|[D_2 h_m^0]^{-1} D_2 \tilde{h}_m\|_0 < 1, \quad \forall (y^0, z^0) \in \Sigma_m^1, \tag{4.6}$$

i.e. by (4.3) $D_2 h_m^\epsilon$ is invertible. ■

Remark 4.1.1. Following Remark 4.0.1 on the geometry of Proposition 2.1, we also see that $Y^0 = \tilde{\Sigma}_m \cap \ker(\nabla_E P(\mathcal{A}_0))$, where $\tilde{\Sigma}_m = \Sigma_m - m$. Using the fact that $\nabla_X H_1^0(m) = \lambda \nabla_X H_2^0(m)$ we also have that Z^0 is the real span of $\nabla_E P(\mathcal{A}_0)$, and $[0, 0, 1]$. Thus Z^0 is the orthogonal complement of Y^0 in $\tilde{\Sigma}_m$. Also, if $x = (u, \phi, J) \in \Sigma_m$, then we can let $y^0(u, \phi, J)$ be the orthogonal projection of $x - m$ to Y^0, and $\beta_j^0(u, \phi, J) = H_j^0(x) - H_j^0(m)$, $j = 1, 2$.

To define coordinates β^0, y^0 on Σ_m, let $\tilde{\Sigma}_m = \Sigma_m - m$. Also let $Y^0 = \tilde{\Sigma}_m \cap \ker \nabla_E P(\mathcal{A})$ and let Z^0 be the real span of $\nabla_E P(\mathcal{A})$, and $[0, 0, 1]$. Note that by $\nabla H_1^0 = \lambda \nabla H_2^0$ the definitions of Y^0, Z^0 coincide with the homonymous subspaces in the proof of Theorem 2.2 (with a slight abuse of notation) and that Z^0 is the orthogonal complement of Y^0 in $\tilde{\Sigma}_m$. Let $x = (u, \phi, J) \in \Sigma_m$. Then we let $y^0(u, \phi, J)$ be the orthogonal projection of $x - m$ to Y^0. Also let $\beta_j^0(u, \phi, J) = H_j^0(x) - H_j^0(m)$, $j = 1, 2$.

Let $g_\epsilon^\tau(y^0, z^0) = g_\epsilon^\tau(y^0, z^0, 0)$, $|\epsilon| < \epsilon_1$, $(y^0, z^0) \in \Sigma_m^1$. Let $[g_\epsilon^\tau(y^0, z^0)]_W$ denote the W-component of $g_\epsilon^\tau(y^0, z^0, 0)$. Also, consider a point (y^0, z^0, w) in a neighborhood of $\Sigma_m^1 \times \{0\}$ in M, and the equation

$$[g_\epsilon^\tau(y^0, z^0)]_W = w \tag{4.7}$$

for $\tau \in \mathbf{R}^s$, i.e. (y^0, z^0, w) is a parameter. We want to find a neighborhood of $\Sigma_m^1 \times \{0\}$ in M where (4.7) has a unique solution $\tau = \tau^\epsilon(y^0, z^0)$. In such a neighborhood we can use the coordinates $(y^0, \beta^\epsilon, \tau^\epsilon)$.

To see that this is possible, let $\{\hat{w}_1, \ldots, \hat{w}_s\}$ be a basis for W. In a neighborhood of $m \in \Lambda$ define the $s \times s$ matrices A, B by

$$A_{ij} = \langle \hat{w}_i, X_j^0 \rangle, \quad B_{ij} = \langle \hat{w}_i, \tilde{X}_j^0 \rangle, \quad i, j \in \{1, \ldots\}. \tag{4.8}$$

The vector fields X_j^0, $j = \{1, \ldots, s\}$, at any $m \in \Lambda$ are along Λ and independent by assumptions AII, AIII. Therefore A at any $m \in \Lambda$ is invertible. By the continuity of A in m we can then choose $M_2, K_3 > 0$ satisfying

$$\sup_{m \in \Lambda} \|A(m)\|_0 < M_2, \quad M_2 < \frac{1}{2} K_3. \tag{4.9}$$

Similarly, there exists $K_4 > 0$ such that

$$\sup_{m \in \Lambda} \left(\sup_{(y^0, z^0) \in \Sigma_m^1} \|B(y^0, z^0, 0)\|_0 \right) < K_4. \tag{4.10}$$

Consider the function $G(w, \tau) = [g_\epsilon^\tau(y^0, z^0)]_W - w$, i.e. compare with (4.7), in a subset of the origin in $\mathbf{R}^s \times \mathbf{R}^s$. Note that $G(0, 0) = 0$. Also, in the case where $D_2 G(0, 0)$ is invertible, define the constant μ_2 by

$$\mu_2 = \|[D_2 G(0, 0)]^{-1}\|_0 \tag{4.11}$$

and consider an $r_2 > 0$ satisfying

$$\|D_2 G(w, \tau) - D_2 G(0, 0)\|_0 < \frac{1}{2\mu_2},$$
$$\forall (w, \tau) \in \mathcal{B}_{\frac{r_2}{2\mu_2}} \times \mathcal{B}_{r_2}. \tag{4.12}$$

By the definition of g_ϵ^τ, $D_2 G$ is continuous near $(0, 0)$ and such an $r_2 > 0$ always exists (i.e. we have assumed that μ_2 is finite).

