Continuation of breathers in the discrete NLS with rapid parametric forcing

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Abstract.

We study the existence breather-type localized solutions in the discrete NLS equation with high frequency time-periodic parametric forcing. The question is formulated as a problem of persistence for breathers of an autonomous averaged equation that approximates the full system. We show that single-peak breathers of the averaged equation with vanishing residual diffraction and small forcing amplitude to forcing frequency ratio can be continued to periodic or quasiperiodic solutions of the full system, provided that the frequency is sufficiently large. We also present numerical results on possible extensions of the theory to wider classes of breathers.

1. Introduction

In this paper we show the existence of localized periodic and quasiperiodic solutions in a discrete NLS equation with high frequency periodic parametric forcing. The equation we consider is also known as the discrete NLS with diffraction management and was originally proposed by [AM] to describe light propagation in an array of coupled waveguides with the experimental geometry studied by [ESMA].

Localized solutions of the parametrically forced discrete NLS model of [AM] have been primarily studied through an autonomous system, referred to as the averaged system, that approximates the full system in the high frequency forcing regime (see [M]). The averaged equation has localized breather solutions (see [M], [P2]), as well as multipeak breather solutions (see [P3]) and a natural question is whether such breather solutions can be continued (in an appropriate sense) to solutions of the full nonautonomous system.

In the present work we consider the continuation question for single-peak breather solutions of the averaged equation with vanishing residual diffraction and small forcing amplitude (β) to forcing frequency (Ω) ratio. We show that such breathers can be continued to periodic and quasiperiodic solutions of the full system, provided that the forcing frequency is sufficiently large. The result can be also interpreted in terms of the time$-T$ map of the nonautonomous full system, where $T = \frac{2\pi}{\Omega}$ is the period of the forcing. In particular, we show the existence of invariant circles of an appropriate iterate of the time$-T$ map of the full system that are $\Omega^{-1}-$close to the breather orbit of the averaged equation.

A related problem is that of continuing breathers of the discrete NLS to solutions of the parametrically forced NLS of [AM] with small amplitude parametric forcing, i.e. with Ω fixed and β small. In the small amplitude forcing regime we have shown (see [P4]) an analogous continuation result for single-peak breathers of the anticontinuous limit NLS by developing a framework that can be used to answer the continuation question for more general breather solutions of the discrete NLS. An extension of the continuation results to a class of multi-peak breathers is presented in [PP] (see also [P4] for some numerical evidence). In the high frequency case we follow a similar plan, considering first the continuation problem for the simplest breather and expecting to extend the results to other breathers in further work.

The continuation of breathers relies on an infinite-dimensional version of a result of [N] and [BG] on the continuation of invariant tori in Hamiltonian systems with additional conserved quantities (see [BV] for another application). In the case of small amplitude time-periodic perturbations of the discrete NLS the idea is to first write the full system as an autonomous Hamiltonian system in an extended phase space and view the parametric forcing term as a perturbation of the discrete NLS Hamiltonian. Breathers of the discrete NLS can be then interpreted as invariant $2-$tori and we want to show the existence of nearby invariant $2-$tori in the perturbed system. To continue the tori we must verify a nonresonance condition on the Floquet spectrum of a linear combination of the Hamiltonian flows of the Hamiltonian, and an additional integral of the discrete NLS. In the high frequency case we arrive at a similar setup by applying a symplectic change of coordinates that transforms the original system to the averaged equation plus a $T-$periodic remainder of size $O(\Omega^{-1})$. (The change of coordinates is defined in a neighborhood of the origin that is independent of Ω for large Ω.) The unperturbed system is then the averaged equation and the $2-$tori correspond to its breather solutions. The goal is to continue the $2-$tori to the system that contains the remainder.

In contrast to the small amplitude case, where the amplitude of the parametric forcing does not appear in the discrete NLS, in the high frequency continuation problem the "small parameter" Ω^{-1} that controls the size of the perturbation also appears in the averaged equation and affects the distance from the resonance. Instead we replace Ω^{-1} in part of the perturbed Hamiltonian by an artificial parameter ϵ and show that the continuation is possible for $|\epsilon| < \epsilon_0$, with an $\epsilon_0 > 0$ that can be chosen independently of Ω for Ω sufficiently large (i.e. for $\Omega > \Omega_0$ for some $\Omega_0 > 0$). The perturbed invariant $2-$tori we find imply the existence of invariant circles of iterates of the time$-T$ map of the perturbed system. The number of iterates is roughly proportional to Ω so that the time required to return to the invariant circle is independent of Ω. As in the small amplitude case, the invariant circles are near the breather orbits of the averaged equation.

The continuation is shown for arbitrary forcing amplitude β and we also discuss the dependence of Ω_0 on β. We speculate that the continuation argument should be valid for a region below a line through the origin in the Ω-β plane. In the current continuation proof the slope of this line is assumed small, however this restriction seems to be technical. The possiblity of extending the continuation results to more breathers is examined numerically.

The paper is organized as follows. In chapter 2 we formulate the breather continuation problem for small amplitude parametric forcing, state a theorem on continuation of tori in equivariant Hamiltonian systems, and apply it to a simple case. In chapter 3 we formulate the breather continuation problem for high frequency parametric forcing and state the main continuation result of the paper. Two preliminary steps are the definition of the symplectic transformation that leads to the averaged equation, and a statement on the existence of single-peak breathers. We then formulate and prove the auxiliary continuation statement that involves the artificial parameter ϵ. We also present some numerical results on possible extensions of the theory. In chapter 4 we show the theorem on continuation of tori in equivariant Hamiltonian systems. In the proof we make explicit the quantities that determine the size of the forcing amplitude threshold ϵ_0 and show that they can be bounded uniformly in Ω, for Ω sufficiently large. Chapter 5 contains some technical lemmas used in chapter 4.

2. Localized solutions for small amplitude forcing

We consider the parametrically forced discrete cubic nonlinear Schrödinger equation

$$\partial_t u = iD(t)\Delta u - 2i\gamma g(u), \quad \text{with} \tag{2.1}$$

$$(\Delta u)_j = u_{j+1} - 2u_j + u_{j-1}, \quad g_j(u) = |u_j|^2 u_j \tag{2.2}$$

and u a complex valued function on the integers $\mathbf{Z}$. (f_j denotes the value of $f : \mathbf{Z} \to \mathbf{C}$ at the site j.) Also, γ is a real constant and D is a $T-$periodic real valued function (for some $T > 0$). We also write

$$D(t) = \overline{D} + \tilde{D}(t), \quad \text{where} \quad \overline{D} = \frac{1}{T}\int_0^T D(\tau)d\tau \tag{2.3}$$

is the average over the period.

Equation (2.1)-(2.2) is a non-autonomous Hamiltonian system in $X = l_2(\mathbf{Z}^d, \mathbf{C})$, the set of square-summable complex valued functions on $\mathbf{Z}$ with the real inner product $\langle u, v\rangle = \mathrm{Re}\sum_{n\in\mathbf{Z}} u_n v_n^*$, and corresponding norm $||.||$. The Hamiltonian structure is specified below. Physically, t in (2.1) is the distance along the waveguides, and u_j is the complex amplitude of (any) one of the components of the electric field at the waveguide j (see [AM], [ESMA]). The initial condition $u(t_0)$ for (2.1) is the emitted light.

Our goal is to examine the existence of localized periodic or quasiperiodic localized solutions of (2.1). The strategy be will be to consider (2.1) as a perturbation of a simpler system with known periodic localized solutions of breather type. The existence of localized periodic or quasiperiodic localized solutions of (2.1) is then formulated as a continuation problem.

This general strategy is applied to two parameter parameter regimes where the systems that approximate (2.1) are different. In this chapter we consider simpler the case where the amplitude of the function $\tilde{D}$ is small. We thus set $\tilde{D} = \epsilon\tilde{d}(\Omega t)$, $\Omega = \frac{2\pi}{T}$, with $\tilde{d}$ a $2\pi-$periodic function with zero average and consider the limit of $\epsilon \to 0$. System (2.1) with $\epsilon = 0$ is referred to as the discrete NLS. Breathers are solutions of the discrete NLS that have the form $u_n = e^{-i\lambda\overline{\mathcal{A}}}$, with $\lambda \in \mathbf{R}$, $\overline{\mathcal{A}} \in X$. There are several results on the existence of breather solutions and we here formulate and show and answer the continuation problem for a simple example that is easily tractable and illustrates the main idea. Extensions are developed elsewhere. The main tool, Theorem 2.2, is also used in the next section where we consider the high frequency forcing case.

To formulate the small amplitude forcing problem we write (2.1)-(2.2) with the initial condition $u(t_0) = v \in X$ as

$$\partial_t u = i\overline{D}\Delta u - 2i\gamma g(u) + \epsilon d(\phi)\Delta u, \quad \dot{\phi} = \Omega, \tag{2.4}$$

$\phi \in S^1$, with the initial condition $u(0) = v$, $\phi(0) = \phi_0$. Equation (2.4) is written as a Hamiltonian system by adding an extra variable $J \in \mathbf{R}$. The phase space will be $X \times S^1 \times \mathbf{R}$. The Hamiltonian H_ϵ is

$$H_\epsilon = -\Omega J + \sum_{j\in\mathbf{Z}} \left((\overline{D} + \epsilon d(\phi))|u_{j+1} - u_j|^2 + \gamma|u_j|^4\right), \tag{2.5}$$

and we formally obtain (2.4) by the first two of Hamilton's equations

$$\partial_t u = -i\frac{\partial H_\epsilon}{\partial u^*}, \quad \dot{\phi} = -\frac{\partial H_\epsilon}{\partial J}, \quad \dot{J} = \frac{\partial H_\epsilon}{\partial \phi}. \tag{2.6}$$

To simplify we assume that $\overline{D} = 0$. Let $\epsilon = 0$. Then for any $n_0 \in \mathbf{Z}$, $A \in \mathbf{C}\setminus\{0\}$, and $\phi_0 \in S^1$ we have the "one-peak" breather solution

$$u_{n_0}(t) = e^{-i\lambda t}A \quad \text{with} \quad \lambda = 2\gamma|A|^2; \quad u_n(t) = 0, \quad \forall t \in \mathbf{R} \quad \text{if} \quad n \neq n_0; \tag{2.7}$$

$$\phi(t) = \Omega t + \phi_0; \quad J(t) = 0, \quad \forall t \in \mathbf{R}. \tag{2.8}$$

The choice $J = 0$ is arbitrary. Let $C(n_0, A, \phi_0)$ be the set of points of $X \times S^1 \times \mathbf{R}$ in the orbit defined by (2.7), (2.8). Also let $\Lambda_0(n_0, A) = \cup_{\phi_0\in S^1} C(n_0, A, \phi_0)$. The set $\Lambda_0(n_0, A)$

is an invariant $2-$torus of the $\epsilon = 0$ system, e.g. the torus is foliated by periodic orbits if $\frac{\lambda}{\Omega}$ is rational. We now consider (2.6) with $|\epsilon|$ small.

Proposition 2.1 Consider $n_0 \in \mathbf{Z}$, $A \in \mathbf{C} \setminus \{0\}$, and the set $\Lambda_0(n_0, A)$ as above. Let r be an integer, $r \geq 2$, and suppose that the $2\pi-$periodic function d above is C^r in $\mathbf{R}$. Assume that $\frac{\lambda}{\Omega} \notin \mathbf{Z}$, where $\lambda = 2|\gamma||A|^2$ (and $\Omega \neq 0$). Then there exist ϵ_0, $\beta_0 > 0$ such that for any ϵ with $|\epsilon| < \epsilon_0$ the corresponding system (2.4) has a C^r $2-$parameter family of invariant $2-$tori $\Lambda_{\epsilon,\beta}$, with $\beta \in (-\beta_0, \beta_0)^2$, and $\Lambda_{0,0} = \Lambda_0(n_0, A)$. The motion on each torus is periodic or quasiperiodic (with two quasiperiods). Also, for any $\epsilon \in (-\epsilon_0, \epsilon_0)$ there is a two-parameter family of C^r functions $f_{\epsilon,\beta} : \Lambda_0(n_0, A) \to X \times S^1$, $\beta \in (-\beta_0, \beta_0)^2$ with $\Lambda_{\epsilon,\beta} = f_{\epsilon,\beta}(\Lambda_0(n_0, A))$, and $f_{0,0}(\Lambda_0(n_0, A)) = \Lambda_0(n_0, A)$.

Proposition 2.1 was shown in [P4] and is also included here. The proof is based on Theorem 2.2 below.

To set up Theorem 2.2, let r be an integer, $r \geq 2$, and consider a real C^r Hilbert manifold M modeled on a real (separable) Hilbert space E with inner product $\langle\, , \rangle$. Assume that M also has a weak symplectic structure ω with corresponding Poisson bracket $\{\ ,\ \}$. Consider s real functions H_1^ϵ, ..., H_s^ϵ on M that have the form $H_j^\epsilon = H_j^0 + \epsilon \tilde{H}_j$, $j = 1$, ..., s. (The parameter ϵ is real and the H_j^0, $\tilde{H}_j$ are independent of ϵ.) The Hamiltonian vector fields of H_j^ϵ, H_j^0, $\tilde{H}_j$ are respectively denoted by X_j^ϵ, X_j^0, $\tilde{X}_j$, $j = 1$, ..., s. We are assuming that s is finite, and in the case of $\dim(M) = 2n$ that also $1 \leq s \leq n$. We further assume that there exists $\tilde{\epsilon} > 0$ such that for any $\epsilon \in (-\tilde{\epsilon}, \tilde{\epsilon})$ the following hold:

A I The Hamiltonian vector fields X_j^ϵ of the H_j^ϵ, $j = 1$, ..., s are C^r, and their time$-t$ maps exist and are C^r in M, $\forall t \in \mathbf{R}$.

A II There exists an $s-$dimensional torus Λ that is invariant under the Hamiltonian flows of the H_j^0, $j = 1$, ..., s. Moreover, Λ is a C^r submanifold of M and has a C^r tubular neighborhood in M.

A III The H_j^ϵ, $j = 1$, ..., s mutually Poisson commute and are functionally independent in a neighborhood of Λ in M.

We are interested on on whether the invariant torus Λ of X_1^0 can be continued to an invariant torus of the perturbed system X_1^ϵ for $|\epsilon|$ sufficiently small. First note that given any $\alpha \in \pi_1(\Lambda) \simeq \mathbf{Z}^s$ there exists a $c = [c_1, \ldots, c_s] \in \mathbf{R}^s$ such that the integral curves of the restriction of the vector field $K_0(\alpha) = \sum_{j=1}^s c_j X_j^0$ to Λ are $1-$periodic orbits that belong to the homotopy class α. Denote the time-1 map of $K_0(\alpha)$ by g_0^c. The Fréchet derivative $Dg_0^c(m)$ of g_0^c at any $m \in \Lambda$ is a bounded linear operator in $E \simeq T_m M$. It is easily seen that the derivatives at two different points of Λ are related by a similarity transformation. The spectrum of $Dg_0^c(m)$ in E is therefore independent of the point $m \in \Lambda$ and will be denoted by $\sigma(Dg_0^c)$.

Theorem 2.2 Consider the functions H_j^ϵ, $j = 1, \ldots, s$ as above and assume that there exists $\alpha \in \pi_1(\Lambda)$ and a corresponding vector $c = c(\alpha) \in \mathbf{R}^s$ with the property that $\sigma(Dg_0^c)$ has exactly s eigenvalues that are unity and that $\sigma(Dg_0^c) \setminus \{1\}$ lies outside an open disc around 1. Then there exist ϵ_0, $\beta_0 > 0$ such that for any ϵ with $|\epsilon| < \epsilon_0$ there exists an $s-$parameter family of $s-$tori $\Lambda_{\epsilon,\beta}$, $\beta \in (-\beta_0, \beta_0)^s$, that are invariant under the flow of each of the X_j^ϵ, $j = 1, \ldots, s$. The motion on each $\Lambda_{\epsilon,\beta}$ is periodic or quasiperiodic (with at most s quasiperiods). The family $\Lambda_{\epsilon,\beta}$ is also C^r in β and there exists $\beta_* \in (-\beta_0, \beta_0)^s$ for which $\Lambda_{0,\beta_*} = \Lambda$.

Theorem 2.2 is a generalization of results of [N], [BG] and is proved in [P4] using the Poincare map construction of [BG]. We here give a different version of the proof. The goal is to understand better the particular case of Proposition 2.1, and especially how ϵ_0, and β_0 depend on the parameters of equation (2.1).