Lemma 4.2. *There exist $\epsilon_2 > 0$, $r_2 > 0$, and nonempty $\Sigma_m^2 \subset \Sigma_m^1$ such that for $|\epsilon| < \epsilon_2$, $(y^0, z^0) \in \Sigma_m^2$, $\|w\| < r_2$ Eq. (4.7) has a unique solution $\tau^\epsilon(y^0, z^0, w)$. Moreover, the function $\chi_m^\epsilon : \Sigma_m^2 \times \mathcal{B}_{r_2} \to Y^0 \times \mathbf{R}^s \times \mathbf{R}^s$ defined by $\chi_m^\epsilon(y^0, z^0, w) = (y^0, \beta^\epsilon, \tau^\epsilon)$ with $\beta^\epsilon = h_m^\epsilon(y^0, z^0)$, τ^ϵ the solution of (4.7), is injective and continuous in $\Sigma_m^2 \times \mathcal{B}_{r_2}$. Also, there exists $\tilde{r}_2 > 0$, $\tilde{r}_2 \le r_2$ for which χ_m^ϵ, restricted to $\Sigma_m^2 \times \mathcal{B}_{\tilde{r}_2}$ is a C^r diffeomorphism.*

Proof. Let $m \in \Lambda$, $(y^0, z^0) \in \Sigma_m^1$. Given w, we seek $\tau = \tau(w)$ for which $G(w, \tau) = 0$. We have $G(0, 0) = 0$ and we want to apply the implicit function theorem. The partial derivative of F with respect to τ at $(w, \tau) = (0, 0)$ is an $s \times s$ matrix with entries

$$[D_2 G(0, 0)]_{i,j} = \langle \hat{w}_i, \partial_{\tau_j} g_\epsilon^\tau(y^0, z^0)|_{\tau=0} \rangle = \tag{4.13}$$
$$= \langle \hat{w}_i, X^{\epsilon j}(y^0, z^0, 0) \rangle = A(y^0, z^0, 0) + \epsilon B(y^0, z^0, 0). \tag{4.14}$$

By the assumption on $A(m)$, we can choose Σ_m^2 so that $A(y^0, z^0, 0)$ is invertible, $\forall (y^0, z^0) \in \Sigma_m^2$. Moreover, (4.10), $|\epsilon| < \epsilon_2$ imply that $D_2 G(0, 0)$ is invertible with $\|[D_2 G(0, 0)]^{-1}\|_0$ bounded by some μ_2, $\mu_2 < 2$. Also, $G(w, 0) = -w$ and we can choose r_2 so that $\|w\| < r_2$ implies that (4.7) has a unique solution $\tau^\epsilon = \tau^\epsilon(w)$ that is defined for all $w \in \mathcal{B}_{r_2} \subset W$ and is C^r in a nontrivial subset of $\mathcal{B}_{r_2}$. Combining with Lemma 4.1 on coordinates (y^0, β^ϵ), the lemma follows. ■

Let $m \in \Lambda$, $|\epsilon| < \epsilon_2$, and $(y^0, z^0) \in \Sigma_m^2$. Define the map $\phi_m^\epsilon : \Sigma_m^2 \to Y^0 \times \mathbf{R}^s$ by $\phi_m^\epsilon(y^0, z^0) = (y^0, \beta^\epsilon)$. Also define the map $f_m^\epsilon : (\phi_m^0)^{-1}(\Sigma_m^2) \to \phi_m^\epsilon(\Sigma_m^2)$ by $f_m^\epsilon = (\phi_m^0)^{-1} \circ \phi_m^\epsilon$. By the lemma f_m^ϵ is a C^r diffeomorphism between the coordinates (y^0, β^0), and (y^0, β^ϵ). Let I_2 be the set of $w \in \mathbf{R}^s$ satisfying $\|w\| < r_2$. We have the following.

Lemma 4.3. *Let $m \in \Lambda$. There exists $\epsilon_3 > 0$ (and $\epsilon_3 \leq \epsilon_2$) such that if $|\epsilon| < \epsilon_3$ then there exists $U_m \subset \Sigma_m^2 \times I_2$ with the property that any solution of cX^ϵ with initial condition $v(0) \in U_m$ satisfies $v(1) \in \Sigma_m^2 \times I_2$.*

The lemma follows from the continuity of the flows g_ϵ^c and $X^\epsilon = X^0 + \epsilon \tilde{X}$.

Let $\Phi_\epsilon = g_\epsilon^c$. Note that the dependence of Φ_ϵ on the homotopy class α and c is not made explicit in this notation.

By Lemma 4.2, for any $(y^0, \beta^0, w) \in \Sigma_m^2 \times I_2$ we can use coordinates y^ϵ, β^ϵ, τ^ϵ defined by $\tau^\epsilon(y^0, \beta^0, w)$, $\beta^\epsilon(y^0, \beta^0, w) = h_m^\epsilon(y^0, \beta^0)$, i.e. as in Lemma 4.2, and $y^\epsilon(y^0, \beta^0, w) = y^0$. Consider the image $\Phi_\epsilon(y^0, \beta^0, w)$ of points of U_m under Φ_ϵ. Using coordinates y^ϵ, β^ϵ, τ^ϵ we define

$$\hat{y}^\epsilon = y^\epsilon(\Phi_\epsilon(y^\epsilon, \beta^\epsilon, \tau^\epsilon)), \qquad \hat{\beta}^\epsilon = \beta^\epsilon(\Phi_\epsilon(y^\epsilon, \beta^\epsilon, \tau^\epsilon)), \quad \hat{\tau}^\epsilon = \tau^\epsilon(\Phi_\epsilon(y^\epsilon, \beta^\epsilon, \tau^\epsilon)). \tag{4.15}$$

By Lemma 4.3, $\hat{y}^\epsilon$, $\hat{\beta}^\epsilon$, $\hat{\tau}^\epsilon$ are well defined in U_m.

Lemma 4.4. *Fix $m \in \Lambda$ and let $(\hat{\beta}^\epsilon, \hat{\tau}^\epsilon, \hat{y}^\epsilon)$ be as above. Then for any $(\beta^\epsilon, \tau^\epsilon, y^\epsilon) \in U_m$ we have* (i) $\hat{\beta}^\epsilon = \beta^\epsilon$, (ii) $\hat{\tau}^\epsilon = \tau^\epsilon + \tau_0^\epsilon$, *where τ_0^ϵ depends on β^ϵ, y^ϵ,* (iii) $\hat{y}^\epsilon$ *is independent of τ^ϵ.*

Proof. Property (i) follows immediately from the fact that the H_j^ϵ are invariant under the maps $g_{j,\epsilon}^t$, and that the $g_{j,\epsilon}^t$ mutually Poisson commute. Property (iii) also follows from the definition of the coordinate τ^ϵ: for any $\mu \in \mathbf{R}^s$,

$$\begin{aligned}\Phi_\epsilon((\beta^\epsilon, \tau^\epsilon + \mu, y^\epsilon)_m) &= \Phi_\epsilon g_\epsilon^\mu((\beta^\epsilon, \tau^\epsilon, y^\epsilon)_m) \\ &= g_\epsilon^\mu \Phi_\epsilon((\beta^\epsilon, \tau^\epsilon, y^\epsilon)_m) \\ &= g_\epsilon^\mu((\hat{\beta}^\epsilon, \hat{\tau}^\epsilon, \hat{y}^\epsilon)_m) \\ &= (\hat{\beta}^\epsilon, \hat{\tau}^\epsilon + \mu, \hat{y}^\epsilon)_m,\end{aligned} \tag{4.16}$$

i.e. the third component does not depend on τ^ϵ. From (4.16) we also see that

$$\hat{\tau}^\epsilon(\beta^\epsilon, \tau^\epsilon + \mu, y^\epsilon) = \hat{\tau}^\epsilon(\beta^\epsilon, \tau^\epsilon, y^\epsilon) + \mu \tag{4.17}$$

and therefore

$$\hat{\tau}^\epsilon(\beta^\epsilon, \tau^\epsilon, y^\epsilon) = \hat{\tau}^\epsilon(\beta^\epsilon, 0, y^\epsilon) + \tau^\epsilon. \tag{4.18}$$

Setting $\tau_0^\epsilon = \hat{\tau}^\epsilon(\beta^\epsilon, 0, y^\epsilon)$ we obtain (ii). ■

Remark 4.4.1. In the case of Proposition 2.1 we note that the time-1 maps of $c_1 X_1^\epsilon$, and $c_2 X_2^\epsilon$ map $X \times \{\phi_0\} \times \mathbf{R} \subset M$ to itself, $\forall \phi_0 \in S^1$, $\epsilon \in \mathbf{R}$, i.e. these sets are invariant under the maps g_ϵ^c, for any choice of homotopy class $\alpha = [n_1, n_2] \in \pi_1(\Lambda)$. ($g_{1,\epsilon}^{c_1}$ advances the second component by $n_2 2\pi = 0 \mathrm{mod}(2\pi)$, while $g_{2,\epsilon}^{c_2}$ does not change it.) Since ϕ_0 does not change under the maps g_ϵ^c, we only need to use the time-τ_2^ϵ map of the flow of X_2 to move points $\Phi_\epsilon(x)$, $x \in \Sigma_m^3$, back to Σ_m^3.