Proof of Proposition 2.1: Let $M = X \times S^1 \times \mathbf{R}$, and $H_1^\epsilon = H_\epsilon$, with H_ϵ as in (2.5). The corresponding Hamiltonian vector field is denoted by X_1^ϵ. Also, let $\Lambda = \Lambda_0(n_0, A)$, with $\Lambda_0(n_0, A)$ as defined by (2.7). Thus is Λ is a $2-$torus that is invariant under the vector field X_1^0. Also let $P = \sum_{j \in \mathbf{Z}} |u_j|^2$ and define a second family of functions H_2^ϵ by $H_2^\epsilon = P$, $\forall \epsilon \in \mathbf{R}$. The corresponding Hamiltonian vector field is denoted by X_2^ϵ. We observe Λ is also invariant under X_2^0 and we check that H_1^ϵ, H_2^ϵ satisfy the conditions AI-AIII of Theorem 2.2. To verify the nonresonance condition for the Floquet map of an appropriate linear combination of X_1^0, X_2^0 we parametrize $\Lambda \in M$ and define the function $\overline{\mathcal{A}} : \mathbf{Z} \to \mathbf{C}$ by $\overline{\mathcal{A}}_{n_0} = A$, $\overline{\mathcal{A}}_n = 0$ for $n \neq n_0$. Then

$$\Lambda = \{[e^{i\theta}\overline{\mathcal{A}}, \phi, 0] \in X \times S^1 \times \mathbf{R} : \quad \theta \in \mathbf{R}, \quad \phi \in S^1\}. \tag{2.9}$$

Also, given any $F : M \to \mathbf{R}$, let g_F^t be the time$-t$ maps of the flows of the Hamiltonian vector field of F. On Λ we then have

$$g_{H_1^0}^t([e^{i\theta}\overline{\mathcal{A}}, \phi, 0]) = [e^{i(\theta - \lambda t)}\overline{\mathcal{A}}, (\phi + \Omega t)\mathrm{mod}2\pi, 0], \tag{2.10}$$

$$g_{H_2^0}^t([e^{i\theta}\overline{\mathcal{A}}, \phi, 0]) = [e^{i(\theta - t)}\overline{\mathcal{A}}, \phi, 0]. \tag{2.11}$$

Also, $g^t_{cF} = g^{ct}_F$, $\forall c \in \mathbf{R}$, therefore the time$-t$ map of the Hamiltonian vector field of $c_1 H_0 + c_2 P$ is $g^{c_1 t}_{H_0} g^{c_2 t}_P$. Using (2.10), (2.11) we therefore see that the condition for the orbits of the Hamiltonian vector field of $c_1 H_0 + c_2 P$ to be $1-$periodic on Λ and to belong to the homotopy class $[n_1, n_2] \in \mathbf{Z}^2$ is that $-c_1\lambda - c_2 = 2\pi n_1$, and $c_1 \Omega = 2\pi n_2$, hence

$$c_1 = n_2 \frac{2\pi}{\Omega}, \quad c_2 = -2\pi n_1 - n_2 \frac{2\pi\lambda}{\Omega}. \tag{2.12}$$

To calculate the Floquet map around any such $1-$periodic orbit, i.e. Dg^c_0, let

$$u = e^{-i(c_1\lambda + c_2)t} e^{i\theta} \overline{\mathcal{A}} + w, \quad \phi = c_1 \Omega t + \phi_0 + \psi, \quad J = I, \tag{2.13}$$

with $\overline{\mathcal{A}}$ as above. Using (2.1), (2.5), (2.6) and keeping only linear terms we obtain the variational equation for $\dot{w}$, $\dot{\psi}$, $\dot{I}$. The variational equation is made autonomous by the change of variables $v = e^{i(c_1\lambda + c_2)t} w$. We obtain

$$\dot{v}_n = i c_1 \lambda v_n, \quad n \in \mathbf{Z} \setminus \{n_0\} \tag{2.14}$$

$$\dot{v}_{n_0} = -i c_1 \lambda (v_{n_0} + v^*_{n_0}), \qquad \dot{\psi} = 0, \quad \dot{I} = 0. \tag{2.15}$$

Since $v = w$ at $t = 1$, the Floquet map around the $1-$periodic orbit coincides with the time-1 map of the equations for $\dot{v}$, $\dot{\psi}$, $\dot{I}$. The spectrum of the time-1 map of the linear system (2.14)-(2.15) is calculated readily since the system is block diagonal with 2×2 blocks: the second and third equations of (2.15) yield two unit eigenvalues. The first equation of (2.15) yields another pair of unit eigenvalues. Finally, equations (2.14) yield the pair of eigenvalues $e^{\pm 2\pi i n_2 \frac{\lambda}{\Omega}}$ for each integer $n \neq n_0$. Choose $[n_1, n_2] = [-1, 1]$. Then, by our assumption that $\frac{\lambda}{\Omega} \notin \mathbf{Z}$ we have exactly 4 unit eigenvalues with the rest of the Floquet spectrum bounded away from unity and the proposition follows from Theorem 2.2.

■

Note that each invariant $2-$torus $\Lambda_{\epsilon,\beta} \subset X \times S^1 \times \mathbf{R}$ of the Hamiltonian system (2.6) obtained by Proposition 2.1 after verifying the nonresonance condition associated to a homotopy class $\alpha = [n_1, n_2]$ projects to an invariant $2-$torus $\overline{\Lambda}_{\epsilon,\beta} \subset X \times S^1$ of the first two equations of (2.6). From the proof of Theorem 2.2 (see also [P4], Proposition 2.3) we see that this projection yields a $1-$parameter family of $2-$tori $\overline{\Lambda}_{\epsilon,\beta}$ (see [P4], Proposition 2.3). (The parameter corresponds to the l_2 norm of the X component.) We furthermore see that the intersection of any $\overline{\Lambda}_{\epsilon,\beta}$ with any hyperplane $X \times \{\phi_0\}$ is an invariant circle of the time$-n_2 T$ map of the flow of (2.1). Each of the circles we obtain by varying ϕ_0 belongs to a $2-$dimensional plane through the origin in X and is centered at the origin of X. The plane generally depends on θ_0. Also the time$-n_2 T$ map of the flow of (2.1) induces a rigid rotation on the circles. A more complete discussion of geometrical interpretation of the tori and the choice of homotopy class is presented in [P4].

The setup of the proof of Proposition 2.1 works for more general breathers: for $u_n = e^{-i\lambda t}\overline{\mathcal{A}}$, $\overline{\mathcal{A}} \in X$, a solution of (2.1) with $\tilde{D} \equiv 0$, $\overline{D} \in \mathbf{R}$ (the discrete NLS), the parametrization of the torus Λ in (2.9), the expressions for the actions in (2.10), (2.11), and the c_1, c_2 in (2.12) are the same as for the example of Proposition 2.1. The variational equation is different: using (2.13), and the change of variables $v = e^{i(c_1\lambda + c_2)t}w$, we obtain

$$\dot{v}_n = c_1[i\lambda v_n + i\overline{D}(\Delta v)_n - 2i\gamma((\overline{\mathcal{A}}_n)^2 v_n^* + 2|\overline{\mathcal{A}}_n|^2 v_n)]. \tag{2.16}$$

Further study the spectrum of the right hand side of (2.16) allows us to extend the continuation results to other types of breathers, e.g. multipeak breathers. Some extensions are presented in [PP] (see also [P4] for a numerical study).

3. Localized solutions for high frequency forcing

We now study the continuation question in the parameter regime where the oscillating part $\tilde{D}$ has high frequency. Specifically, let $\tilde{D}(t) = \beta\tilde{d}(\Omega t)$, with $\tilde{d}$ a $2\pi-$periodic function, $\beta > 0$, and $\Omega = \frac{2\pi}{T} > 0$. Thus, in contrast to the parameter range of Proposition 2.1, we here fix β, and the function $\tilde{d}$ and consider Ω large.

To set up the continuation problem we first consider (2.1) and define the new variable b by

$$u(t) = \tilde{L}_t b(t), \quad \text{with} \quad \tilde{L}_t = e^{i\tilde{\Lambda}(t)\Delta}, \quad \text{and} \quad \tilde{\Lambda}(t) = \int_0^t \beta\tilde{d}(\Omega\tau)d\tau. \tag{3.1}$$

By (3.1), (2.1) the evolution equation for b is then

$$\partial_t b = i\overline{D}\Delta b - 2i\gamma\tilde{L}_t^\dagger g(\tilde{L}_t b), \tag{3.2}$$

with the initial condition $b(t_0) = b_0 = \tilde{L}_{t_0}^{-1}u(0)$. Note that $\tilde{L}_t^{-1} = \tilde{L}_t^\dagger = e^{i\tilde{\Lambda}(t)\Delta}$. System (3.2) is equivalent to the autonomous system

$$\partial_t b = i\delta\Delta b - 2i\gamma L_\phi^\dagger g(L_\phi b), \quad \dot{\phi} = \Omega, \tag{3.3}$$

where $\delta = \overline{D}$, and $L_\phi = \tilde{L}_{\frac{\phi}{\Omega}}$. The initial condition is $b(0) = b_0$, $\phi(0) = \phi_0$. The fact that $\tilde{L}_t$ is $T-$periodic in t implies that L_ϕ, and the cubic nonlinearity in (3.3) are $2\pi-$periodic in ϕ. In the range where δ, γ are $O(1)$ and Ω is large we can heuristically approximate (3.3) by the averaged system

$$(3.4) \qquad \partial_t a = i\delta\Delta a - 2i\gamma\overline{g}_L(a), \quad \dot{\phi} = \Omega \quad \text{with} \quad \overline{g}_L(a) = \frac{1}{2\pi}\int_0^{2\pi} L_\theta^\dagger g(L_\theta a)d\theta.$$

The dependence on the angle ϕ has been therefore averaged out of the nonlinearity.

Remark 3.0.1 The averaged equation can be also defined without introducing the angle ϕ. In particular, we have $\overline{g}_L(a) = \frac{1}{T}\int_0^T \tilde{L}_\tau^\dagger g(\tilde{L}_\tau a)d\tau$. We also note that the rescaling $\beta \to s\beta$, $\Omega \to s\Omega$ leaves $\overline{g}_L$ invariant and that $\frac{\beta}{\Omega} \to 0$ implies that $\sup_{t\in\mathbf{R}}|\tilde{\Lambda}(t)| \to 0$. Thus, for any fixed $2\pi-$periodic function $\tilde{d}$, the nonlinearity $\overline{g}_L(v)$ depends only on the ratio $\frac{\beta}{\Omega}$.

Denote the set of bounded linear operators in X by $B(X)$ and let $||\cdot||_0$ denote the operator norm on $B(X)$. For $\tilde{\Lambda}$ bounded, the exponential in (3.1) is well defined and $L_t \in B(X)$, for all $t \in \mathbf{R}$. The fact that $\tilde{\Lambda}(t)\Delta$ is a one-parameter family of bounded operators also implies that the map $t \mapsto L_t$ from $\mathbf{R}$ to $B(X)$ is norm continuous in $\mathbf{R}$. This follows from basic results on the abstract Cauchy problem (see e.g. [F], ch. 7). The operators L_t also depend on Ω (this is not explicit in the notation at this point). Fixing the $2\pi-$periodic function $\tilde{d}$ and scaling Ω outside the integral of (3.1), we see that for any fixed $t \in \mathbf{R}$, the maps $\Omega \mapsto L_t$ from $\mathbf{R}^+$ to $B(X)$ are norm continuous in $\mathbf{R}^+$. Since L_t is $T-$periodic in t, the continuity is uniform in $t \in \mathbf{R}$.

Both (3.2) and (3.3) can be made formally into a Hamiltonian system in the standard way, adding a real variable J as before. For instance, the Hamiltonian $\overline{H}$ for (3.4) is

$$(3.5) \qquad \overline{H} = -\Omega J + \sum_{j\in\mathbf{Z}}\left(\delta|a_{j+1} - a_j|^2 + \frac{\gamma}{2\pi}\int_0^{2\pi}|(L_\psi a)_j|^4 d\psi\right),$$

The Hamiltonian structure is as in chapter 2 and its meaning can be made rigorous using the boundedness of the operators L_t and the norm continuity of the family L_t in t mentioned above. Also, we can show that there exists a symplectic change of variables which transforms (3.2) into (3.3) plus a small term that depends on ϕ. Let $\mathcal{B}_R$ denote the ball of radius R in X. We have:

Proposition 3.1 Assume that $\tilde{d}$ is piecewise continuous, $|\delta|$, $|\gamma| < C_1$. Let $b(t)$ be a solution of (3.3). There exists $\Omega_1 > 0$ such that if $\Omega > \Omega_1$ there exist $\rho > 0$ independent of Ω, and a symplectic map $\mathcal{T} : \mathcal{B}_\rho \times S^1 \times \mathbf{R} \to X$ for which the variable a defined by $b = \mathcal{T}(a)$, $a \in \mathcal{B}_\rho$, satisfies

$$\partial_t a = i\delta\Delta a - 2i\gamma\overline{g}_L(a) + \frac{1}{\Omega}\tilde{g}_1(a,\phi), \quad \dot{\phi} = \Omega, \tag{3.6}$$

where $\tilde{g}_1(a,\phi)$ is $2\pi-$periodic in ϕ and satisfies

$$||\tilde{g}_1(v,\phi)|| < C, \quad \forall a \in \mathcal{B}_\rho, \quad \forall \phi \in S^1 \tag{3.7}$$

with $C > 0$ that depends on C_1, and ρ (and is independent of Ω, and β). In the case where $\tilde{d}$ is C^r, $r \geq 2$, then g_1 is C^r in $\mathcal{B}_\rho \times S^1 \times \mathbf{R}$. Furthermore, let $\tilde{H} = H \circ \mathcal{T}$ be the Hamiltonian in the new coordinates, and $\tilde{P} = P \circ \mathcal{T}$. Then $\tilde{P} = P$.

The proposition follows from the normal form arguments in [P1] and we only sketch the formal part. We work in $X \times \mathbf{R} \times \mathbf{R}$, the covering space of the extended phase space $X \times S^1 \times \mathbf{R}$. The Hamiltonian H corresponding to (3.3) is written as $H = h_0 + h_2 + h_4$, where $h_0 = -\Omega J$, with h_2, h_4 the quadratic, and quartic parts (in u) respectively. The time-1 map $\mathcal{T}$ of the the Hamiltonian flow of S changes H to

$$H \circ \mathcal{T} = -\Omega J + h_2 + h_4 + [S, -\Omega J] + R, \tag{3.8}$$

where the remainder R contains the remaining terms. We want to choose S so that $h_4 + [S, -\Omega J] = \overline{h}_4$, where $\overline{h}_4$ is the quartic term of $\overline{H}$ in (3.5), and [,] is the Poisson bracket in the extended phase space. The solution is

$$S(u,\phi) = \frac{1}{\Omega}\int_0^\phi \left[h_4(u,\psi) - \overline{h}_4(u)\right] d\psi. \tag{3.9}$$

The fact that S is independent of J implies that $\phi \circ \mathcal{T} = \phi$. The new Hamiltonian can be written as

$$H \circ \mathcal{T} = -\Omega J + h_2 + \overline{h}_4 + R, \quad \text{with} \tag{3.10}$$

$$R = ((\exp S)h_2 - h_2) + ((\exp S)h_4 - h_4) + ((\exp S)h_0 - h_0 - [S, h_0])\,. \tag{3.11}$$

The term $\Omega^{-1}\tilde{g}_2$ in (3.6) is the Hamiltonian vector field of R. The size of each of the three terms of R is determined respectively by the size of $[h_2, S]$, $[h_4, S]$, and $[[S, h_0], S] = [\overline{h}_4, S]$, each of $O(\Omega^{-1})$. The rigorous estimates of the Hamiltonian vector field of R follow from analogous estimates of $\mathcal{T}$ and its derivative in [P1] (and can be also reconstructed from the the lemmas of chapter 5). We also see that we can choose ρ is independent of Ω. The C^r regularity of the remainder vector field can be shown starting from the expression for R in (3.11).

Also, (3.9), the fact that P is independent of ϕ, and $[h_4, P] = [\overline{h}_4, P] = 0$ imply that $[P, S] = 0$. Therefore $P \circ \mathcal{T} = P$.

We are interested in continuing breather solutions of the averaged equation. There are several existence results for breather solutions of (3.4), and we first consider the case $\delta = 0$, where we also have multi-peak breather solutions. Let $n_0 \in \mathbf{Z}$, $A \in \mathbf{C} \setminus \{0\}$ and set $\lambda = -\gamma|A|^2$. Define $\overline{\mathcal{A}} : \mathbf{Z} \to \mathbf{C}$ by $\overline{\mathcal{A}}_{n_0} = A$, $\overline{\mathcal{A}}_n = 0$ if $n \neq n_0$. (The dependence of $\overline{\mathcal{A}}$ on n_0, A is not made explicit here.)

Proposition 3.2 There exists $\Omega_2 > \Omega_1$ such that the averaged equation (3.4) with $\Omega > \Omega_2$ has a unique breather solution $a = \mathcal{A}e^{-i\lambda t}$ that satisfies $||\mathcal{A} - \overline{\mathcal{A}}|| \to 0$ as $\Omega \to \infty$. The breather amplitude $\mathcal{A}$ depends on Ω, n_0, and A.

Remark 3.2.1 In Proposition 3.2, β is assumed fixed. By Remark 3.0.1, the breather solutions for the averaged equation also exist for $\frac{\beta}{\Omega}$ sufficiently small.

We now combine the above statements to set up the continuation problem. We consider the system

$$\partial_t v = i\delta\Delta v - 2i\gamma\overline{g}_L(v) + \epsilon\tilde{g}_1(v, \phi), \quad \dot{\phi} = \Omega, \quad \dot{J} = -\frac{\partial \tilde{H}}{\partial \phi}. \tag{3.12}$$

By Lemma 3.1, sufficiently near the origin, and for $\epsilon = \Omega$ with $\Omega > \Omega_1$, (3.12) is equivalent to the full system. System (3.12) is Hamiltonian, for all real ϵ, with Hamiltonian $\tilde{H}^\epsilon = h_0 + h_2 + \overline{h}_4 + \epsilon R$. Moreover, since P Poisson commutes with both $\tilde{H}^{\Omega^{-1}}$, and $h_0 + h_2 + \overline{h}_4$, we see that P Poisson commutes with $\tilde{H}^\epsilon$, for all real ϵ. Therefore $\tilde{H}^\epsilon$, and $\tilde{P}$ satisfy conditions AI-AIII of chapter 2.

For $\epsilon = 0$, $\tilde{H}^0$ is independent of ϕ and Hamilton's equation for $\tilde{H}$ yield the averaged system (3.4), and $\dot{\phi} = \Omega$, $\dot{J} = 0$. By Proposition 3.2, given any $n_0 \in \mathbf{Z}$, $A \in \mathbf{C} \setminus \{0\}$, and assuming that $|\Omega| > \max\{\Omega_1, \Omega_2\}$, the $\epsilon = 0$ system has the breather solution

(3.13). $\quad a(t) = e^{-i\lambda t}\mathcal{A} \quad \text{with} \quad \lambda = 2\gamma|A|^2; \quad \phi(t) = \Omega t + \phi_0; \quad J(t) = 0, \quad \forall t \in \mathbf{R}.$

Let $C(\mathcal{A}, \phi_0)$ be the set of points of $X \times S^1 \times \mathbf{R}$ in the orbit defined by (3.13). Also let $\Lambda_0(\mathcal{A}) = \cup_{\phi_0 \in S^1} C(\mathcal{A}, \phi_0)$. The set $\Lambda_0(\mathcal{A})$ is an invariant 2$-$torus of the $\epsilon = 0$ system. By

Proposition 3.2, there exists $A_0 > 0$ (and a corresponding λ_0), and $\Omega_3 > \max\{\Omega_1, \Omega_2\}$ for which the breather $\mathcal{A}$ corresponding to A with any $A \in (0, A_0)$ and any $n_0 \in \mathbf{Z}$ yields a set $\Lambda_0(\mathcal{A}_\Omega)$ that satisfies $\Lambda_0(\mathcal{A}) \subset \mathcal{B}_\rho \times S^1 \times \mathbf{R}$.