By Lemma 4.4 the component $\hat{y}^\epsilon$ of Φ_ϵ depends on β^ϵ, y^ϵ and we write $\hat{y}^\epsilon = \hat{y}^\epsilon(\beta^\epsilon, y^\epsilon)$. We now use the condition on the spectrum of the derivative of Φ_0.

Let $m \in \Lambda$, $|\epsilon| < \epsilon_3$. Let Σ_m^3, I_3 be nonempty subsets of Σ_m^2, I_2 respectively, with the property that $\Sigma_m^3 \times I_3 \subset U_m$. Let $V_m = \chi_m^0(\Sigma_m^3 \times \{0\})$. For $(y^0, \beta^0) \in V_m$, define the functions β, and $\hat{y}$ by

$$\beta(\epsilon, y^0, \beta^0) = \beta^\epsilon(y^0, \beta^0), \tag{4.19}$$

$$\hat{y}(\epsilon, y^0, \beta^0) = \hat{y}^\epsilon(y^0, \beta(\epsilon, y^0, \beta^0)) = \hat{y}^\epsilon(\beta^\epsilon(y^0, \beta^0)). \tag{4.20}$$

Also, let

$$F(\epsilon, y^0, \beta^0) = \hat{y}^\epsilon(y^0, \beta^\epsilon) - y^0 = \hat{y}(\epsilon, y^0, \beta(\epsilon, y^0, \beta^0)) - y^0,$$

i.e. F is defined for $(y^0, \beta^0) \in V_m$, $|\epsilon| < \epsilon_3$. We may assume without loss of generality that $\beta^0(m) = 0$, $\forall m \in \Lambda$, i.e. by adding appropriate constants to the H_j^0, $j = 1, \ldots, s$. Viewing F as a function of the two variables $\mathbf{x} = (\epsilon, \beta^0)$, $\mathbf{y} = y^0$, we then have $F(0, 0) = 0$. The function F is C^r in its domain by the C^r regularity of the flows Φ_ϵ. Also, by Lemma 4.1 and the construction of the coordinates (y^0, β^0), F is C^r in $m \in \Lambda$.

Let $\|(\mathbf{x}, \mathbf{y})\|_0 = (|\epsilon|^2 + \|(y^0, \beta^0)\|^2)^{\frac{1}{2}}$. In the case where $[D_2 F(0, 0)]^{-1} \in B(Y^0)$, let

$$\mu_3 = \|[D_2 F(0, 0)]^{-1}\|_0. \tag{4.21}$$

Lemma 4.5. *Let $m \in \Lambda$, χ_m^ϵ, Φ_ϵ as above, and $|\epsilon| < \epsilon_3$. There exists $\epsilon_0 > 0$ ($\epsilon_0 \leq \epsilon_3$), and for each ϵ with $|\epsilon| < \epsilon_0$ a β_*^ϵ so that if $|\epsilon| < \epsilon_0$, $\beta^\epsilon \in \mathcal{B}_{\beta_*^\epsilon}$ then the equation $\hat{y}^\epsilon(y^0, \beta^\epsilon) = y^0$ has a unique solution $y^0 = \rho_m^\epsilon(\beta^\epsilon)$. The map $\rho_m^\epsilon : \mathcal{B}_{\beta_*^\epsilon} \to Y^0$ is C^r in a nontrivial subset of its domain. Also, the maps ρ_m^ϵ depend on m in a C^r way, $\forall \epsilon \in (-\epsilon_0, \epsilon_0)$, $\beta^\epsilon \in \mathcal{B}_{\beta_*^\epsilon}$.*

Proof. We want to solve $F(\mathbf{x}, \mathbf{y}) = 0$, i.e. find $\mathbf{y}(\mathbf{x})$ for $\mathbf{x}$ near the origin. We have $F(0, 0) = 0$. Also,

$$D_2 F = \frac{\partial \hat{y}^\epsilon}{\partial \beta} \frac{\partial \beta}{\partial y^0} + \frac{\partial \hat{y}}{\partial y^0} - I. \tag{4.22}$$

From $\beta(0, y^0, \beta^0) = \beta^0$ and the continuity of $\frac{\partial \beta}{\partial y^0}$ at the origin we have that

$$\frac{\partial \beta}{\partial y^0}(0, 0, 0) = 0. \tag{4.23}$$

Also, at the origin,

$$\frac{\partial \hat{y}}{\partial y^0} = \frac{\partial \hat{y}^0}{\partial y^0}(0, 0). \tag{4.24}$$