Theorem 3.3 Let $n_0 \in \mathbf{Z}$, $A < A_0$, $\lambda < \lambda_0$, and consider the corresponding invariant tori $\Lambda_0(\mathcal{A})$ of system (3.12) with $\epsilon = 0$, as above. Then there exist an $\Omega_0 > \Omega_3$, $\beta_0' > 0$ such for every $\Omega > \Omega_0$ system (3.12) with $\epsilon = \Omega^{-1}$, has a $2-$parameter family of invariant $2-$tori $\Lambda_{\Omega^{-1},\beta}$, $\beta \in (-\beta_0', \beta_0')^2$. The motion on each $\Lambda_{\epsilon,\beta}$ is periodic or quasiperiodic (with two quasiperiods). Also, there exists $\beta_*' > 0$ for which $\Lambda_{0,\beta_*'} = \Lambda_0(\mathcal{A})$.

Theorem 3.3 is proved in two steps. First we consider (3.12) and show that we can apply Theorem 2.2 for some pair ϵ_0, $\beta_0 > 0$. This is done below, using Lemmas 3.4-3.8. Next, we show that we can choose ϵ_0, $\beta_0 > 0$ that are independent of Ω. This is the content of Lemma 3.9 below (whose proof is the next chapter). Theorem 3.3 then follows by setting $\Omega_0 = {\epsilon_0}^{-1}$, and applying Theorem 2.2 to (3.12) with $\Omega > \Omega_0$, and $\epsilon = \Omega^{-1}$. The corollary also follows from the proof of Lemma 3.9 where we examine the geometry of the proof Theorem 2.2 in the particular problem.

We start by reducing the continuation problem to a study of the Floquet spectrum of an appropriate variational equation.

Let $M = \mathcal{B}_\rho \times S^1 \times \mathbf{R}$, and $H_1^\epsilon = \tilde{H}^\epsilon$, $H_2^\epsilon(u, \phi, J) = P(u) = ||u||^2$. The corresponding Hamiltonian vector fields are denoted by X_1^ϵ, X_2^ϵ respectively. Also, let $\Lambda = \Lambda_0(\mathcal{A})$. We check that we have the set-up for Theorem 2.2, and in particular that H_1^ϵ, H_2^ϵ satisfy the conditions AI-AIII. The regularity properties follow from the regularity of the cubic nonlinearity and the observations above on the family L_t. To continue the torus Λ we examine the Floquet map of an appropriate linear combination of X_1^0, X_2^0. We parametrize Λ as

$$\Lambda = \{[e^{i\theta}\overline{\mathcal{A}}, \phi, 0] \in X \times S^1 \times \mathbf{R} : \quad \theta \in \mathbf{R}, \quad \phi \in S^1\}. \tag{3.14}$$

Let g_F^t be the time$-t$ map of the flow of the Hamiltonian vector field of F, where $F : M \to \mathbf{R}$. On Λ we then have

$$g_{H_1^0}^t([e^{i\theta}\overline{\mathcal{A}}, \phi, 0]) = [e^{i(\theta-\lambda t)}\overline{\mathcal{A}}, (\phi + \Omega t)\mathrm{mod}2\pi, 0], \tag{3.15}$$

$$g_{H_2^0}^t([e^{i\theta}\overline{\mathcal{A}}, \phi, 0]) = [e^{i(\theta-t)}\overline{\mathcal{A}}, \phi, 0]. \tag{3.16}$$

Also, $g^t_{cF} = g^{ct}_F$, $\forall c \in \mathbf{R}$, therefore the time$-t$ map of the Hamiltonian vector field of $c_1 H_0 + c_2 P$ is $g^{c_1 t}_{H_0} g^{c_2 t}_P$. Using (3.15), (3.16), the condition for the orbits of the Hamiltonian vector field of $c_1 H_0 + c_2 P$ to be $1-$periodic on Λ and to belong to the homotopy class $[n_1, n_2] \in \mathbf{Z}^2$ is that

$$-c_1\lambda - c_2 = 2\pi n_1, \quad c_1\Omega = 2\pi n_2, \tag{3.17}$$

or

$$c_1 = n_2\frac{2\pi}{\Omega}, \quad c_2 = -2\pi n_1 - n_2\frac{2\pi\lambda}{\Omega}. \tag{3.18}$$

To calculate the Floquet map around the $1-$periodic orbit, we let

$$u_n = e^{-i(c_1\lambda + c_2)t} e^{i\theta}\mathcal{A} + w, \quad \phi = c_1\Omega t + \phi_0 + \psi, \quad J = I, \tag{3.19}$$

and linearize around the periodic solution. The variational equation can be made autonomous by the change of variables $v = e^{i(c_1\lambda + c_2)t} w$ and we obtain

$$\partial v_n = c_1 \mathcal{L}(v), \quad \dot{\psi} = 0, \quad \dot{I} = 0, \quad \text{where} \tag{3.20}$$

$$\mathcal{L}(v) = i\lambda v + i\delta\Delta v - 2i\gamma(G(v)), \tag{3.21}$$

$$(G(v))_n = \frac{1}{2\pi}\int_0^{2\pi} \sum_{m\in\mathbf{Z}} [L^\dagger_\psi]_{nm} \left((L_\psi\mathcal{A})^2_m (L_\psi v)^*_m + 2|(L_\psi\mathcal{A})_m|^2 (L_\psi v)_m\right) d\psi, \tag{3.22}$$

$n \in \mathbf{Z}$, and $[M]_{nm}$ is the (n, m) matrix element of the operator M in the standard basis of X. By (3.19), at $t = 1$ we have $w(t) = v(t)$. Therefore the Floquet map around the $1-$periodic orbit coincides with the time-1 map of the system generated by (3.20)-(3.22).

The formal calculations leading to (3.20)-(3.22) above are justified using the properties of L_t and the cubic nonlinearity. In particular, the Fréchet derivative of the Hamiltonian vector field of $c_1 H^0_1 + c_2 H^0_2$ at each point of the periodic orbit exists and is a bounded operator in X, moreover the variational equation leading to (3.20)-(3.21) has a unique solution for all real t. In the autonomous problem (3.20)-(3.21), G, $\mathcal{L}$ are bounded linear operators in X. Therefore $\mathcal{L}$ generates a norm continuous semigroup.

The operators G, $\mathcal{L}$ depend on Ω, through L_t, and also through the breather amplitude $\mathcal{A}$ of Proposition 3.2. To make this dependence explicit we write $\mathcal{G}_\Omega$, $\mathcal{L}_\Omega$, $\mathcal{A}_\Omega$. Also, define the operator $\mathcal{L}_\infty$ by

$$(\mathcal{L}_\infty(v))_n = i\lambda v_n, \quad \text{if} \quad n \in \mathbf{Z}\setminus\{n_0\}; \quad (\mathcal{L}_\infty(v))_{n_0} = -i\lambda(v_{n_0} + v^*_{n_0}). \tag{3.23}$$

The spectrum of the time-1 map of $\partial_t v = c_1 \mathcal{L}_\infty$, $\dot\psi = 0$, $\dot I = 0$ is calculated readily since the system is block diagonal with 2×2 blocks: the equations for $\dot\psi$, $\dot I$ yield two unit eigenvalues. and we also obtain a double unit eigenvalue from the n_0 block of the first equation. Also each integer $n \neq n_0$ yields a pair of eigenvalues $e^{\pm 2\pi i n_2 \frac{\lambda}{\Omega}}$.

Lemma 3.4 The map $\Omega \mapsto \mathcal{L}_\Omega$ from $[\Omega_2, \infty)$ to $B(X)$ is norm continuous in $[\Omega_2, \infty)$. Moreover, as $\Omega \to \infty$, $\mathcal{L}_\Omega$ converges to $\mathcal{L}_\infty$ in the norm.

We remark that by (3.18) $c_1 = 2\pi \frac{n_2}{\Omega}$, i.e. it depends on Ω. As $\Omega \to \infty$, $\mathcal{L}_\Omega \to \mathcal{L}_\infty$ in the norm and we expect that the Floquet spectrum of $\partial_t v = c_1 \mathcal{L}_\Omega(v)$ should (in some sense) approach the Floquet spectrum of $\partial_t v = c_1 \mathcal{L}_\infty(v)$, computed above. If n_2 is kept fixed, we have that $c_1 \to 0$ and we expect that the Floquet spectrum of $\partial_t v = c_1 \mathcal{L}_\Omega(v)$ collapses to unity as $\Omega \to \infty$. Note however that the perturbation is also proportional to c_1, i.e. by Proposition 3.2 it is of $O(\Omega^{-2})$. Roughly, we therefore have an $O(\Omega^{-1})$ "unperturbed part" (i.e. distance from the resonance) with an $O(\Omega^{-2})$ perturbation. The continuation may be therefore possible. To see this we will instead vary n_2 with Ω and keep c_1 of $O(1)$, i.e. we will consider an equivalent continuation problem with an $O(1)$ unperturbed part and an $O(\Omega^{-1})$ perturbation.

To define n_2 as a function of Ω, consider $\Omega_3 > \Omega_2$ that also satisfies

$$\frac{|\lambda|}{\Omega_3} \leq \frac{1}{8}, \quad \Omega_3 > 1. \tag{3.24}$$

Also let $\Omega_4 = \Omega_3 + \frac{1}{2}$. By (3.24) it is easy to check that there exists a positive integer N_3 satisfying

$$2\pi N_3 \frac{|\lambda|}{\Omega_3} \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right]. \tag{3.25}$$

Let also $N_4 = N_3$. Let $\mathcal{C}_j$, $j = 3$, 4 be the set of all $\Omega \geq \Omega_j$ that can be written as

$$\Omega = \Omega_j + k_j + x_j, \quad k_j \in \{0, 1, 2, \ldots\}, \quad x_j \in \left[0, \frac{3}{4}\right]. \tag{3.26}$$

Clearly, any $\Omega \geq \Omega_3$ belongs to either $\mathcal{C}_3$ or $\mathcal{C}_4$ (or both). Moreover, given any $\Omega \in \mathcal{C}_j$, $j = 3$, 4, the corresponding k_j, x_j are determined uniquely. Finally, each $\mathcal{C}_j$, $j = 3$, 4 is a disjoint union of intervals $I_k^j = [\Omega_j + k_j, \Omega_j + k_j + \frac{2}{3}]$, $k_j \in \{0, 1, 2, \ldots\}$. Using the above, let $\Omega \in \mathcal{C}_j$, $j = 3$, 4, and define c_1 by

$$c_1 = c_1(\Omega) = n_2(\Omega) \frac{2\pi}{\Omega}, \quad \text{with} \quad n_2(\Omega) = N_j(\Omega) + k_j(\Omega). \tag{3.27}$$

For $\Omega \in \mathcal{C}_3 \cap \mathcal{C}_4$ the scalar $c_1(\Omega)$ can have two values. To avoid extra notation we first consider the case $\Omega \in \mathcal{C}_3$ and use the value $n_2(\Omega) = N_3(\Omega) + k_3(\Omega)$. We then proceed to prove Theorem 3.3 for $\Omega \in \mathcal{C}_3$. We can then repeat the same arguments for $\Omega \in \mathcal{C}_4$, this time choosing $n_2(\Omega) = N_4(\Omega) + k_4(\Omega)$. The following is elementary.

Lemma 3.5 Let $\Omega \in \mathcal{C}_3$ with $c_1(\Omega)$ as above. Then

$$|c_1(\Omega)\lambda| \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]. \tag{3.28}$$

With $\Omega \in \mathcal{C}_3$, and $c_1(\Omega)$ as above, denote the time-1 map of $\partial_t v = c_1(\Omega)\mathcal{L}_\Omega(v)$ by $\mathcal{M}_\Omega$. Also, let $\overline{\mathcal{M}}_\Omega$ be the time-1 map of $\partial_t v = c_1(\Omega)\mathcal{L}_\infty(v)$.

Lemma 3.6 Let $\Omega \in \mathcal{C}_3$ with $c_1(\Omega)$, $\mathcal{M}_\Omega$, $\overline{\mathcal{M}}_\Omega$ as above. The maps $\Omega \mapsto \mathcal{M}_\Omega$, and $\Omega \mapsto \overline{\mathcal{M}}_\Omega$, both from $\mathcal{C}_3$ to $B(X)$, are norm continuous in $\mathcal{C}_3$. Moreover, for every $\epsilon > 0$ there exists $M > 0$ for which $\Omega > M$ implies $||\mathcal{M}_\Omega - \overline{\mathcal{M}}_\Omega||_0 < \epsilon$.

By the proof of Proposition 2.1, the spectrum of $\overline{\mathcal{M}}_\Omega$ consists of a double unit eigenvalue, and the two infinite multiplicity eigenvalues $e^{\pm i c_1(\Omega)\lambda}$. We want to show that $\sigma(\mathcal{M})$ is in some sense similar. Let Γ be a circle of radius r_0 around unity in the complex plane, with $r_0 \in (0, \sin\frac{\pi}{6})$. The radius r_0 will be kept fixed as we vary $\Omega \in \mathcal{C}_3$. Given any $\mathcal{N} \in B(X)$ we will say that Γ *separates* $\sigma(\mathcal{N})$ if $\mathcal{N}$ has two simple eigenvalues λ_1, λ_2 inside Γ and $\sigma(\mathcal{N}) \setminus \{\lambda_1, \lambda_2\}$ lies outside Γ. (λ_1, λ_2 are allowed to coincide, in which case we are assuming a spectral projection of rank two.) Clearly, if $\Omega \in \mathcal{C}_3$ as in Lemma 3.5, then the circle Γ separates $\sigma(\overline{\mathcal{M}}_\Omega)$.

Lemma 3.7 Suppose that $\overline{\mathcal{N}} \in B(X)$, and that Γ separates $\sigma(\overline{\mathcal{N}})$. Then there exists $\epsilon_S = \epsilon_S(\overline{\mathcal{N}}, \Gamma) > 0$ such that $\mathcal{N} \in B(X)$ and $||\mathcal{N} - \overline{\mathcal{N}}||_0 < \epsilon_S$ imply that Γ separates $\sigma(\mathcal{N})$.

The lemma is a special case of a standard result on stability properties of spectral projections, (see [K], ch. 4, par. 3). In our case Γ is fixed and $\epsilon_S(\overline{\mathcal{M}}_\Omega, \Gamma)$ depends only on $\overline{\mathcal{M}}_\Omega$ and therefore on Ω. We see that we can choose ϵ_S that is independent of Ω:

Lemma 3.8 Suppose that Γ separates $\sigma(\overline{\mathcal{M}}_\Omega)$ for all $\Omega \in \mathcal{C}_3$. Then there exists $\epsilon_S > 0$ such that if $\mathcal{N} \in B(X)$ and $||\mathcal{N} - \overline{\mathcal{M}}_\Omega||_0 < \epsilon_S$ for some $\Omega \in \mathcal{C}_3$, then Γ separates $\sigma(\mathcal{N})$

Proof: From the definition of c_2, $\overline{\mathcal{M}}_\Omega + k = \overline{\mathcal{M}}_\Omega$, $\forall k \in \{0, 1, 2, \ldots\}$. It is therefore sufficient to show the lemma for $\overline{\mathcal{M}}_\Omega$ with $\Omega \in I_k^3$ for some non-negative integer k. Let $I = I_k^3$. Consider $\omega_0 \in I$, and an $\epsilon_S(\omega_0) > 0$ such that for any $\mathcal{N} \in B(X)$ with $||\mathcal{N} - \overline{\mathcal{N}}||_0 <$

$\epsilon_S(\omega_0)$ we have that Γ separates $\sigma(\mathcal{N})$. By Lemma 3.7 such $\epsilon_S(\omega_0)$ exists. Let

$$\epsilon_{\omega_0}(\Omega) = \epsilon_S(\omega_0) - ||\overline{\mathcal{M}}_\Omega - \overline{\mathcal{M}}_{\omega_0}||_0. \tag{3.29}$$

There exists a neighborhood $E_{\omega_0} \subset I$ of ω_0 where, by Lemma 3.6, ϵ_{ω_0} is bounded away from zero. Moreover, suppose that a bounded operator $\mathcal{N}$ satisfies $||\mathcal{N} - \overline{\mathcal{M}}_\Omega||_0 < \tilde{\epsilon}_{\omega_0}(\Omega)$ for some $\Omega \in E_{\omega_0}$. Then, using (3.29),

$$||\mathcal{N} - \overline{\mathcal{M}}_{\omega_0}||_0 \leq ||\mathcal{N} - \overline{\mathcal{M}}_\Omega||_0 + ||\overline{\mathcal{M}}_\Omega - \overline{\mathcal{M}}_{\omega_0}||_0 < \epsilon_S(\omega_0) \tag{3.30}$$

and therefore Γ separates $\sigma(\mathcal{N})$. Thus for $\Omega \in E_{\omega_0}$ we can choose the function $\epsilon_S(\overline{\mathcal{M}}_\Omega, \Gamma)$ of Lemma 3.6 to be bounded away from zero in I (Γ is fixed). We can repeat the construction around any other point of I. By the compactness of I we have a finite set of intervals E_1, …, E_m, $I \subset E_1 \cup \ldots E_m$, and functions $\epsilon_j : E_j \to \mathbf{R}^+$ that are bounded away from zero in E_j and satisfy the property that if $\mathcal{N} \in B(X)$ is at most $\epsilon_j(\Omega)$ away from $\overline{\mathcal{M}}_\Omega$ in the norm for some $\Omega \in E_j$, then Γ separates $\sigma(\mathcal{N})$. Choosing lower bounds $\epsilon_S^j > 0$ of the functions ϵ_j on each E_j, $j = 1$, …, m, we have the lemma with $\epsilon_S = \min_{j \in 1,\ldots,m} \epsilon_S^j$.

■

By the second part of Lemma 3.5, setting $\epsilon = \epsilon_S$, there exists $M_3 > \Omega_3$ such that $\Omega > M_3$ implies $||\mathcal{M}_\Omega - \overline{\mathcal{M}}_\Omega||_0 < \epsilon_S$. By Lemma 3.8, for any $\Omega \in \mathcal{C}_3$ with $\Omega > M_3$, Γ separates $\sigma(\mathcal{M}_\Omega)$.

Lemmas 3.5, 3.6, and 3.8 are also valid for $\Omega \in \mathcal{C}_4$ with the choice $c_1(\Omega) = k_4(\Omega) + x_4(\Omega)$. ($\epsilon_S$ may be different.) We therefore see that there exists an $M_4 > \Omega_4$ with the property that for any $\Omega \in \mathcal{C}_4$ with $\Omega > M_4$, Γ separates $\sigma(\mathcal{M}_\Omega)$. Consequently, Γ separates $\sigma(\mathcal{M}_\Omega)$, for any $\Omega > \Omega_5 = \max\{M_3, M_4\}$. By the Hamiltonian structure of the flow that defines $\mathcal{M}_\Omega$, the part of $\sigma(\mathcal{M}_\Omega)$ inside Γ consists of a double unit eigenvalue.

Lemma 3.9 Consider the invariant torus $\Lambda(\mathcal{A})$ of system (3.12). Assume that $\Omega > \Omega_0$ for some $\Omega_0 \geq \Omega_5$. Then there exist β_0, $\epsilon_0 > 0$ such that for any ϵ with $|\epsilon| < \epsilon_0$ system (3.12) has a two parameter family of invariant tori $\Lambda_{\epsilon,\beta}$ with $\beta \in (-\beta_0, \beta_0)^2$, and $\Lambda_{0,0} = \Lambda(\mathcal{A})$. Moreover, the constants β_0, ϵ_0 can be chosen to be independent of Ω.

The first part of Lemma 3.9, on the existence of invariant the 2−parameter families of invariant 2−tori follows from our discussion of the Floquet spectrum around the breathers, and Theorem 2.2, by choosing n_2 as in (3.27). It remains to show that that β_0, and ϵ_0 can be chosen to be independent of Ω. This is shown in the next chapter.

Remark 3.9.1 Theorem 3.3 applies to any $\beta > 0$, and a natural question is how Ω_0 depends on β. We conjecture that we can choose Ω_0 linear in β so that the continuation is valid in a region that is to the right of some ω_0 and below some $\frac{\beta}{\Omega} = \alpha$ for some $\alpha > 0$ (see Figure 1).

To see that this is plausible, first consider (3.20), (3.21): c_1 depends on n_2, and Ω, with n_2 chosen so that c_1 belongs to an interval that is independent of Ω for Ω sufficiently large. On the other hand, the operator $\mathcal{L}$ of (3.21) depends on λ and $\frac{\beta}{\Omega}$ (by Remark 3.0.1). Since λ is fixed the distance of $\sigma(e^{c_1 \mathcal{L}}) \setminus \{1\}$ from unity only depends on $\frac{\beta}{\Omega}$. Thus Ω_5 is a linear function of β. Also, in the proof of Lemma 3.9 we observe that for Ω sufficiently large, and $\frac{\beta}{\Omega}$ sufficiently small, all estimates depend on quantities that are independent of Ω, $\frac{\beta}{\Omega}$ (this is shown in chapters 4, 5). In particular, the distance from the resonance, the geometry of the unperturbed torus, and the size of the neighborhood of the unperturbed torus where we define the modified Poincare map in the proof of Theorem 2.2 depend only on $\frac{\beta}{\Omega}$. For Ω sufficiently large and $\frac{\beta}{\Omega}$ sufficiently small the other quantities affecting the size of ϵ_0, and β_0, i.e. norms giving information how much the unperturbed system changes in a neighborhood of the unperturbed torus and on the size of the perturbation, are independent of Ω, and β.

We conclude this chapter with some numerical calculations of the Floquet spectra of breathers of the averaged equation (3.4). The argument used to prove Proposition 3.2 (in [P3]) can be also used to show the existence of multipeak breathers. For $U \subset \mathbf{Z}$, $|U| = k$, the $k-$peak breathers of the vanishing diffraction averaged system

$$\partial_t a = -2i\gamma \overline{g}_L(a), \tag{3.31}$$

with $\overline{g}_L$ as in (3.4), are continuations of solutions of the system $\partial_t a = -2i\gamma g(a)$, g as in (2.1), that have the form $a_n(t) = e^{-i\lambda t} A_n$, where $A_n = \pm A$, if $n \in U$, even; $A_n = \pm iA$, if $n \in U$, odd, with λ, $A \in \mathbf{R}$, $\lambda = 2\gamma A^2$. The continuation is valid for $\frac{\beta}{\Omega}$ sufficiently small (see [P3]). Recall that at $\beta = 0$, $\overline{g}_L$ becomes g.

In the numerical experiments we fix λ and find breather solutions $a = \mathcal{A}e^{-i\lambda t}$ of (3.31) using minpack routines. We use values of $\frac{\beta}{\Omega}$ between 10^{-3} and 1.5. Also, $\gamma = 1$, $\lambda = 10$, $\Omega = 13$. As before, given any homotopy class α, we have a $c = [c_1, c_2] \in \mathbf{R}^2$ for which the breathers of (3.4) correspond to $1-$periodic solutions of

$$u_t = c_1[i\overline{D}\Delta u - 2\gamma i \overline{g}(u)] + c_2[-iu], \quad \dot{\phi} = c_1 \Omega, \quad \dot{J} = 0. \tag{3.32}$$

The modified Floquet map is obtained by integrating numerically the variational equation around these $1-$periodic solutions. The criterion for continuation is that the spectrum of the numerically computed nontrivial block of the modified Floquet map have exactly two unit eigenvalues.

We first consider single-peak breathers. Figures 2(a), 3(a), 4(a) show the behavior of the spectra of the nontrivial Floquet blocks as we increase the parameter $\frac{\beta}{\Omega}$. The corresponding breathers are in Figures 2(b), 3(b), 4(b). The homotopy class is $[n_1, n_2] = [-1, 1]$ with the corresponding $c = [c_1, c_2]$ of (2.12). For $\beta = 0$, we see a double unit eigenvalue and the multiple eigenvalues $e^{\pm 2\pi i \frac{\lambda}{\Omega}}$, as expected. As we increase $\frac{\beta}{\Omega}$ away from the origin we see a double unit eigenvalue, moreover the rest of the spectrum remains on the unit circle and accumulates at the points $e^{\pm 2\pi i \frac{\lambda}{\Omega}}$, i.e. as expected, it stays off 1 for small $\frac{\beta}{\Omega}$.

In the $k-$peak case with $\beta = 0$, the nontrivial Floquet block has $2k$ unit eigenvalues, and the multiple eigenvalues $e^{\pm 2\pi i \frac{\lambda}{\Omega}}$. As we increase $\frac{\beta}{\Omega}$ away from the origin we see $2k - 2$ eigenvalues that move off 1 along the unit circle. The rest of the spectrum remains on the unit circle and accumulates at the points $e^{\pm 2\pi i \frac{\lambda}{\Omega}}$, away from 1. This is indicated in the spectra of Figures 5(a), 6(a) that correspond to the $2-$peak, and $3-$peak breathers of Figures 5(b), 6(b) respectively. For peaks that are further apart, the phenomenon is less pronounced and the $2k - 2$ extra eigenvalues remain close to unity.

Increasing n_2 with Ω can control the apparent accumulation points of the spectrum but not the $2k - 2$ eigenvalues that move off 1 for nonzero $\frac{\beta}{\Omega}$. Since the distance of these $2k - 2$ eigenvalues from 1 vanishes as $\Omega \to \infty$, it does not seem likely that the $k-$peak breathers of the averaged equation can be continued by the argument of Theorem 3.3. The strategy may still work by continuing from solutions of higher averaged equations and using asymptotics of the eigenvalues. Another possible approach is suggested by Remark 4.5.2.

4. The equivariant continuation theorem

To prove Theorem 2.2 we follow the notation used in setting up the theorem in section 2. Let r be a fixed integer, with $r \geq 2$. Also let $g^t_{i,\epsilon}$ denote the time$-t$ map of the vector field X^ϵ_i. Assumption AI implies that the maps $g^t_{i,\epsilon}$ are C^r, $\forall t \in \mathbf{R}$. Letting $c = [c_1, \ldots, c_s] \in \mathbf{R}^s$, $t \in \mathbf{R}$, we also use the notation $g^{ct}_\epsilon = g^{c_1 t}_{1,\epsilon} \ldots g^{c_s t}_{s,\epsilon}$. By assumption AIII the maps $g^t_{i,\epsilon}$, $g^{t'}_{j,\epsilon}$ mutually commute, for all i, $j \in \{1, \ldots, s\}$, and t, $t' \in \mathbf{R}$. Also, $B(Z, Y)$ denotes the bounded linear operators from a Banach space Z to Banach space Y.

The operator norm in $B(Z, Y)$ is denoted by $||.||_0$ (the spaces Z, Y will be clear from the context in each case). The ball of radius r around the point x is denoted by $\mathcal{B}_r(x)$, and $\mathcal{B}_r$ if x is the origin (the spaces and norms will be clear from the context).

The plan is to use the maps g_ϵ^{ct} to construct a version of the Poincare map from a neighborhood of (each) $m \in \Lambda$ in Σ_m to Σ_m, where Σ_m is a set of codimension s in M that is transverse to the unperturbed torus. This map will be further restricted to the intersection of the level hypersurfaces of the functions H_j^0, $j = 1, \ldots, s$. The Poincare map will be then defined on a set of codimension $2s$ in M. The directions we remove correspond to the eigenvectors of the unit eigenvalues of the Floquet map Dg_0^c so that by the nonresonance condition the linearization of the restricted Poincare map will be invertible and we can continue the fixed point of the unperturbed restricted map uniquely. The invariant $s-$torus will be the set of fixed points obtained for each $m \in \Lambda$ and we obtain an $s-$parameter family by varying the level hypersurfaces of the H_j^0.

The construction of the Poincare map from a subset of Σ_m to Σ_m relies on the definition of a suitable coordinate system in a neighborhood of m. This coordinate system allows us to identify the image of a neighborhood of m in Σ_m under g_ϵ^c with points on Σ_m. This is done in Lemmas 4.1-4.4. The Poincare map is defined after Lemma 4.3. The existence of fixed points of the Poincare map is shown in Lemma 4.5. Lemma 4.6 states that the fixed points of the restricted Poincare are the invariant tori we seek. (The proof makes a repeated application of the implicit function theorem, stated as Lemma 4.7.)

In the continuation problems we are studying the construction of the Poincare map is simpler, and the map can be defined directly without reference to the general construction described in Lemmas 4.1-4.4. Similarly, the construction of the invariant tori from the fixed points of the restricted Poincare map can be done directly, without Lemma 4.6. Consequently we state these lemmas without proof (see [P4] for proofs). The direct construction of the restricted Poincare map and the invariant tori is detailed in the proof of Lemma 3.9.

We first define a system of coordinates around the set Λ. Note that for any $m \in \Lambda$ there exists a C^r Hilbert submanifold Σ_m of M that has codimension s and is transverse to Λ at m. Moreover, by assumption AII on the existence of a C^r tubular neighborhood around Λ, we can choose a family $\{\Sigma_m\}_{m\in\Lambda}$ of such submanifolds that constitutes a C^r foliation of a neighborhood U of Λ (see [L]).

Letting $m \in \Lambda$ we can assume that a neighborhood of Σ_m in M has been identified with a neighborhood of the origin of $T_m\Sigma_m \simeq E$ by a C^r chart (that we do not make explicit). Also, let $h_m^0 = H^0|_{\Sigma_m}$. Then, in a neighborhood of m in Σ_m we can use coordinates (y^0, z^0), where $y^0 \in Y^0$, the nullspace of Dh_m^0, and $z^0 \in Z^0$, the orthogonal complement of Y^0 in $T_m\Sigma_m$. Note that Y^0 splits in $T_m\Sigma_m$, and that m has coordinates $(0, 0)$. Also, in a neighborhood of $m \in \Lambda$ we can use C^r coordinates $(y^0, z^0, w) \in Y^0 \times Z^0 \times W$, $W \simeq \mathbf{R}^s$, where W is the orthogonal complement of Σ_m in $T_m\Sigma_m$. Points with coordinates $(y^0, z^0, 0)$ belong to Σ_m. The dependence of the coordinates of a given point on m is not made explicit in this notation.

Consider $H^\epsilon = H^0 + \epsilon\tilde{H}$, where H^0, $\tilde{H}$, H^ϵ are $s-$component vectors with $j-$th components H_j^0, $\tilde{H}_j$, H_j^ϵ respectively. Let $\tilde{h}_m = \tilde{H}|_{\Sigma_m}$. By AI, the derivatives $D_2h_m^0(y^0, z^0)$, $D_2\tilde{h}_m(y^0, z^0)$ are elements of $B(Z^0, \mathbf{R}^s)$, i.e. $s \times s$ matrices, for any $(y^0, z^0) \in \Sigma_m$. By the independence of the components of H^0, and the definition of Σ_m, $D_2h_m^0(y^0, z^0)$ is invertible, $\forall (y^0, z^0) \in \Sigma_m$.

Lemma 4.1 There exist an $\epsilon_1 > 0$ and and nonempty $\Sigma_m^1 \subset \Sigma_m$ such that for $|\epsilon| < \epsilon_1$, and $(y^0, z^0) \in \Sigma_m^1$, the map $(y^0, z^0) \mapsto (y^0, \beta^\epsilon)$ defined by $\beta^\epsilon = h_m^\epsilon(y^0, z^0)$ is a C^r diffeomorphism in Σ_m^1, i.e. defines a new C^r coordinate system in Σ_m^1.

Let $g_\epsilon^\tau(y^0, z^0) = g_\epsilon^\tau(y^0, z^0, 0)$, $|\epsilon| < \epsilon_1$, $(y^0, z^0) \in \Sigma_m^1$. Let $[g_\epsilon^\tau(y^0, z^0)]_W$ denote the $W-$component of $g_\epsilon^\tau(y^0, z^0, 0)$. Also, consider a point (y^0, z^0, w) in a neighborhood of $\Sigma_m^1 \times \{0\}$ in M, and the equation

$$[g_\epsilon^\tau(y^0, z^0)]_W = w \tag{4.1}$$

for $\tau \in \mathbf{R}^s$, i.e. (y^0, z^0, w) is a parameter. We want to find a neighborhood of $\Sigma_m^1 \times \{0\}$ in M where (4.1) has a unique solution $\tau = \tau^\epsilon(y^0, z^0)$. In such a neighborhood we can use the coordinates $(y^0, \beta^\epsilon, \tau^\epsilon)$.

Consider now the function $G(w, \tau) = [g_\epsilon^\tau(y^0, z^0)]_W - w$, i.e. compare with (4.1), in a subset of the origin in $\mathbf{R}^s \times \mathbf{R}^s$. Note that $G(0, 0) = 0$.

Lemma 4.2 There exist $\epsilon_2 > 0$, $r_2 > 0$, and nonempty $\Sigma_m^2 \subset \Sigma_m^1$ such that for $|\epsilon| < \epsilon_2$, $(y^0, z^0) \in \Sigma_m^2$, $||w|| < r_2$ equation (4.1) has a unique solution $\tau^\epsilon(y^0, z^0, w)$. Moreover, the function $\chi_m^\epsilon : \Sigma_m^2 \times \mathcal{B}_{r_2} \to Y^0 \times \mathbf{R}^s \times \mathbf{R}^s$ defined by $\chi_m^\epsilon(y^0, z^0, w) = (y^0, \beta^\epsilon, \tau^\epsilon)$ with $\beta^\epsilon = h_m^\epsilon(y^0, z^0)$, τ^ϵ the solution of (4.1), is injective and continuous in $\Sigma_m^2 \times \mathcal{B}_{r_2}$. Also, there exists $\tilde{r}_2 > 0$, $\tilde{r}_2 \leq r_2$ for which χ_m^ϵ, restricted to $\Sigma_m^2 \times \mathcal{B}_{\tilde{r}_2}$ is a C^r diffeomorphism.

Let $m \in \Lambda$, $|\epsilon| < \epsilon_2$, and $(y^0, z^0) \in \Sigma^2_m$. Define the map $\phi^\epsilon_m : \Sigma^2_m \to Y^0 \times \mathbf{R}^s$ by $\phi^\epsilon_m(y^0, z^0) = (y^0, \beta^\epsilon)$. Also define the map $f^\epsilon_m : (\phi^0_m)^{-1}(\Sigma^2_m) \to \phi^\epsilon_m(\Sigma^2_m)$ by $f^\epsilon_m = (\phi^0_m)^{-1} \circ \phi^\epsilon_m$. By the lemma f^ϵ_m is a C^r diffeomorphism between the coordinates (y^0, β^0), and (y^0, β^ϵ). Let I_2 be the set of $w \in \mathbf{R}^s$ satisfying $||w|| < \frac{r_2}{2\mu_2}$. We have the following.

Lemma 4.3 Let $m \in \Lambda$. There exists $\epsilon_3 > 0$ (and $\epsilon_3 \leq \epsilon_2$) such that if $|\epsilon| < \epsilon_3$ then there exists $U_m \subset \Sigma^2_m \times I_2$ with the property that any solution of cX^ϵ with initial condition $v(0) \in U_m$ satisfies $v(1) \in \Sigma^2_m \times I_2$.

The lemma follows from the continuity of the flows g^c_ϵ and $X^\epsilon = X^0 + \epsilon\tilde{X}$.

Let $\Phi_\epsilon = g^c_\epsilon$. Note that the dependence of Φ_ϵ on the homotopy class α and c is not made explicit in this notation.

By Lemma 4.2, for any $(y^0, \beta^0, w) \in \Sigma^2_m \times I_2$ we can use coordinates y^ϵ, β^ϵ, τ^ϵ defined by $\tau^\epsilon(y^0, \beta^0, w)$ as in Lemma 4.2, $\beta^\epsilon(y^0, \beta^0, w) = h^\epsilon_m(y^0, \beta^0)$, and $y^\epsilon(y^0, \beta^0, w) = y^0$. Consider the image $\Phi_\epsilon(y^0, \beta^0, w)$ of points of U_m under Φ_ϵ. Using coordinates y^ϵ, β^ϵ, τ^ϵ we define $\hat{y}^\epsilon$, $\hat{\beta}^\epsilon$, $\hat{\tau}^\epsilon$ by

$$(4.2) \qquad \hat{y}^\epsilon = y^\epsilon(\Phi_\epsilon(y^\epsilon, \beta^\epsilon, \tau^\epsilon)), \quad \hat{\beta}^\epsilon = \beta^\epsilon(\Phi_\epsilon(y^\epsilon, \beta^\epsilon, \tau^\epsilon)), \quad \hat{\tau}^\epsilon = \tau^\epsilon(\Phi_\epsilon(y^\epsilon, \beta^\epsilon, \tau^\epsilon)).$$

By Lemma 4.3, $\hat{y}^\epsilon$, $\hat{\beta}^\epsilon$, $\hat{\tau}^\epsilon$ are well defined in U_m.