Therefore

$$D_2 F(0, 0) = \frac{\partial \hat{y}^0}{\partial y^0}(0, 0) - I. \tag{4.25}$$

Using the coordinates $(y^\epsilon, \beta^\epsilon, \tau^\epsilon)$ for $\epsilon = 0$, the derivative of Φ_0 is

$$D\Phi_0 = \begin{pmatrix} \partial_{y^0} \hat{y} & \partial_{\beta^0} \hat{y} & \partial_{\tau^0} \hat{y} \\ \partial_{y^0} \hat{\beta}^0 & \partial_{\beta^0} \hat{\beta}^0 & \partial_{\tau^0} \hat{\beta}^0 \\ \partial_{y^0} \hat{\tau}^0 & \partial_{\beta^0} \hat{\tau}^0 & \partial_{\tau^0} \hat{\tau}^0 \end{pmatrix}. \tag{4.26}$$

By Lemma 4.4, at $\beta^0 = 0$, $y^0 = 0$, $\tau_0 = 0$ we have

$$D\Phi_0(0, 0, 0) = \begin{pmatrix} \partial_{y^0} \hat{y}(0, 0) & \partial_{\beta^0} \hat{y}(0, 0) & 0 \\ 0 & I_s & 0 \\ \partial_{y^0} \hat{\tau}^0(0, 0) & \partial_{\beta^0} \hat{\tau}^0(0, 0) & I_s \end{pmatrix}, \tag{4.27}$$

where I_s is the $s \times s$ identity matrix. The block 1, 1 is the operator $\frac{\partial \hat{y}}{\partial y^0}(0,0)$ of (4.26). By the block triangular structure of $D\Phi_0(0,0)$, i.e. swap the first and second components, the spectrum of $D\Phi_0(0,0)$ is the union of the spectra of $\frac{\partial \hat{y}}{\partial y^0}(0,0)$ and I_s. Therefore $\sigma(D\Phi_0(0,0))$ contains at least $2s$ unit eigenvalues, moreover by the assumption on $\sigma(D\Phi_0(0,0))$, the spectrum of $\frac{\partial \hat{y}^0}{\partial y^0}(0,0)$ belongs to the complement of a disk around 1. The operator $\frac{\partial F}{\partial y^0}(0,0)$ of (4.26) has thus a bounded inverse in Y and there exists some $\mu_3 > 0$ that satisfies (4.21). By the C^r regularity of the flows Φ_ϵ we can apply the implicit function theorem, i.e. there exists an $r_1 > 0$ such that for $(\epsilon, \beta^0) \in \mathcal{B}_{r_1}$ we have a unique map $(\epsilon, \beta^0) \mapsto \rho_m(\epsilon, \beta^0) \in Y^0$, with $F(\epsilon, \beta^0, \rho_m(\epsilon, \beta^0)) = 0$. The map is C^r in a nontrivial subset of $\mathcal{B}_{r_1}$. Note that there exist $\epsilon > 0$, β^0_* such that $(-\epsilon_0, \epsilon_0) \times (-\beta^0_*, \beta^0_*)^s \subset \mathcal{B}_{r_1}$, moreover, given any ϵ with $|\epsilon| < \epsilon_0$, there exists a $\beta^\epsilon_* > 0$ such that the set $V^\epsilon_m = (-\beta^\epsilon_*, \beta^\epsilon_*) \times \{0\} \subset \phi^\epsilon_m(\Sigma^3_m) \subset Z^0 \times Y^0$ is mapped to V^0_m by $(f^\epsilon_m)^{-1}$. We then define $\rho^\epsilon_m : V^\epsilon_m \to Y^0$ by

$$\rho^\epsilon_m(\beta^\epsilon) = \rho(\epsilon, (f^\epsilon_m)^{-1}(\beta^\epsilon, 0)), \tag{4.28}$$

i.e. $\rho^\epsilon_m(\beta^\epsilon) = \rho(\epsilon, \beta^0)$ with $\beta^0 = (f^\epsilon_m)^{-1}(\beta^\epsilon, 0)$. By the definition of the map ρ_m, and letting $\beta^0 = (f^\epsilon_m)^{-1}(\beta^\epsilon, 0)$, we have

$$\begin{aligned}&\hat{y}^\epsilon(\beta^\epsilon, \rho^\epsilon_m(\beta^\epsilon)) - \rho^\epsilon_m(\beta^\epsilon)\\ &\quad= \hat{y}(\epsilon, \beta(\epsilon, \beta^0, \rho_m(\epsilon, \beta^0)), \rho_m(\epsilon, \beta^0)) - \rho_m(\epsilon, \beta^0)\\ &\quad= F(\epsilon, \beta^0, \rho_m(\epsilon, \beta^0)) = 0,\end{aligned} \tag{4.29}$$

as required. Also, for $|\epsilon| < \epsilon_0$, the maps ρ^ϵ_m are C^r in β^ϵ, for all β^ϵ in some nontrivial subset of V^ϵ_m. The C^r smoothness of the ρ^ϵ_m in m follows from the C^r regularity of ρ_m on m, and the C^r regularity of the map from the variables (y^0, β^ϵ) to the variables (y^0, β^0). ■

Now, let $m \in \Lambda$ and define the map $\sigma^\epsilon_{\beta^\epsilon} : \Lambda \to \cup_{m\in\Lambda} U^\epsilon_m$ by

$$\chi^\epsilon_m(\sigma^\epsilon_{\beta^\epsilon}(m)) = (\beta^\epsilon, 0, \rho^{\beta^\epsilon}_m). \tag{4.30}$$

Also, let

$$\Lambda_{\epsilon,\beta^\epsilon} = \cup_{m\in\Lambda}(\beta^\epsilon, 0, \rho^\epsilon_m(\beta^\epsilon))_m = \sigma^\epsilon_{\beta^\epsilon}(\Lambda). \tag{4.31}$$

We will see that the set $\Lambda_{\epsilon,\beta^\epsilon}$ is invariant under the flow of the X^ϵ_j, and is diffeomorphic to the torus Λ, i.e. $\Lambda_{\epsilon,\beta^\epsilon}$ is the invariant torus we seek. To do this, first let

$$M^{\epsilon,\beta^\epsilon}_m = \{p \in V^\epsilon_m : \chi^\epsilon_m(p) = (\beta^\epsilon, \tau^\epsilon, \rho^\epsilon_m)\}. \tag{4.32}$$

We have the following.

Lemma 4.6. *Let $|\epsilon| \le \epsilon_0$, $\beta^\epsilon \in (-\beta^\epsilon_*, \beta^\epsilon_*)$. Then $M^{\epsilon,\beta^\epsilon}_m \subset \Lambda_{\epsilon,\beta^\epsilon}$, $\forall m \in \Lambda$.*

Proof. Consider $m \in \Lambda$, and $p \in M^{\epsilon,\beta^\epsilon}_m$. Then

$$p = (\chi^\epsilon_m)^{-1}((\beta^\epsilon, \tau^\epsilon, y^\epsilon)), \quad \text{with } y^\epsilon = \rho^\epsilon_m(\beta^\epsilon). \tag{4.33}$$

If $\tau^\epsilon = 0$ then $p \in \Lambda_{\epsilon,\beta^\epsilon}$, i.e. what we need to show. Suppose that $\tau^\epsilon \neq 0$. There exists some $m_1 \in \Lambda$ such that $p = (\chi^\epsilon_{m'})^{-1}((\beta^\epsilon_1, 0, y^\epsilon_1))$ and we want to show that $y^\epsilon_1 = \rho^\epsilon_{m_1}(\beta^\epsilon_1)$. Note that $\beta^\epsilon_1 = \beta^\epsilon$. Let

$$\begin{aligned}\hat{p} = \Phi_\epsilon(p) &= (\chi^\epsilon_m)^{-1}((\hat{\beta}^\epsilon, \hat{\tau}^\epsilon, \hat{y}^\epsilon))\\ &= (\chi^\epsilon_{m_1})^{-1}((\hat{\beta}^\epsilon, \hat{\tau}^\epsilon_1, \hat{y}^\epsilon_1)).\end{aligned} \tag{4.34}$$

We have that

$$g_\epsilon^{-(\hat{\tau}^\epsilon - \tau^\epsilon)}(\hat{p}) = p = (\chi^\epsilon_{m_1})^{-1}((\beta^\epsilon, 0, y^\epsilon_1)). \tag{4.35}$$

Also,

$$g_\epsilon^{-(\hat{\tau}^\epsilon - \tau^\epsilon)}(\hat{p}) = (\chi^\epsilon_{m_1})^{-1}((\beta^\epsilon, \hat{\tau}^\epsilon_1 - (\hat{\tau}^\epsilon - \tau^\epsilon), \hat{y}^\epsilon_1)) \tag{4.36}$$

since g^τ_ϵ does not change the third component. Comparing the third components in (4.35) and (4.36) we obtain that $\hat{y}^\epsilon_1 = y^\epsilon_1$. By Lemma 4.3 we therefore have that $y^\epsilon_1 = \rho^\epsilon_{m_1}(\beta^\epsilon)$. ■

Lemma 4.7. *The set $\Lambda_{\epsilon,\beta^\epsilon}$ is invariant under g^τ_ϵ, $\forall\tau \in \mathbf{R}^s$ and therefore invariant under the flow of the X^ϵ_j, $j = 1, \ldots, s$. Moreover, $\Lambda_{\epsilon,\beta^\epsilon}$ is C^r diffeomorphic to Λ.*

Proof. Let $p \in \Lambda_{\epsilon,\beta^\epsilon}$. Then there exists some $m \in \Lambda$ such that $p \in M^{\epsilon,\beta^\epsilon}_m$. By the invariance of $M^{\epsilon,\beta^\epsilon}_m$ under g^τ_ϵ, $\forall\tau$ (sufficiently near the origin), and Lemma 4.4 we have

$$g^\tau_\epsilon(p) \in M^{\epsilon,\beta^\epsilon}_m \subset \Lambda_{\epsilon,\beta^\epsilon}.$$

Therefore $\Lambda_{\epsilon,\beta^\epsilon}$ is invariant under g^τ_ϵ locally and hence globally in τ.