Lemma 4.4 Fix $m \in \Lambda$ and let $(\hat{\beta}^\epsilon, \hat{\tau}^\epsilon, \hat{y}^\epsilon)$ be as above. Then for any $(\beta^\epsilon, \tau^\epsilon, y^\epsilon)_m \in U_m$ we have (**i**) $\hat{\beta}^\epsilon = \beta^\epsilon$, (**ii**) $\hat{\tau}^\epsilon = \tau^\epsilon + \tau^\epsilon_0$, where τ^ϵ_0 depends on β^ϵ, y^ϵ, (**iii**) $\hat{y}^\epsilon$ is independent of τ^ϵ.

By Lemma 4.2 the component $\hat{y}^\epsilon$ of Φ_ϵ depends on β^ϵ and y^ϵ and we write $\hat{y}^\epsilon = \hat{y}^\epsilon(\beta^\epsilon, y^\epsilon)$. We now use the condition on the spectrum of the derivative of Φ_0.

Let $m \in \Lambda$, $|\epsilon| < \epsilon_3$. Let Σ^3_m, I_3 be non-empty subsets of Σ^2_m, I_2 respectively, with the property that $\Sigma^3_m \times I_3 \subset U_m$. Let $V_m = \chi^0_m(\Sigma^3_m \times \{0\})$. For $(y^0, \beta^0) \in V_m$, define the functions β, and $\hat{y}$ by

$$(4.3) \qquad \beta(\epsilon, y^0, \beta^0) = \beta^\epsilon(y^0, \beta^0),$$

$$(4.4) \qquad \hat{y}(\epsilon, y^0, \beta^0) = \hat{y}^\epsilon(y^0, \beta(\epsilon, y^0, \beta^0)) = \hat{y}^\epsilon(\beta^\epsilon(y^0, \beta^0)).$$

Also, let

$$F(\epsilon, y^0, \beta^0) = \hat{y}^\epsilon(y^0, \beta^\epsilon) - y^0 = \hat{y}(\epsilon, y^0, \beta(\epsilon, y^0, \beta^0)) - y^0$$

We may assume without loss of generality that $\beta^0(m) = 0$, $\forall m \in \Lambda$, i.e. by adding appropriate constants to the H^0_j, $j = 1, \ldots, s$. Viewing F as a function of the two variables $\mathbf{x} = (\epsilon, \beta^0)$, $\mathbf{y} = y^0$, we then have $F(0,0) = 0$. The function F is C^r in its domain by the C^r regularity of the flows Φ_ϵ. Also, by Lemma 4.1 and the construction of the coordinates (y^0, β^0), F is C^r in $m \in \Lambda$.

Let

$$||(\mathbf{x}, \mathbf{y})||_1 = (|\epsilon|^2 + ||(y^0, \beta^0)||^2)^{\frac{1}{2}}. \tag{4.5}$$

Assume that there exist positive reals μ_3, Λ_1, Λ_2 satisfying respectively

$$\mu_3 = ||[D_2 F(0,0)]^{-1}||_0. \tag{4.6}$$

$$||D_2 F(\mathbf{x}, \mathbf{y}) - D_2 F(0,0)||_0 < \Lambda_1 ||(\mathbf{x}, \mathbf{y})||_1, \quad \forall (\mathbf{x}, \mathbf{y}) \in I_3 \times V_m, \tag{4.7}$$

$$||F(\mathbf{x}, 0)|| < \Lambda_2 ||(\mathbf{x}, 0)||_1, \quad \forall (\mathbf{x}, 0) \in I_3 \times V_m \tag{4.8}$$

Lemma 4.5 Let $m \in \Lambda$, χ^ϵ_m, Φ_ϵ as above, and $|\epsilon| < \epsilon_3$. There exists $\epsilon_0 > 0$, $\epsilon_0 \leq \epsilon_3$, that depends on Λ_1, Λ_2, μ_3, such that for every $\epsilon \in (-\epsilon_0, \epsilon_0)$ there exists a $\beta^\epsilon_* > 0$ for which $|\epsilon| < \epsilon_0$, $\beta^\epsilon \in \mathcal{B}_{\beta^\epsilon_*}$ imply that the equation $\hat{y}^\epsilon(y^0, \beta^\epsilon) = y^0$ has a unique solution $y^0 = \rho^\epsilon_m(\beta^\epsilon)$. The map $\rho^\epsilon_m : \mathcal{B}_{\beta^\epsilon_*} \to Y^0$ is C^r in a nontrivial subset of its domain. Also, the maps ρ^ϵ_m depend on m in a C^r way, $\forall \epsilon \in (-\epsilon_0, \epsilon_0)$, $\beta^\epsilon \in \mathcal{B}_{\beta^\epsilon_*}$.

Remark 4.5.1 In the case where one of the μ_3, Λ_1, Λ_2 is not well defined, the conclusions of Lemma 4.5 are still valid for some $\epsilon_0 > 0$. The more detailed version and additional assumptions here allows us to understand the dependence of ϵ_0 on some of the parameters of the problem, especially Ω; this is used the proof of Lemma 3.9 below.

Proof of Lemma 4.5: We want to solve $F(\mathbf{x}, \mathbf{y}) = 0$, i.e. find $\mathbf{y}(\mathbf{x})$ for $\mathbf{x}$ near the origin. We have $F(0,0) = 0$. Also,

$$D_2 F = \frac{\partial \hat{y}^\epsilon}{\partial \beta} \frac{\partial \beta}{\partial y^0} + \frac{\partial \hat{y}}{\partial y^0} - I, \tag{4.9}$$

From $\beta(0, y^0, \beta^0) = \beta^0$ and the continuity of $\frac{\partial \beta}{\partial y^0}$ at the origin we have that

$$\frac{\partial \beta}{\partial y^0}(0,0,0) = 0. \tag{4.10}$$

Also, at the origin,

$$\frac{\partial \hat{y}}{\partial y^0} = \frac{\partial \hat{y}^0}{\partial y^0}(0,0). \tag{4.11}$$

Therefore

$$D_2F(0,0) = \frac{\partial \hat{y}^0}{\partial y^0}(0,0) - I. \tag{4.12}$$

Using the coordinates $(y^\epsilon, \beta^\epsilon, \tau^\epsilon)$ for $\epsilon = 0$, the derivative of Φ_0 is

$$D\Phi_0 = \begin{pmatrix} \partial_{y^0}\hat{y} & \partial_{\beta^0}\hat{y} & \partial_{\tau^0}\hat{y} \\ \partial_{y^0}\hat{\beta}^0 & \partial_{\beta^0}\hat{\beta}^0 & \partial_{\tau^0}\hat{\beta}^0 \\ \partial_{y^0}\hat{\tau}^0 & \partial_{\beta^0}\hat{\tau}^0 & \partial_{\tau^0}\hat{\tau}^0 \end{pmatrix}. \tag{4.13}$$

By Lemma 4.4, at $\beta^0 = 0$, $y^0 = 0$, $\tau_0 = 0$ we have

$$D\Phi_0(0,0,0) = \begin{pmatrix} \partial_{y^0}\hat{y}(0,0) & \partial_{\beta^0}\hat{y}(0,0) & 0 \\ 0 & I_s & 0 \\ \partial_{y^0}\hat{\tau}^0(0,0) & \partial_{\beta^0}\hat{\tau}^0(0,0) & I_s \end{pmatrix}, \tag{4.14}$$

where I_s is the $s \times s$ identity matrix. The block 1, 1 is the operator $\frac{\partial \hat{y}}{\partial y^0}(0,0)$ of (4.13). By the block triangular structure of $D\Phi_0(0,0)$, i.e. swap the first and second components, the spectrum of $D\Phi_0(0,0)$ is the union of the spectra of $\frac{\partial \hat{y}}{\partial y^0}(0,0)$ and I_s. Therefore $\sigma(D\Phi_0(0,0))$ contains at least $2s$ unit eigenvalues, moreover by the assumption on $\sigma(D\Phi_0(0,0))$, the spectrum of $\frac{\partial \hat{y}^0}{\partial y^0}(0,0)$ belongs to the complement of a disk around 1. The operator $\frac{\partial F}{\partial y^0}(0,0)$ of (4.13) has thus a bounded inverse in Y and there exists some $\mu_3 > 0$ that satisfies (4.6). To apply the implicit function theorem, consider some $r > 0$ with the properties that $(\mathbf{x}, \mathbf{y}) \in \mathcal{B}_r \times \mathcal{B}_r$ implies that $(\mathbf{x}, \mathbf{y}) \in I_3 \times V_m$, and

$$r < (2\sqrt{2}\Lambda_1\mu_3)^{-1}. \tag{4.15}$$

Let

$$r_2 = r; \quad r_1 = (2\Lambda_1\mu_3)^{-1} \quad \text{if} \quad 2\Lambda_1\mu_3 \geq 1, \quad r_1 = r \quad \text{otherwise}. \tag{4.16}$$

We check that the conditions for the implicit function theorem are satisfied for $(\mathbf{x}, \mathbf{y}) \in \mathcal{B}_{r_1} \times \mathcal{B}_{r_1}$. Therefore for $(\epsilon, \beta^0) \in \mathcal{B}_{r_1}$ we have a unique map $(\epsilon, \beta^0) \mapsto \rho_m(\epsilon, \beta^0) \in Y^0$, with $F(\epsilon, \beta^0, \rho_m(\epsilon, \beta^0))) = 0$. The map is C^r in a nonempty subset of $\mathcal{B}_{r_1}$. Note that there exist $\epsilon > 0$, β^0_* such that $(-\epsilon_0, \epsilon_0) \times (-\beta^0_*, \beta^0_*)^s \subset \mathcal{B}_{r_1}$, moreover, given any ϵ with $|\epsilon| < \epsilon_0$, there exists a $\beta_*^\epsilon > 0$ such that the set $V_m^\epsilon = (-\beta_*^\epsilon, \beta_*^\epsilon) \times \{0\} \subset \phi_m^\epsilon(\Sigma_m^3) \subset Z^0 \times Y^0$ is mapped to V_m^0 by $(f_m^\epsilon)^{-1}$. We then define $\rho_m^\epsilon : V_m^\epsilon \to Y^0$ by

$$\rho_m^\epsilon(\beta^\epsilon) = \rho_m(\epsilon, (f_m^\epsilon)^{-1}(\beta^\epsilon, 0)), \tag{4.17}$$

i.e. $\rho_m^\epsilon(\beta^\epsilon) = \rho(\epsilon, \beta^0)$ with $\beta^0 = (f_m^\epsilon)^{-1}(\beta^\epsilon, 0)$. By the definition of the map ρ_m, and letting $\beta^0 = (f_m^\epsilon)^{-1}(\beta^\epsilon, 0)$, we have

$$\hat{y}^\epsilon(\beta^\epsilon, \rho_m^\epsilon(\beta^\epsilon)) - \rho_m^\epsilon(\beta^\epsilon) = \hat{y}(\epsilon, \beta(\epsilon, \beta^0, \rho_m(\epsilon, \beta^0)), \rho_m(\epsilon, \beta^0)) - \rho_m(\epsilon, \beta^0) = \tag{4.18}$$

$$= F(\epsilon, \beta^0, \rho_m(\epsilon, \beta^0)) = 0,$$

as required. Also, for $|\epsilon| < \epsilon_0$, the maps ρ_m^ϵ are C^r in β^ϵ, for all β^ϵ in some nontrivial subset of V_m^ϵ. The C^r smoothness of the ρ_m^ϵ in m follows from the C^r regularity of ρ_m on m, and the C^r regularity of the map from the variables y^0, β^ϵ) to the variables y^0, β^0.

■

Remark 4.5.2 In the case where the multiplicity of the unit eigenvalue of $D\Phi_\epsilon$ is greater than $2s$ and finite one may be able to analyze the continuation question using Lyapunov-Schmidt reduction. This approach may be useful in the multipeak case.

Now, let $m \in \Lambda$ and define the map $\sigma_{\beta^\epsilon}^\epsilon : \Lambda \to \cup_{m\in\Lambda} U_m^\epsilon$ by

$$\chi_m^\epsilon(\sigma_{\beta^\epsilon}^\epsilon(m)) = (\beta^\epsilon, 0, \rho_m^{\beta^\epsilon}). \tag{4.19}$$

Also, let

$$\Lambda_{\epsilon,\beta^\epsilon} = \cup_{m\in\Lambda}(\beta^\epsilon, 0, \rho_m^\epsilon(\beta^\epsilon))_m = \sigma_{\beta^\epsilon}^\epsilon(\Lambda). \tag{4.20}$$

Lemma 4.6 The set $\Lambda_{\epsilon,\beta^\epsilon}$ is C^r diffeomorfic to Λ and is invariant under g_ϵ^τ, $\forall \tau \in \mathbf{R}^s$ (and therefore invariant under the flow of the X_j^ϵ, $j = 1, \ldots, s$). Moreover the motion on $\Lambda_{\epsilon,\beta^\epsilon}$ is conditionally periodic.

The following is the version of the implicit function theorem used in Lemma 4.5 (see e.g. [Z]).

Lemma 4.7 Let $\mathbf{X}$, $\mathbf{Y}$ be Banach spaces, $(\mathbf{x}_0, \mathbf{y}_0)$ a point in $\mathbf{X}\times\mathbf{Y}$, and U a neighborhood of $(\mathbf{x}_0, \mathbf{y}_0)$ in $\mathbf{X}\times\mathbf{Y}$. Consider a function $F : U \to \mathbf{Y}$, that satisfies $F(\mathbf{x}_0, \mathbf{y}_0) = 0$.

Assume that F is continuous in U, that D_2F exists and is continuous in U, and that $[D_2F](\mathbf{x}_0,\mathbf{y}_0) \in B(\mathbf{Y})$ has a bounded inverse. Let $M_2 > 0$ satisfy

$$||([D_2F](\mathbf{x}_0,\mathbf{y}_0))^{-1}||_0 < M_2 \tag{4.21}$$

and consider r_1, $r_2 > 0$, $\mathcal{B}_{r_1}(\mathbf{x}_0) \times \mathcal{B}_{r_2}(\mathbf{y}_0) \subset U$, satisfying

$$\sup_{(\mathbf{x},\mathbf{y})\in\mathcal{B}_{r_1}(\mathbf{x}_0)\times\mathcal{B}_{r_2}(\mathbf{y}_0)} ||D_2F(\mathbf{x},\mathbf{y}) - D_2F(\mathbf{x}_0,\mathbf{y}_0)||_0 < \frac{1}{2M_2}, \tag{4.22}$$

$$M_2 \sup_{\mathbf{x}\in\mathcal{B}_{r_1}(\mathbf{x}_0)} ||F(\mathbf{x},\mathbf{y}_0)||_Y < \frac{1}{2} r_2. \tag{4.23}$$

Then there exists a unique function $g : \mathcal{B}_{r_1}(\mathbf{x}_0) \to \mathbf{Y}$ with $g(\mathbf{x}_0) = \mathbf{y}_0$, and $F(\mathbf{x}, g(\mathbf{x})) = 0$, $\forall \mathbf{x} \in \mathcal{B}_{r_1}(\mathbf{x}_0)$. Also $g(\mathcal{B}_{r_1}(\mathbf{x}_0)) \subset \mathcal{B}_{r_2}(\mathbf{y}_0)$. If in addition, F is C^r in U, $r \geq 1$, then g is C^r for x in some $\mathcal{B}_{\tilde{r}_1}(\mathbf{x}_0)$, where $\tilde{r}_1 > 0$.

We now proceed with the proof of Lemma 3.9. We follow the steps of the Poincare map construction.

We use the following notation: let $h : X \to \mathbf{R}$ be C^r in X and define $\nabla h : X \to \mathbf{R}$ by $\langle \nabla h(u), v\rangle = Dh(u)v$. Let $\mathcal{J}v = -iv$, $v \in X$. $\mathcal{J}$ defines a symplectic structure in X. Also let $V_h = \mathcal{J}\nabla h$.

Similarly, let $E = X \times \mathbf{R} \times \mathbf{R}$ (the covering space of M) with the inner product $\langle\ ,\ \rangle_E$ defined by $\langle (u,\phi,J),(u',\phi',J')\rangle_E = \langle u,u'\rangle + \phi\phi' + JJ'$. Let $h : E \to \mathbf{R}$ be C^r in E and define the gradient $\nabla_E h$ of h as before using the inner product on E. We also use the notation $\tilde{\nabla} = \nabla_E$. Also let $\tilde{\mathcal{J}}$ be the tensor product of $\mathcal{J}$ with the standard symplectic structure in $\mathbf{R}^2$. $\tilde{\mathcal{J}}$ is a symplectic structure on E and we let $\tilde{V}_h = \tilde{\mathcal{J}}\tilde{\nabla}h$.

Furthermore denote the orthogonal projections from E to X, $\mathbf{R}$ (angle), $\mathbf{R}$ (action) by P_1, P_2, P_3 respectively. Let $\tilde{V}_h^1$, $\tilde{V}_h^2$, $\tilde{V}_h^3$ denote the corresponding components of $\tilde{V}_h$.

We first define the hypersurfaces Σ_m. Let $M = \mathcal{B}_\rho \times S^1 \times \mathbf{R}$, with ρ as in Proposition 3.1. Thus ρ is independent of Ω, for $\Omega > \Omega_1$. Let $S_\mathcal{A}$ be the set of points that belong to the periodic orbit $e^{-i\lambda t}\mathcal{A}$, where $\mathcal{A} \subset \mathcal{B}_\rho$. Note that $S_\mathcal{A}$ depends on Ω. Let $\Lambda = S_\mathcal{A} \times S^1 \times \{0\} \subset M$ be the invariant torus of the $\epsilon = 0$ problem. Let $m = (p, \phi_0, 0) \in \Lambda$, i.e. $p \in S_\mathcal{A}$ and we may assume without loss of generality that $p = \mathcal{A}$. Let $\mathcal{P}_\mathcal{A} \subset X$ be the hyperplane through $\mathcal{A}$ that is normal to $S_\mathcal{A}$ in X. Also let $\mathcal{P}_\mathcal{A}^k$ be the set of points $a \in \mathcal{P}_\mathcal{A}$ that satisfy $||a - \mathcal{A}|| \leq k$.

It is easy to check that the set $S_{\mathcal{A}}$ belongs to a $2-$(real)dimensional subspace $\mathcal{E}_{\mathcal{A}}$ of X, in particular it is a circle of radius $||\mathcal{A}||$ in $\mathcal{E}_{\mathcal{A}}$, centered at the origin. Therefore, choosing $k_0 > 0$ with $k_0 \leq \min\{\rho - ||\mathcal{A}||, ||\mathcal{A}||\}$, the sets $\mathcal{P}_{\mathcal{A}}^{k_0}$ corresponding to different $p \in S_{\mathcal{A}}$ are disjoint. Moreover, by Proposition 3.2, we have that $||\mathcal{A}|| \to A$ as $\Omega \to \infty$. Since A is a fixed parameter, i.e. it is independent of Ω, K_0 can be chosen independently of Ω for $\Omega > \Omega_1$. Let Σ_m be the set of points $(\varpi(p), \phi_0, J)$ with $\varpi(p) \in \mathcal{P}_{\mathcal{A}}^{k_0}$, $\phi_0 \in S^1$, $J \in \mathbf{R}$. By the definition of Σ_m there exists some $\tilde{\rho} > 0$ independent of Ω, and m with the property that the intersection of Σ_m and $\mathcal{B}_{\tilde{\rho}}(m)$ is nonempty, for every $\Omega > \Omega_1$, i.e. for Ω sufficiently large the size of Σ_m is independent of m and Ω.

Remark 4.7.1 Since the averaged equation, and the breather $\mathcal{A}$ depend only on $\frac{\beta}{\Omega}$, the size of Σ_m is also independent of $\frac{\beta}{\Omega}$ for $\frac{\beta}{\Omega}$ sufficiently small.

Given the defintion of Σ_m above the Poincare map construction can be outlined as follows. We observe that the time-1 map of $c_1 X_1^\epsilon$, $\epsilon \in \mathbf{R}$, maps $X \times \{\phi_0\} \times \mathbf{R} \subset M$, $\phi_0 \in S^1$, to itself, i.e. the angle is advanced by $n_2 2\pi$. Since the flow of X_2^ϵ does not change the $\phi-$component, the set $X \times \{\phi_0\} \times \mathbf{R}$ is also invariant under the time-1 map of $c_1 X_1^\epsilon + c_2 X_2^\epsilon$, $\epsilon \in \mathbf{R}$. Since for $\epsilon = 0$, $m \in \Lambda$ is also mapped to itself, we expect that for $\epsilon \neq 0$, $|\epsilon|$ sufficiently small the time-1 map of $c_1 X_1^\epsilon + c_2 X_2^\epsilon$ maps a neighborhood of m in Σ_m to points near $\mathcal{P}_{\mathcal{A}} \times \phi_0 \times \mathbf{R}$. These points can be further identified with points on Σ_m via the flow of X_2^ϵ; this will define the Poincare map from a subset of Σ_1 to Σ_1.

To define coordinates β^0, y^0 on Σ_m, let $\tilde{\Sigma}_m = \Sigma_m - m \simeq T_m \Sigma_m$. Also let $Y^0 = \tilde{\Sigma}_m \cap \ker(\nabla_E P(\mathcal{A}))$ and let Z^0 be the real span of $\nabla_E P(\mathcal{A})$, and $[0, 0, 1]$. By $\nabla H_1^0(\mathcal{A}) = \lambda \nabla H_2^0(\mathcal{A})$ the Y^0, Z^0 here coincide with the ones in the proof of Theorem 2.2. Also Z^0 is the orthogonal complement of Y^0 in $\tilde{\Sigma}_m$. Let $x = (u, \phi, J) \in \Sigma_m$. Then we let $y^0(u, \phi, J)$ be the orthogonal projection of $x - m$ to Y^0. Also let $\beta_j^0(u, \phi, J) = H_j^0(x) - H_j^0(m)$, $j = 1$, 2.

To define the domain of the restricted Poincare map, let

$$K = \{(u, \phi, J) : u = e^{i\theta} x, x \in P_1 \Sigma_m, |\theta| < \frac{\pi}{16}, \phi = \phi(m), J \in P_3 \Sigma_m\} \tag{4.24}$$

Lemma 4.8 Let $m \in \Lambda$, Σ_m, K as above. There exist $\Omega_0 > 0$, $\epsilon_3 > 0$, and a neighborhood Σ_m^3 of m in Σ_m for which $\Phi_\epsilon(\Sigma_m^3) \subset K$, for all $\epsilon \in (-\epsilon_3, \epsilon_3)$, $\Omega > \Omega_0$.

Note that for $\Omega > \Omega_0$, the set Σ_m^3 , and ϵ_3 do not depend on Ω. Lemma 4.8 follows from:

Lemma 4.9 Let $m \in \Lambda$. There exists an $\Omega_0 > 0$ such that for every neighborhood $\tilde{U}_m$ of m in M we can choose a neighborhood U_m of m in M, $U_m \subset \tilde{U}_m$, and $\epsilon_3 > 0$, both independent of Ω for $\Omega > \Omega_0$ with the property that $\Phi_\epsilon(U_m) \subset \tilde{U}_m$, for all $\epsilon \in (-\epsilon_3, \epsilon_3)$, $\Omega > \Omega_0$. (U_m independent of Ω means that it contains a ball that is independent of Ω.)

Proof: We show that there exist $\epsilon_3 > 0$ and a subset $U_m \subset \Sigma_m^2 \times I_2$, both independent of Ω, and $\frac{\beta}{\Omega}$, such that for any ϵ, $|\epsilon| < \epsilon_3$, any trajectory $v(t)$ of $c \cdot X^\epsilon = c_1 X_1^\epsilon + c_2 X_2^\epsilon$ with initial condition $v(0) \in U_m$ satisfies $v(1) \in \Sigma_m^2 \times I_2$. Let $u(t)$ be trajectory of $c \cdot X^0$ on Λ, i.e. $u(0) = u(1) = m$. Let $v(t)$ be a trajectory of $c \cdot X^\epsilon$, $v(t) \in \mathcal{B}(\rho) \times S^1 \times \mathbf{R}$. Let P_1, P_2, P_3 denote the projections onto X, S^1 (the angle coordinate), and $\mathbf{R}$ (the action coordinate) respectively. Let $\zeta_1(t) = ||P_1 v(t) - P_1 u(t)||$, $\zeta_j(t) = |P_j v(t) - P_j u(t)|$, $j = 2, 3$. Note that by $u(1) = u(m)$, $\zeta_1(1) = ||P_1 v(t) - P_1 m||$, we have $\zeta_j(1) = |P_j v(1) - P_j m|$, $j = 2, 3$. From the definition of $v(t)$, $u(t)$ we have

$$(4.25)\quad \zeta_1(t) \leq \zeta_1(0) + \int_0^t ||P_1 c \cdot X^0(u(s)) - P_1 c \cdot X^0(v(s))|| ds + |\epsilon| \int_0^t ||P_1 c \cdot \tilde{X}(v(s))|| ds.$$

From the discussion of section 3 (see also Lemma 5.4), $|c_1|$, $|c_2|$ are bounded independently of Ω for Ω sufficiently large. By Lemma 5.3 the Lipschitz constant for $P_1 X^0$ in $\mathcal{B}_\rho$ can be chosen to be independent of Ω for Ω sufficiently large. Also, by Proposition 3.1, $P_1 \tilde{X}$ is bounded by a constant independent of Ω (and β). Thus, for $t \in \mathbf{R}$, we have

$$(4.26)\qquad \zeta_1(t) \leq \zeta_1(0) + \int_0^t C_2 \zeta_1(s) ds + |\epsilon| C_3$$

with C_2, C_3 constants that are independent of Ω (and β). The functions $cH^\epsilon = c_1 H_1^\epsilon + c_2 H_2^\epsilon$, $H_2^\epsilon = P$ are constant along the trajectories of cX^ϵ. Comparing the values of H_1^ϵ at $t = 0$, 1 we have

$$(4.27)\qquad \zeta_3(1) = |J(1) - J(0)| = \frac{1}{\Omega}\left(|H_1^0(v_\epsilon(1)) - H_1^0(m)| + |\epsilon||\tilde{H}_1(v_\epsilon(1)) - \tilde{H}_1(m)|\right)$$

By Proposition 3.1, and Lemma 5.2 we then have

$$(4.28)\qquad \zeta_3(1) < C_5 \zeta_3(0) + C_6 |\epsilon|,$$

with C_5, C_6 independent of m, Ω, and $\frac{\beta}{\Omega}$ for Ω sufficiently large. Therefore

$$(4.29)\qquad \zeta_1(1) \leq (\zeta(0) + |\epsilon| C_3)(1 + C_2 e^{C_2}), \quad \zeta_2(1) = \zeta_2(0), \quad \zeta_3(1) \leq \zeta_3(0) + |\epsilon| C_4$$

with C_2, C_3, C_4 independent of Ω, and $\frac{\beta}{\Omega}$, therefore, given any $\delta > 0$, there exist d_1, d_2, d_3, $\tilde{\epsilon}_3 > 0$ independent of Ω, and $\frac{\beta}{\Omega}$ such that $\zeta_j(0) < d_j$, $j = 1, 2, 3$, $|\epsilon| < \tilde{\epsilon}_3$ imply that $\zeta_j(1) < \delta$, $j = 1, 2, 3$, as required.

$\blacksquare$

In Σ_m^3 we further let $\overline{\psi}(\beta^0, y^0) = (u, \phi, J)$, i.e. $\overline{\psi}$ takes us from the β^0, y^0 to the u, ϕ, J coordinates in Σ_m^3. Also let $\overline{\psi}_j$, $j = 1, 2, 3$ denote the respective u, ϕ, J components of $\overline{\psi}$.

In addition, let $p = (u, \phi(m), J) \in K$. Then there exists a unique pair of $\tau \in \mathbf{R}$, and $p' = (u', \phi(m), J) \in \Sigma_m^3$ such that the time$-\tau$ map of $\tilde{V}_P = -i\nabla_E P$ takes p' to p. We then let $\psi(u, \phi(m), J) = (\beta^0(u', \phi(m), J), y^0(u', \phi(m), J))$. Let ψ_2 be the y^0-component of ψ.

Remark 4.9.1 Note that τ corresponds to τ_2 in (the general setup of) Lemma 4.4.

We then define the restricted Poincare map F by

$$F(\epsilon, \beta^0, y^0) = \psi_2(\Phi_\epsilon(\overline{\psi}(\beta^0, y^0))). \tag{4.30}$$

By Lemma 4.8, and the definition of ψ, $\overline{\psi}$, the map F can be defined for all $\epsilon \in (-\epsilon_3, \epsilon_3)$ and all β^0, y^0 corresponding to points in Σ_m^3. By the definition of β^0, y^0, and the fact that Σ_m^3 contains a ball that is independent of Ω for $\Omega > \Omega_0$, the range of $\beta^0(x)$, and $y^0(x)$, with $x \in \Sigma_m^3$ is independent of Ω for $\Omega > \Omega_0$. Thus the domain of F is independent of Ω for $\Omega > \Omega_0$.

Remark 4.9.2 By the β^0, y^0, and the fact that Σ_m^3 contains a ball that is independent of $\frac{\beta}{\Omega}$ for $\Omega > \Omega_0$ and $\frac{\beta}{\Omega}$ sufficiently small we similarly see that the domain of F is independent of $\frac{\beta}{\Omega}$ for $\Omega > \Omega_0$, and $\frac{\beta}{\Omega}$ sufficiently small.

Letting $\mathbf{x} = (\epsilon, \beta^0)$, $\mathbf{y} = y^0$, and defining the norm $||\cdot||_1$ as in (4.5), we complete the proof of Lemma 3.9 by showing that μ_3, Λ_1, Λ_2 of (4.6)-(4.8), and hence ϵ_0 in Lemma 4.5 can be chosen independently of Ω for Ω sufficiently large. The estimates below are also seen to be also independent of $\frac{\beta}{\Omega}$ for $\Omega > \Omega_0$ and $\frac{\beta}{\Omega}$ sufficiently small.

By (4.6), the definition of $\frac{\partial \hat{y}^0}{\partial y^0}$ in the proof of Lemma 4.5, and Lemmas 3.7-3.8 we have

$$\mu_3 = ||[D_2 F(0,0)]^{-1}||_0 = ||[\frac{\partial \hat{y}^0}{\partial y^0}(0,0)]^{-1}||_0 < r_0, \tag{4.31}$$

with r_0 independent of Ω for $\Omega > \Omega_0 \geq \Omega_5$. By Remark 3.9.1 r_0 is independent of $\frac{\beta}{\Omega}$ for $\Omega > \Omega_0$ and $\frac{\beta}{\Omega}$ sufficiently small. We further have:

Lemma 4.10 Let $m \in \Lambda$, $|\epsilon| < \epsilon_3$. Then there exists an $\Omega_0 > 0$ such that for $\Omega > \Omega_0$ the constants Λ_1, Λ_2 can be chosen independently of Ω, and $m \in \Lambda$.

Proof: Let $m \in \Lambda$ and ψ, $\overline{\psi}$ as above. Let Φ_ϵ be the time-1 map of the vector field $c \cdot X^\epsilon = c_1 X_1^\epsilon + c_2 X_2^\epsilon$. We first estimate Λ_2. Since

$$F(\epsilon, \beta^0, 0) = \psi_1(\Phi_\epsilon(\overline{\psi}(\beta^0, 0))), \tag{4.32}$$

to show (4.8) with Λ_2 independent of Ω, m it is enough to find $K_1, K_2, \ldots, K_6$ independent of Ω, m, such that for $|\epsilon| < \epsilon_3$ we have

$$||\overline{\psi}(\beta^0, 0)||_E < K_5||\beta^0||_{\mathbf{R}^2} + K_6|\epsilon|, \quad \forall (\beta^0, 0) \in \Sigma_m^3; \tag{4.33}$$

$$||\Phi_\epsilon(v - m)||_E < K_1||v - m||_E + K_2|\epsilon|, \quad \forall v \in U_m; \tag{4.34}$$

$$||\psi_2(v - m)||_{Y^0} < K_3||v - m||_E + K_4|\epsilon|, \quad \forall v \in K. \tag{4.35}$$

To prove (4.33), write $m = \mathcal{A}e^{i\theta}$, i.e. as in the breather equation, and let $z \in \Sigma_m^3$. Then $\beta_j^0 = H_j^z - H_j(m)$. Also, let $\tilde{u} = P_1(z - m) \in X$, $\tilde{J} = P_3(z - m) \in \mathbf{R}$. Then

$$\beta_2^0 = 2\mathrm{Re}\langle \mathcal{A}e^{i\theta}, u\rangle_c + ||\tilde{u}||, \tag{4.36}$$

with $\langle u, v\rangle_c = \sum_{n\in\mathbf{Z}} u_n v_n^*$ the standard Hermitian inner product in $l_2(\mathbf{Z}, \mathbf{C})$. Since $y^0 = 0$, $z \in \Sigma_m^3$, the variable u can only have components along $\nabla_E H_1(m)$, $\nabla_E H_2(m)$. Moreover, by the breather equation with $\lambda \in \mathbf{R}$, $\nabla H_1(m)$, $\nabla H_2(m)$ are colinear. Therefore $\mathcal{A}e^{i\theta}$, and $\tilde{u}$ are colinear and (4.36) becomes

$$\beta_2^0 = \pm 2||\mathcal{A}||||\tilde{u}|| + ||u||^2 \tag{4.37}$$

with the $+$, $-$ signs corresponding to $\beta_2^0 > 0$, $\beta_2^0 < 0$ respectively. In the $+$ case we have

$$||\tilde{u}|| \leq \frac{|\beta_2^0|}{2||\mathcal{A}|| + ||\tilde{u}||} \leq \frac{|\beta_2^0|}{2||\mathcal{A}||}. \tag{4.38}$$

Recall that by the discussion of the tubular neighborhood around Λ we have $||\tilde{u}|| < ||\mathcal{A}||$. Then, in the $-$ case we have

$$||\tilde{u}|| \leq \frac{|\beta_2^0|}{2||\mathcal{A}||||\tilde{u}||} \leq \frac{|\beta_2^0|}{||\mathcal{A}||}. \tag{4.39}$$

By the discussion on the tubular neighborhood around Λ, $||\mathcal{A}||$ is bounded away from the origin as $\Omega \to \infty$ and therefore (4.38), (4.39) imply that

$$||\tilde{u}|| < C||\beta_2^0||, \tag{4.40}$$

with C independent of Ω for Ω sufficiently large. Furthermore,

$$\beta_1^0 = -\Omega\tilde{J} + \overline{h}_4(z) - \overline{h}_4(m), \tag{4.41}$$

hence

$$\tilde{J} = \frac{1}{\Omega}\left(-\beta_1^0 - \overline{h}_4(z) + \overline{h}_4(m)\right). \tag{4.42}$$

By Lemma 5.2 we then have that, for Ω sufficiently large, there exist C_1, C_2 independent of Ω such that

$$|\tilde{J}| < C_1|\beta_1^0| + C_2||\tilde{u}||. \tag{4.43}$$

In addition, $|\phi| = |P_3(z-m)| = 0$. Combining this with (4.40), (4.43), we obtain (4.33).

To show (4.34) observe that $v = v_\epsilon(1) - m$, i.e. with v as in (4.34), v_ϵ as in of Lemma 4.9. Then, (4.34) follows from the arguments of Lemma 4.9.