Furthermore, by (4.33) the function $\sigma^{\beta^\epsilon}_\epsilon$ maps Λ onto $\Lambda_{\epsilon,\beta^\epsilon}$ and is also injective, for $m_1 \neq m_2$ implies that $\Sigma_{m_1} \cap \Sigma_{m_2}$ is empty. Also, let $p \in \Lambda_{\epsilon,\beta^\epsilon}$. Then

$$p = (\chi^\epsilon_m)^{-1}((\beta^\epsilon, 0, \rho^\epsilon_m(\beta^\epsilon))), \quad m \in \Lambda.$$

Since $(\chi^\epsilon_m)^{-1}$ and ρ^ϵ_m are C^r on m by Lemmas 4.1 and 4.3 respectively, the map defined by the right-hand side is C^r on m. But this map is $\sigma^{\beta^\epsilon}_\epsilon$, i.e. it is a C^r diffeomorphism of Λ onto its image.

Lemma 4.8. *The motion on each torus $\Lambda_{\epsilon,\beta^\epsilon}$ above is conditionally periodic.*

To prove the lemma we show that for each torus we can construct "angular coordinates" $\boldsymbol{\phi}$ in which the flow on the torus is $\dot{\boldsymbol{\phi}} = \omega$. The frequency vector $\omega \in \mathbf{R}^n$ depends on $\Lambda_{\epsilon,\beta^\epsilon}$ (the construction is in [2], Ch. 10).

We now combine Lemmas 4.1–4.8, and Remarks 4.0.1, 4.1.1 and 4.4.1 to show Proposition 2.3.

Proof of Proposition 2.3. Consider ϵ, β as in the hypothesis. Let $(\mathcal{A}_{\epsilon,\beta}(\phi_0), \phi_0, J_{\epsilon,\beta}) \in \Lambda_{\epsilon,\beta}$ be the unique fixed point of Φ_ϵ that satisfies $(\mathcal{A}_{\epsilon,\beta}(\phi_0)) = (\mathcal{A}_0, \phi_0, 0) = m \in \Lambda$. Therefore $(\mathcal{A}_{\epsilon,\beta}(\phi_0), \phi_0) \in X \times \{\phi_0\}$ is a fixed point of $P_{1,2}\Phi_\epsilon$, with $P_{1,2}$ the projection to the first two components.

Consider the case where $\Phi_\epsilon = g^c_\epsilon$, with $c = c(\alpha)$, and $\alpha = [n_1, 1]$ for some $n_1 \in \mathbf{Z}$. Since the vector field X^ϵ_1 is equivariant under the flow of X^ϵ_2, the points $(e^{i\theta}\mathcal{A}_{\epsilon,\beta}(\phi_0), \phi_0)$, $\theta \in S^1$, are also invariant under Φ_ϵ. Moreover $e^{i\theta}\mathcal{A}_{0,0}(\phi_0) = e^{i\theta}\mathcal{A}_0$ by the uniqueness of the continuation. Thus $\overline{\Lambda}_{\epsilon,\beta} \cap X \times \{\phi_0\} =$

$\{e^{i\theta}\mathcal{A}_{\epsilon,\beta}(\phi_0) : \theta \in \mathbf{R}\}$. Observe that this set is a circle lying on a plane through the origin in X, and that the center of the circle is also at the origin. Also, since $\overline{\Lambda}_{\epsilon,\beta} \cap X \times \{\phi_0\}$ is invariant under $P_{1,2}g^c_\epsilon$, $P_{1,2}g^{c_2}_{2,\epsilon}$, it is also invariant under $P_{1,2}g^{c_1}_{1,\epsilon}$, the time-$T$ map of the flow of (2.4). The above arguments are valid for any $\phi_0 \in S^1$. The proposition then follows by setting $S^1(\epsilon, \beta, \phi_0) = \{e^{i\theta}\mathcal{A}_{\epsilon,\beta}(\phi_0) : \theta \in \mathbf{R}\}$. ■