To show (4.35) we write

$$\psi_2(v-m) = P_{Y^0}(g_2^\tau(v) - m) = P_{Y^0\cap X}P_1(g_2^\tau(v) - m), \quad \tau = \tau(v, \Sigma_m), \tag{4.44}$$

where P_{Y^0} is the orthogonal projection onto Y^0, and $g_2^\tau = g_{2,\epsilon}^\tau$. We have

$$||P_{Y^0\cap X}||_0 \le 1 + ||P_{Z^0\cap X}||_0 \le 1 + ||||\mathcal{A}||^{-1}\mathcal{A}\langle\mathcal{A},\cdot\rangle||_0 \le 1 + ||\mathcal{A}|| \le 1 + \rho, \tag{4.45}$$

with ρ independent of Ω. To estimate $P_1(g_2^\tau(v) - m)$ in (4.44), let $v_1 = P_1 v$, $m_1 = P_1 m$, and $\theta = \tau(v, \Sigma_m)$. We have

(4.46)
$$||P_1(g_2^\tau(v) - m)|| = ||e^{i\theta}v_1 - m_1|| \le ||v_1 - m_1|| + |\int_0^\theta P_1 X_2^\epsilon(v_1(s))ds| \le ||v_1 - m_1|| + \rho|\theta|.$$

To estimate θ consider X_c, i.e. $l_2(\mathbf{Z}, \mathbf{C})$ with the complex Hermitian structure $\langle\cdot,\cdot\rangle_c$, and apply Gram-Schmidt to obtain a new basis $\{b_n\}_{n\in\mathbf{Z}}$ in which $-i\nabla P(\mathcal{A}) = ||\mathcal{A}||b_1$. Let U be the corresponding unitary operator. The action of pointwise mutiplication by $e^{i\theta}$ in the standard basis of X_c commutes with U, $U^\dagger$ and therefore has the same diagonal representation in the new basis. Note also that $\tilde{\Sigma}_m$ is the direct sum of the Hermitian complement of $-i\nabla P(\mathcal{A})$, viewed as a (real) subspace of X, and the (real) span of $\nabla P(\mathcal{A})$. Then θ is the angle between $\tilde{v}_1$, and $\mu = \nabla P(\mathcal{A})$, where $\tilde{v}_1 = \langle v_1, b_1\rangle_c$. Since both $\nabla P(\mathcal{A})$, v_1 belong to the same real $2-$plane, and $|\theta| < \frac{\pi}{16}$ by $v \in K$, we have

$$|\theta| \leq \frac{\pi}{2}\sin|\theta| \leq \frac{\pi}{2}\frac{|\tilde{\mu}-\mu|}{|\mu|}, \tag{4.47}$$

where $\tilde{\mu}$ is the point along $\tilde{v}_1$ which makes the segment $\tilde{\mu}$, μ normal to $\tilde{v}_1$. Therefore, (4.47) implies

$$|\theta| \leq \frac{\pi}{2}\frac{|\tilde{v}_1-\mu|}{|\mu|} \leq \frac{\pi}{2}\frac{||v_1-m_1||}{|m_1|} \leq \frac{\pi}{2\rho}||v-m||_E. \tag{4.48}$$

(4.35) is obtained by combining (4.44)- (4.46), (4.48).

To estimate Λ_1, write

$$D_2F(\epsilon,\beta^0,y^0) = \frac{\partial}{\partial y^0}\psi_2(\Phi_\epsilon(\overline{\psi}(\beta^0,y))) - I = \tag{4.49}$$

$$= \frac{\partial\psi_2}{\partial u}\frac{\partial\Phi_\epsilon^1}{\partial y^0} + \frac{\partial\psi_2}{\partial\phi}\frac{\partial\Phi_\epsilon^2}{\partial y^0} + \frac{\partial\psi_2}{\partial J}\frac{\partial\Phi_\epsilon^3}{\partial y^0} - I$$

where I is the identity in Y^0 and Φ_ϵ^1, Φ_ϵ^2, Φ_ϵ^3 are the u, ϕ, and J components of Φ_ϵ respectively. We observe that $\frac{\partial\psi_2}{\partial\phi}$, $\frac{\partial\psi_2}{\partial J}$ vanish identically. Moreover

$$\frac{\partial\Phi_\epsilon^1}{\partial y^0} = \frac{\partial\Phi_\epsilon^1}{\partial u}\frac{\partial\overline{\psi}_1}{\partial y^0} + \frac{\partial\Phi_\epsilon^1}{\partial\phi}\frac{\partial\overline{\psi}_2}{\partial y^0} + \frac{\partial\Phi_\epsilon^1}{\partial J}\frac{\partial\overline{\psi}_3}{\partial y^0} \tag{4.50}$$

where $\frac{\partial\overline{\psi}_2}{\partial y^0}$, $\frac{\partial\overline{\psi}_3}{\partial y^0}$ also vanish identically. Therefore (4.49) reduces to

$$D_2F(\epsilon,\beta^0,y) = AB_\epsilon C - I \tag{4.51}$$

with

$$A = D_1\psi_2 \quad \text{at} \quad \Phi_\epsilon(\overline{\psi}(\beta^0,y)), \quad B_\epsilon = D_1\Phi_\epsilon^1 \quad \text{at} \quad \overline{\psi}(\beta^0,y), \tag{4.52}$$

$$C = D_2\overline{\psi}_1 \quad \text{at} \quad (\beta^0,y).$$

Then

$$D_2F(\epsilon,\beta^0,y) - D_2F(0,0,0) = A(B_\epsilon - B_0)C. \tag{4.53}$$

Note that $A \in B(X, Y^0)$, $B_\epsilon \in B(X, X)$, $C \in B(Y^0, X)$, $D_2F \in B(Y^0, Y^0)$.

To show that Λ_1 can be chosen independently of Ω for Ω sufficiently large it is then enough to show that for $|\epsilon| < \epsilon_3$, $(\beta^0, y^0) \in \Sigma_m^3$ and Ω sufficiently large we can choose L_B, M_1, M_3 independent of Ω, and $m \in \Lambda$ for which

$$||B_\epsilon - B_0||_0 < L_B||(\epsilon, \beta^0, y)||_1 \tag{4.54}$$

$$||A||_0 < M_1, \quad ||C||_0 < M_3, \tag{4.55}$$

where $||\cdot||_0$ denotes the operator norm in the respective spaces.

To bound $||A||_0$, we have that by (4.52), (4.44)

$$A = \frac{\partial}{\partial u}\psi_2(u, \phi, J) = P_{Y^0 \cap X}\frac{\partial}{\partial u}P_1 g_2^\tau(u)), \quad \tau = \tau(v, \Sigma_m), \tag{4.56}$$

since $P_{Y^0 \cap X}$ is constant. By (4.45), (4.56) implies

$$||A||_0 < C||\frac{\partial}{\partial u}P_1 g_2^\tau(u))||_0, \tag{4.57}$$

with C independent of Ω for Ω large. Letting $\theta = \tau(u, \Sigma_m)$ we also have

$$[\frac{\partial}{\partial u}P_1 g_2^\tau(u))]v = \langle\nabla\theta, v\rangle i e^{i\theta}u + e^{i\theta}v. \tag{4.58)}$$

Employing the change of basis we used in showing (4.35), and letting $u_1 = \langle u, b_1\rangle_c$, we observe that θ is the angle between u_1, $\mu = \nabla P(\mathcal{A})$ and therefore depends on u_1 alone. Then, letting $u_1 = [x, y] \in \mathbf{R}^2$, i.e. the complex span of b_1, we have

$$||\langle\nabla\theta, v\rangle|| \leq (\theta_x^2 + \theta_y^2)||v||. \tag{4.59}$$

From (4.58) we therefore have that

$$||\frac{\partial}{\partial u}P_1 g_2^\tau(u)||_0 \leq 1 + \rho. \tag{4.60}$$

By (4.57), (4.60) we can choose $M_1 = c(1+\rho)$ in (4.55), i.e. is idependent of Ω.

To bound $||C||_0$ in (4.55), we note that u, the first coordinate of point in Σ_m, is given by $u = P_1(z + y)$, with $z \in Z^0$, $y \in Y^0$. Thus we can choose $M_3 = 1$.

To show (4.54), let $s \in [0,1]$, $u_j(s) \in M$ a trajectory of cX^{ε_j}, with $|\varepsilon_j| < \epsilon_3$, $j = 1, 2$. Then $D\Phi_{\epsilon_j}(u_j(0)) = \mathbf{V}_j(1)$ where $\mathbf{V}_j(t)$ satisfies

$$\mathbf{V}_j(t) = \mathbf{V}_j(0) + \int_0^t D(c \cdot X^{\varepsilon_j}(u_j(s)))\mathbf{V}_j(s)ds, \quad \mathbf{V}_j(0) = I, \tag{4.61}$$

with I the identity in X. Letting $\varepsilon_2 = \epsilon$, $|\epsilon| < \epsilon_3$, and $\varepsilon_1 = 0$ we have

$$||B_\epsilon - B_0||_0 = ||\mathbf{V}_2(1) - \mathbf{V}_1(1)||_0 \tag{4.62}$$

with

$$\begin{aligned} ||\mathbf{V}_2(t) - \mathbf{V}_1(t)||_0 \leq & \int_0^t ||D_1 P_1 c \cdot X^0(u_2(s))||_0 ||\mathbf{V}_2(s) - \mathbf{V}_1(s)||_0 ds + \\ & + \int_0^t ||D_1 P_1 c \cdot X^0(u_2(s)) - D_1 P_1 c \cdot X^0(u_1(s))||_0 ||\mathbf{V}_1(s)||_0 ds + \\ & |\epsilon||c_2| \int_0^t ||D_1 P_1 \tilde{X}(u_2(s))||_0 ||\mathbf{V}_2(s)||_0 ds, \end{aligned} \tag{4.63}$$

with $||\cdot||_0$ the operator norm in $B(X,X)$, $t \in [0,1]$. Recall that $u_j(s) \in \mathcal{B}(\rho) \times S^1 \times \mathbf{R}$, $\forall s \in [0,1]$, $j = 1, 2$, and that by the argument of Lemma 4.9 we have

$$||P_1 u_2(s) - P_1 u_1(s)||_{\tilde{X}} \leq K_4''(||u_2(0) - u_1(0)||_X + |\epsilon|), \quad \forall s \in [0,1], \tag{4.64}$$

with K_4'' independent of Ω for Ω sufficiently large.

Choosing Ω sufficiently large we have the following. First, by $D_1 P_1 c \cdot X^{\varepsilon_1} = D_1 P_1 c \cdot X^0$, expression (4.61) for $\mathbf{V}_1(t)$, Lemma 5.5, and Gronwall we obtain

$$||\mathbf{V}_1(t)||_0 < K_5, \quad \forall t \in [0,1], \tag{4.65}$$

with K_5 independent of Ω. Also, by $D_1 P_1 c \cdot X^{\varepsilon_2} = D_1 P_1 c \cdot (X^0 + \epsilon \tilde{X})$, expression (4.61) for $V_2(t)$, Lemmas 5.1, 5.5, and Gronwall we have

$$||\mathbf{V}_2(t)||_0 < K_7, \quad \forall t \in [0,1], \tag{4.66}$$

with K_7 independent of Ω. By Lemma 5.5 we also have

$$||D_1 P_1 c_1 X_1^0(u_2(s))||_0 < K_3, \quad \forall s \in [0,1], \tag{4.67}$$

with K_3 independent of Ω. Also, by Lemma 5.1, and (4.64) we have

$$||D_1 P_1 c \cdot X^0(u_2(s)) - D_1 P_1 c \cdot X^0(u_1(s))||_0 < K_4' K_4''(||u_2(0) - u_1(0)||_E + |\epsilon|), \tag{4.68}$$

$\forall s \in [0,1]$. Furthermore by (4.64) we have

$$||\mathbf{V}_1(s)||_0 < K_5, \quad \forall s \in [0,1], \tag{4.69}$$

with K_5 independent of Ω.

Combining (4.65)-(4.69), and (4.63) we then have

$$||\mathbf{V}_2(t) - \mathbf{V}_1(t)||_0 < \int_0^t K_3 ||\mathbf{V}_2(s) - \mathbf{V}_1(s)||_0 ds + \tag{4.70}$$

$$+ \int_0^t K_4{}' K_4{}'' K_5 (||P_1 u_2(0) - P_1 u_1(0)||_X + |\epsilon|) ds + |\epsilon| \int_0^t K_6 K_7 ds.$$

By (4.70), and Gronwall we therefore have

$$||B_\epsilon - B_0||_0 = ||\mathbf{V}_2(t) - \mathbf{V}_1(t)||_0 < K_8(||u_2(0) - u_1(0)||_E + |\epsilon|), \tag{4.71}$$

with K_8 independent of Ω. By $u_1(0) = m$, and setting $u = u_2(0) - m$, (4.71) then yields

$$||B_\epsilon - B_0||_0 < K_8(||u||_E + |\epsilon|) \leq K_8{}' ||(\epsilon, \beta^0, y)||_1 \tag{4.72}$$

with $K_8{}'$ independent of Ω. Setting $L_B = K_8{}'$ we obtain (4.54).

■

5. Some auxiliary lemmas

Lemma 5.1 Let $u \in \mathcal{B}(\rho) \times S^1 \times \mathbf{R}$, $|| \cdot ||_0$ the operator norm in $B(X,X)$. Then

$$||D_1 P_1 \tilde{X}(u)||_0 < K_6,$$

with K_6 independent of Ω, and $\frac{\beta}{\Omega}$ for Ω sufficiently large and $\frac{\beta}{\Omega}$ sufficiently small.

Proof: Let $z \in M$. We will use the following expression for the remainder:

$$\frac{1}{\Omega}\tilde{X}(z) = R_I(z) + R_{II}(z) + R_{III}(z), \quad \text{with} \tag{5.1}$$

$$R_I(z) = \tilde{V}_{h_4 \circ \mathcal{T}}(z) - \tilde{V}_{h_4}(z), \tag{5.2}$$

$$R_{II}(z) = \tilde{V}_{h_2 \circ \mathcal{T}}(z) - \tilde{V}_{h_2}(z), \tag{5.3}$$

$$R_{III}(z) = \tilde{V}_{h_0 \circ \mathcal{T} - h_0 - [S, h_0]}(z). \tag{5.4}$$

We want to bound ΩR_I, …, ΩR_{III} by quantities that are independent of Ω for Ω sufficiently large.

First we recall that for $h : M \to \mathbf{R}$, and $\mathcal{G}$ a symplectic transformation in M,

$$\tilde{V}_{h \circ \mathcal{G}}(z) = ([D\mathcal{G}](z))^{-1} \tilde{V}_h(\mathcal{G}(z)), \tag{5.5}$$

by the chain rule. Moreover, by the definition of S, $\overline{h}_4$ we have

$$R_{III}(z) = \int_0^1 \{([D\mathcal{T}^\alpha](z))^{-1} \tilde{V}_{\tilde{h}_4}(\mathcal{T}^\alpha(z)) - \tilde{V}_{\tilde{h}_4}(z)\} d\alpha, \quad \text{with} \quad \tilde{h}_4 = \overline{h}_4 - h_4. \tag{5.6}$$

Also, $[D\mathcal{T}^\alpha(z)]^{-1} = D\mathcal{T}^{-\alpha}(\mathcal{T}^\alpha(z))$, $\forall \alpha \in [0,1]$. We further observe that $D_1 \mathcal{T}_2^\alpha$, $D_1 \mathcal{T}_3^\alpha$, $D_3 \mathcal{T}_1^\alpha$ vanish identically, $\forall \alpha \in [0,1]$, and that $([D\mathcal{T}^\alpha(z)]^{-1})_{1,1} = D_1 \mathcal{T}_\infty{}^{-\alpha}(\mathcal{T}^\alpha(z))$, $\forall \alpha \in [0,1]$, where $(\dot{})_{1,1}$ denotes the X, X block. These simplifications follow from the fact that S depends only on the first two components (see [P1], p. 233).

To estimate $R_I(z)$, let $f(z) = \tilde{V}_{h_4}(z)$. From (5.2), (5.5)

$$R_I(z) = [D\mathcal{T}^{-1}(\mathcal{T}(z))](f(\mathcal{T}(z)) - f(z)) + ([D\mathcal{T}^{-1}(\mathcal{T}(z))] - I) f(z). \tag{5.7}$$

Using the simplifications above we have

$$P_1 R_I(z) = [D_1 \mathcal{T}_1{}^{-1}(\mathcal{T}(z))](f_1(\mathcal{T}(z)) - f_1(z)) + ([D_1 \mathcal{T}_1{}^{-1}(\mathcal{T}(z))] - I) f_1(z) \tag{5.8}$$

(in (5.8) I is the identity in X). Differentiating, using the vanishing of $D_1 \mathcal{T}_2$, $D_1 \mathcal{T}_3$, and taking norms we have

$$||D_1 P_1 R_I(z)||_0 \leq ||D_1^2 \mathcal{T}_1^{-1}(\mathcal{T}(z))||_{0,0} ||D_1 \mathcal{T}_1(z)||_0 ||f_1(\mathcal{T}(z)) - f_1(z)|| + \tag{5.9}$$

$$+ ||D_1 \mathcal{T}_1^{-1}(\mathcal{T}(z))||_0 ||D_1(f_1(\mathcal{T}(z)) - f_1(z))||_0 + ||D_1^2 \mathcal{T}_1^{-1}(\mathcal{T}(z))||_{0,0} ||D_1 \mathcal{T}_1(z)||_0 ||f_1(z)|| +$$

$$+ ||D_1 \mathcal{T}_1^{-1}(\mathcal{T}(z)) - I||_0 ||D_1 f_1((z))||_0,$$

where $||\cdot||_{0,0}$ is the operator norm in $B(X, B(X,X))$, $||\cdot||_0$ is the operator norm in $B(X,X)$, and $||\cdot||$ is the norm in X.

Similarly, differentiating (5.6) and noting that $P_1 R_{III}$ is C^1 in X, we have

$$||D_1 P_1 R_{III}(z)||_0 \leq \int_0^1 ||A_{III}(z,\alpha)||_0 d\alpha \tag{5.10}$$

with

$$\begin{aligned} ||A_{III}(z,\alpha)||_0 \leq\ & ||D_1^2 \mathcal{T}_1^{-\alpha}(\mathcal{T}^\alpha(z))||_{0,0}||D_1\mathcal{T}_1^\alpha(z)||_0||f_1(\mathcal{T}^\alpha(z)) - f_1(z)||+ \\ & +||D_1\mathcal{T}_1^{-\alpha}(\mathcal{T}^\alpha(z))||_0||D_1(f_1(\mathcal{T}^\alpha(z)) - f_1(z))||_0+ \\ & +||D_1^2\mathcal{T}_1^{-\alpha}(\mathcal{T}^\alpha(z))||_{0,0}||D_1\mathcal{T}_1^\alpha(z)||_0||f_1(z)||+ \\ & +||D_1\mathcal{T}_1^{-\alpha}(\mathcal{T}^\alpha(z)) - I||_0||D_1 f_1(\mathcal{T}^\alpha(z))||_0, \end{aligned} \tag{5.11}$$

with $f(z) = \tilde{V}_{\tilde{h}_4}$, and the notation of (5.9) for the norms.

We estimate the various terms in (5.9), (5.11). To avoid repetition we assume that Ω is large enough, and $\frac{\beta}{\Omega}$ small enough for the $\Omega-$, and $\frac{\beta}{\Omega}-$ independent bounds of Lemmas 5.2-5.9 to hold.