Note that the proof of Theorem 2.2 implies the existence of a unique family of invariant tori $\Lambda_{\epsilon,\beta}$ for any $\alpha \in \pi_1(\Lambda)$ for which we can check the nonresonance condition on the Floquet map. The s-tori $\Lambda_{\epsilon,\beta}$, and ϵ_0, β_0 will in general depend on α (this was not made explicit in the notation so far). In the case of Proposition 2.1, the condition $\frac{\lambda}{\Omega} \neq \mathbf{Z}$ allows us to verify the nonresonance condition for any $\alpha = [n_1, 1]$, $n_1 \in \mathbf{Z}$, since (2.16)–(2.18) are independent of n_1. However the tori $\Lambda_{\epsilon,\beta}([n_1, 1])$, and $\epsilon_0([n_1, 1])$, $\beta_0([n_1, 1])$ obtained for different choices of n_1 coincide. This follows from the fact that g^c_ϵ, $c = c([n_1, 1])$, is independent of $n_1 \in \mathbf{Z}$. Also, $\Lambda_{\epsilon,\beta}([0, 1])$ is invariant under the maps g^c_ϵ, $c = c(\alpha)$, for any $\alpha \in \pi_1(\Lambda)$. This holds because $g^{c_1}_{1,\epsilon}$, $c_1 = c_1([n_1, n_2])$, is the n_2-th iterate of $g^{\tilde{c}_1}_{1,\epsilon}$, where $\tilde{c}_1 = c_1([n_1, 1])$, regardless of the fact that the nonresonance may not be satisfied for some choice of n_2. In such a case g^c_ϵ, with $c = c([n_1, n_2])$, may have other families of invariant tori that are continued from Λ.

It is also clear from the proof of Theorem 2.2 that in the case where the Floquet map obtained for some $\alpha \in \pi_1(\Lambda)$ has an eigenvalue of finite multiplicity greater than $2s$, one may be able to replace the implicit function argument of Lemma 4.5 by a Lyapunov–Schmidt reduction statement and obtain the existence of perturbed invariant tori by analyzing a bifurcation equation. This may be an alternative (and more general) approach for analyzing the continuation of the multi-peak breathers of the anticontinuous limit.

We also include the implicit function theorem (see e.g. [15], Ch. 4):

Lemma 4.9. *Let* $\mathbf{X}$, $\mathbf{Y}$ *be Banach spaces,* $(\mathbf{x}_0, \mathbf{y}_0)$ *a point in* $\mathbf{X} \times \mathbf{Y}$, *and* U *a neighborhood of* $(\mathbf{x}_0, \mathbf{y}_0)$ *in* $\mathbf{X} \times \mathbf{Y}$. *Consider a function* $F : U \to \mathbf{Y}$, *that satisfies* $F(\mathbf{x}_0, \mathbf{y}_0) = 0$. *Assume that* F *is continuous in* U, *that* D_2F *exists and is continuous in* U, *and that* $[D_2F](\mathbf{x}_0, \mathbf{y}_0) \in B(\mathbf{Y})$ *has a bounded inverse. Let* $M_2 > 0$ *satisfy*

$$\|([D_2F](\mathbf{x}_0, \mathbf{y}_0))^{-1}\|_0 < M_2 \tag{4.37}$$

and consider r_1, $r_2 > 0$, $\mathcal{B}_{r_1}(\mathbf{x}_0) \times \mathcal{B}_{r_2}(\mathbf{y}_0) \subset U$, *satisfying*

$$\sup_{(\mathbf{x},\mathbf{y}) \in \mathcal{B}_{r_1}(\mathbf{x}_0) \times \mathcal{B}_{r_2}(\mathbf{y}_0)} \|D_2F(\mathbf{x}, \mathbf{y}) - D_2F(\mathbf{x}_0, \mathbf{y}_0)\|_0 < \frac{1}{2M_2}, \tag{4.38}$$

$$M_2 \sup_{\mathbf{x} \in \mathcal{B}_{r_1}(\mathbf{x}_0)} \|F(\mathbf{x}, \mathbf{y}_0)\|_Y < \frac{1}{2} r_2. \tag{4.39}$$

Then there exists a unique function $g : \mathcal{B}_{r_1}(\mathbf{x}_0) \to \mathbf{Y}$ *satisfying* $g(\mathbf{x}_0) = \mathbf{y}_0$, *and* $F(\mathbf{x}, g(\mathbf{x})) = 0$, $\forall \mathbf{x} \in \mathcal{B}_{r_1}(\mathbf{x}_0)$. *Also* $g(\mathcal{B}_{r_1}(\mathbf{x}_0)) \subset \mathcal{B}_{r_2}(\mathbf{y}_0)$. *If in addition,* F *is* C^r *in* U, $r \geq 1$, *then* g *is* C^r *for* x *in some* $\mathcal{B}_{\tilde{r}_1}(\mathbf{x}_0)$, *where* $\tilde{r}_1 > 0$.

5. Discussion

We have seen that several types of breathers of the discrete NLS can be continued to solutions of the discrete NLS with weak diffraction management. The continuation argument we gave applies to single-peak breathers of the anticontinuous limit and we also show numerical evidence that a similar continuation should be possible for multi-peak breathers of the discrete NLS with small residual diffraction. A topic for further work is to obtain asymptotics of the Floquet spectra for multi-peak breathers. The continuation problem is also interesting for other types of breathers and in the present work we considered breathers whose existence and Floquet spectra seem tractable by perturbation arguments.

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