First, in (5.9), and (5.11),

$$||P_1 f(z)|| < T_3, \quad ||D_1 P_1 f(z)||_0 < T_6, \tag{5.12}$$

with T_3, T_6 independent of Ω, $\frac{\beta}{\Omega}$ by Lemma 5.6. Similarly, by Lemma 5.6, and $\mathcal{T}^\alpha(z) \in \mathcal{B}(\rho) \times S^1 \times \mathbf{R}$,

$$||P_1 f(\mathcal{T}^\alpha(z)) - P_1 f(z)|| < T_7, \quad \forall \alpha \in [0,1], \tag{5.13}$$

with T_7 independent of Ω, $\frac{\beta}{\Omega}$.

Also, $D_1\mathcal{T}_1(z) = \mathbf{V}(1)$ where $\mathbf{V}(t)$ satisfies

$$\mathbf{V}(t) = \mathbf{V}(0) + \int_0^t [D_1 \tilde{V}_S^1(z(\tau))]\mathbf{V}(\tau)d\tau, \quad \mathbf{V}(0) = I, \tag{5.14}$$

with $z(\tau)$ the image of $z = z(0)$ under the time$-\tau$ map of the flow of $\tilde{V}_S$. By Lemma 5.7 and Gronwall we then have

$$||D_1\mathcal{T}_1(z)||_0 < T_2, \tag{5.15}$$

with T_2 independent of Ω, $\frac{\beta}{\Omega}$.

Similarly, $D_1\mathcal{T}_1^{-1}(\mathcal{T}(z)) = \tilde{\mathbf{V}}(1)$, where $\tilde{\mathbf{V}}(t)$ satisfies

$$\tilde{\mathbf{V}}(t) = \tilde{\mathbf{V}}(0) + \int_0^t [D_1\tilde{V}^1_{-S}(\tilde{z}(\tau))]\tilde{\mathbf{V}}(\tau)d\tau, \quad \tilde{\mathbf{V}}(0) = I, \tag{5.16}$$

with $\tilde{z}(\tau)$ the image of $\mathcal{T}(z) = \tilde{z}(0)$ under the time$-\tau$ map of the flow of $\tilde{V}_{-S}$. By Lemma 5.7, and Gronwall we then obtain

$$||D_1\mathcal{T}_1^{-1}(\mathcal{T}(z)) - I||_0 < \frac{T_9}{\Omega}, \tag{5.17}$$

$$||D_1\mathcal{T}_1^{-1}(\mathcal{T}(z))||_0 < T_4, \tag{5.18}$$

with T_9, T_4 independent of Ω, $\frac{\beta}{\Omega}$.

Also,

$$||D_1(f_1(\mathcal{T}(z)) - f_1(z))||_0 \leq ||D_1f_1(\mathcal{T}(z))||_0||D_1\mathcal{T}_1(z) - I||_0 + \tag{5.19}$$

$$||D_1f_1(\mathcal{T}(z)) - D_1f_1(z)||_0.$$

Using the variational equation for $D\mathcal{T}(z)$, Gronwall, and Lemma 5.7 we have

$$||D_1\mathcal{T}_1(z) - I||_0 < \frac{T_{10}}{\Omega}, \tag{5.20}$$

with T_{10} independent of Ω, $\frac{\beta}{\Omega}$, moreover, by Lemma 5.8, the variational equation for $D_1\mathcal{T}_1(z)$, and Gronwall

$$||D_1f_1(\mathcal{T}(z)) - D_1f_1(z)||_0 < \frac{T_{11}}{\Omega}, \tag{5.21}$$

with T_{11} independent of Ω, $\frac{\beta}{\Omega}$. By (5.19)-(5.21), and (5.12) we therefore obtain

$$||D_1(f_1(\mathcal{T}(z)) - f_1(z))||_0 < \frac{T_8}{\Omega}, \tag{5.22}$$

with T_8 independent of Ω, $\frac{\beta}{\Omega}$.

To estimate $D_1^2\mathcal{T}_1^{-1}(\mathcal{T}(z))$ we look at the equation for the second variation and obtain that $D_1^2\mathcal{T}_1^{-1}(\mathcal{T}(z)) = \mathbf{W}(1)$ where $\mathbf{W}(t)$ satisfies

$$\mathbf{W}(t)h = \mathbf{W}(0)h + \int_0^t \{D_1\tilde{V}^1_{-S}(u(s))\mathbf{W}(s)h + [D_1^2\tilde{V}^1_{-S}(u(s))\mathbf{V}(s)h]\mathbf{V}(s)\}ds, \tag{5.23}$$

$\mathbf{W}(0) = 0$, where $h \in X$ and $\mathbf{V}(t)$ satisfies (5.14). Furthermore

$$||D_1 \tilde{V}^1_{-S}(u(s))\mathbf{W}(s)||_{0,0} \leq ||D_1 \tilde{V}^1_{-S}(u(s))||_0 ||\mathbf{W}(s)||_{0,0}, \tag{5.24}$$

$$||[D_1^2 \tilde{V}^1_{-S}(u(s))\mathbf{V}(s)h]\mathbf{V}(s)||_0 \leq ||D_1^2 \tilde{V}^1_{-S}(u(s))\mathbf{V}(s)h||_0 ||\mathbf{V}(s)||_0 \leq$$

$$\leq ||D_1^2 \tilde{V}^1_{-S}(u(s))||_{0,0} ||\mathbf{V}(s)h|| ||\mathbf{V}(s)||_0, \tag{5.25}$$

hence

$$||[D_1^2 \tilde{V}^1_{-S}(u(s))\mathbf{V}(s)(\cdot)]\mathbf{V}(s)||_{0,0} \leq ||D_1^2 \tilde{V}^1_{-S}(u(s))||_{0,0}(||\mathbf{V}(s)||_0)^2. \tag{5.26}$$

Then, by Lemmas 5.7, 5.9, (5.23)-(5.26), and Gronwall we have

$$||D_1^2 \mathcal{T}_1^{-1}(\mathcal{T}(z))||_{0,0} < \frac{T_1}{\Omega}, \tag{5.27}$$

with T_1 independent of Ω, $\frac{\beta}{\Omega}$.

Collecting the bounds (5.12), (5.13), (5.15), (5.17), (5.18), (5.22), (5.27) for the terms of (5.9) we have

$$||D_1 P_1 R_I(z)||_0 < \frac{C_1}{\Omega}, \tag{5.28}$$

with C_1 independent of Ω, $\frac{\beta}{\Omega}$ for Ω sufficiently large, $\frac{\beta}{\Omega}$ sufficiently small. To estimate $D_1 P_1 R_{II}$ we use (5.9) with $f(z) = \tilde{V}_{h_2}(z)$. Since h_2 is independent of β, ω we similarly have

$$||D_1 P_1 R_{II}(z)||_0 < \frac{C_2}{\Omega}, \tag{5.29}$$

with C_2 independent of Ω, $\frac{\beta}{\Omega}$ for Ω sufficiently large, $\frac{\beta}{\Omega}$ sufficiently small.

In (5.11) we also have the terms of (5.9) with $\mathcal{T}$, $\mathcal{T}^{-1}$, replaced by $\mathcal{T}^{\alpha}$, $\mathcal{T}^{-\alpha}$ respectively. We argue as above, using Gronwall up to a time α. The bounds are similar, and uniform in $\alpha \in [0, 1]$, so that (5.10), (5.11) yield

$$||D_1 P_1 R_{III}(z)||_0 < \frac{C_3}{\Omega}, \tag{5.30}$$

with C_3 independent of Ω, $\frac{\beta}{\Omega}$ for Ω sufficiently large, $\frac{\beta}{\Omega}$ sufficiently small.

■

We sketch the proofs of the Lemmas 5-2-5.9 usd above.

Lemma 5.2 Let $u,\ v\in\mathcal{B}(\rho)\subset X$. Then $|\overline{h}_4(u)-\overline{h}_4(v)|\le k||u-v||$ with k independent of Ω, $\phi\in S^1$, and $\frac{\beta}{\Omega}$.

Proof: Let $a,\ b\in X$, $h_4(a)=\sum_{n\in\mathbf{Z}}|a_n|^4$. We easily obtain

$$|h_4(a)-h_4(b)|\le(||a||^2+||b||^2)(||a||+||b||)||a-b||. \tag{5.31}$$

Recall that L_ψ is norm continuous in ψ, and an isometry in X, $\forall\psi\in[0,2\pi]$. Then by (5.31) we have that for $u,\ v\in\mathcal{B}(\rho)$

$$|\overline{h}_4(a)-\overline{h}_4(b)|\le\frac{|\gamma|}{2\pi}\int_0^{2\pi}(||L_\psi u||^2+||L_\psi v||^2)(||L_\psi u||+||L_\psi v||)||L_\psi u-L_\psi v||d\psi\le \tag{5.32}$$

$$\le|\gamma|(||u||^2+||v||^2)(|u||+||v||)||u-v||\le 4|\gamma|\rho^3||u-v||.$$

■

Lemma 5.3 Let $u,\ v\in\mathcal{B}(\rho)$. Then

$$||P_1\tilde{V}_{\overline{h}_4}(u)-P_1\tilde{V}^1_{\overline{h}_4}(v)||<C||u-v||, \tag{5.33}$$

with C independent of Ω, $\phi\in S^1$, and $\frac{\beta}{\Omega}$.

Proof: Let $a,\ b\in X$, $(g(a))_n=|a_n|^2a_n$. We estimate

$$||g(a)-g(b)||\le(||a||^4+||b||^4+||a||^2||b||^2)^{\frac{1}{2}}||a-b||. \tag{5.34}$$

Using the continuity and isometry properties of L_ψ, as in Lemma 5.2, (5.34) implies that, for $u,\ v\in\mathcal{B}(\rho)$,

$$||P_1\tilde{V}_{\overline{h}_4}(u)-P_1\tilde{V}_{\overline{h}_4}(v)||\le \tag{5.35}$$

$$\le\frac{|\gamma|}{2\pi}\int_0^{\pi}(||L_\psi u||^4+||L_\psi v||^4+||L_\psi u||^2||L_\psi v||^2)^{\frac{1}{2}}||L_\psi u-L_\psi v||d\psi\le 2\sqrt{3}|\gamma|\rho^2||u-v||.$$

■

Lemma 5.4 Let u, $v \in \mathcal{B}(\rho) \times S^1 \times \mathbf{R}$, $||\cdot||_0$ the operator norm in $B(X, X)$. Then

$$(5.36) \qquad ||D_1 P_1 c \cdot X^0(u) - D_1 P_1 c \cdot X^0(v)||_0 < K_4{}'||u - v||,$$

with $K_4{}'$ independent of Ω, $\phi \in S^1$, and $\frac{\beta}{\Omega}$ for Ω sufficiently large, and $\frac{\beta}{\Omega}$ sufficiently small.

Proof: We have $c \cdot X^0 = c_1 X_1^0 + c_2 X_2^0$, with c_1, c_2 as in (3.18). By Lemma 2.5

$$(5.37) \qquad |\lambda c_1(\Omega)| \leq \frac{\pi}{2}, \quad \forall \Omega > \Omega_3,$$

and since $n_1 = -1$, $|c_2|$ is bounded independently of Ω, and $\frac{\beta}{\Omega}$, provided that $\Omega > \Omega_3$. Similarly,

$$(5.38) \qquad |c_1| = \frac{1}{|\lambda|}|n_2 \frac{2\pi}{\Omega} \lambda| \leq \frac{\pi}{2|\lambda|}, \quad \forall \Omega > \Omega_3,$$

with λ fixed, i.e. independent of Ω, and $\frac{\beta}{\Omega}$. Also, for $u \in X$, the derivative of X_2^0 is

$$(5.39) \qquad D_1 P_1 X_2^0(u) = D_1 u = I,$$

i.e. independent of u. It therefore remains to estimate the Lipschitz constant for $D_1 P_1 X_1^0$. For u, $v \in X$, we have (see also (3.21), (3.22))

$$(5.40) \qquad [D\overline{g}_L(u)]v = \frac{1}{2\pi} \int_0^{2\pi} L_\psi^\dagger [Dg(L_\psi u)] L_\psi v d\psi, \quad \text{with}$$

$$(5.41) \qquad ([Dg(u)]v)_n = 2|u_n|^2 v_n + u_n^2 v_n^*, \quad n \in \mathbf{Z}.$$

For a, b, $v \in X$, we then estimate that

$$(5.42) \qquad ||[Dg(a) - Dg(b)]v|| \leq \sqrt{20}(||a||^2 + ||b||^2)||a - b||||v||.$$

Using (5.39)-(5.42) and the properties of L_ψ we have that for u, $v \in \mathcal{B}(\rho)$, $w \in X$,

$$(5.43) \qquad ||[D_1 P_1 X_1^0(u) - D_1 P_1 X_1^0(v)]w|| \leq$$

$$\leq \sqrt{20}\frac{|\gamma|}{2\pi} \int_0^{2\pi} ||L_\psi^\dagger||_0 (||L_\psi u||^2 + ||L_\psi v||^2)^{\frac{1}{2}} ||L_\psi u - L_\psi v|| d\psi \leq$$

$$\leq \sqrt{20}|\gamma|(||u||^2 + ||v||^2)^{\frac{1}{2}} ||u - v||||w|| \leq \sqrt{40}|\gamma|\rho||u - v||||w||.$$

■

Lemma 5.5 Let $u \in \mathcal{B}(\rho) \times S^1 \times \mathbf{R}$, $||\cdot||_0$ the operator norm in $B(X,X)$. Then

$$(5.44) \qquad ||D_1 P_1 c \cdot X^0(u)||_0 < K_3,$$

with K_3 independent of Ω, $\phi \in S^1$, and $\frac{\beta}{\Omega}$ for Ω sufficiently large, and $\frac{\beta}{\Omega}$ sufficiently small.

Proof: Applying Lemma 5.4 with $v = 0$ we have that for $u \in \mathcal{B}(\rho)$

$$(5.45) \qquad ||D_1 P_1 c \cdot X^0(u)||_0 \le |c_1| ||D_1 X^0_1(u)||_0 + |c_2| \le \sqrt{10}\frac{\pi|\gamma|}{2|\lambda|}\rho^2 + 3\pi,$$

where the bound on $|c_2|$ used (3.18), (5.37).

■

Lemma 5.6 Let h be one of h_2, h_4, $\tilde{h}_4$, $z \in \mathcal{B}(\rho) \times S^1 \times \mathbf{R}$. Let $||\cdot||_0$ be the operator norm in $B(X,X)$. Then $||\tilde{V}^1_h(z)||$, $||D_1\tilde{V}^1_h(z)||_0$ are bounded by constants that are independent of Ω, $\phi \in S^1$, and $\frac{\beta}{\Omega}$ for Ω sufficiently large, and $\frac{\beta}{\Omega}$ sufficiently small.

Proof: For $h = h_4$, $\tilde{h}_4$ we estimate the Lipschitz constant for $\tilde{V}^1_h$, $D_1\tilde{V}^1_h$ as in Lemmas 5.3, 5.4 respectively and bound the norms as in Lemma 5.5.

■

Lemma 5.7 Let $z \in \mathcal{B}(\rho) \times S^1 \times \mathbf{R}$. Let $||\cdot||_0$ be the operator norm in $B(X,X)$. Then

$$(5.46) \qquad ||\tilde{V}^1_S(s)|| < \frac{t_8}{\Omega}, \quad ||D_1\tilde{V}^1_S(z)||_0 < \frac{t_2}{\Omega},$$

with t_8, t_2 independent of Ω, $\phi \in S^1$, and $\frac{\beta}{\Omega}$.

Proof: From the expression for S in (3.9) we have that for u, $v \in X$, $\phi \in S^1$,

$$(5.47) \qquad \tilde{V}^1_S(u,\phi) = -i\frac{2\gamma}{\Omega}\int_0^\phi \left(L^\dagger_\psi g(L_\psi u) - \overline{g}(u)\right) d\psi,$$

$$(5.48) \qquad [D_1\tilde{V}^1_S(u,\phi)]v = -i\frac{2\gamma}{\Omega}\int_0^\phi \left(L^\dagger_\psi [Dg(L_\psi u) - \overline{g}(u)]L_\psi v - D\overline{g}(u)v\right) d\psi.$$

Using the properties of L_ψ, (5.47), and arguing as in Lemma 5.3 we have that for $u \in \mathcal{B}(\rho)$, $\phi \in S^1$,

$$||\tilde{V}_S^1(u,\phi)|| \leq \frac{2|\gamma|}{\Omega}\int_0^\phi \left(||L_\psi^\dagger g(L_\psi u)|| + ||\overline{g}(u)||\right) d\psi \leq \frac{8\pi|\gamma|\rho^3}{\Omega} \tag{5.49}$$

Similarly, using (5.48), and arguing as in Lemma 5.4, we have that for $v \in X$, $u \in \mathcal{B}(\rho)$, $\phi \in S^1$,

(5.50)

$$||[D_1\tilde{V}_S^1(u,\phi)]v|| \leq \frac{2|\gamma|}{\Omega}\int_0^\phi \left(||L_\psi^\dagger Dg(L_\psi u)L_\psi v|| + ||[D\overline{g}(u)]v||\right) d\psi \leq \frac{4\sqrt{20}\pi|\gamma|\rho^2}{\Omega}||v||.$$

■

The proof of Lemma 5.8 below uses the arguments of Lemma 5.4 and is omitted.

Lemma 5.8 Let f be one of $\tilde{V}_{h_2}$, $\tilde{V}_{h_4}$, $\tilde{V}_{\tilde{h}_4}$ above, u, $v \in \mathcal{B}(\rho) \times S^1 \times \mathbf{R}$. Let $||\cdot||_0$ be the operator norm in $B(X,X)$. Then

$$||D_1P_1f(u) - D_1P_1f(v)||_0 < L||u-v||, \tag{5.51}$$

with $L = L(f)$ independent of Ω, $\phi \in S^1$, and $\frac{\beta}{\Omega}$.

Lemma 5.9 Let $z \in \mathcal{B}(\rho)\times S^1\times\mathbf{R}$. Let $||\cdot||_{0,0}$ be the operator norm in $B(X, B(X,X)$ Then

$$||D_1^2\tilde{V}_S^1(z)||_{0,0} < \frac{t_1}{\Omega}, \tag{5.52}$$

with t_1 independent of Ω, $\phi \in S^1$, and $\frac{\beta}{\Omega}$.

Proof: Let a, w, $v \in X$. We compute

$$([D^2g(a)](w,v))_n = 2(a_n + a_n^*)w_nv_n + 2a_nw_nv_n^*, \quad n \in \mathbf{Z}, \tag{5.53}$$

$$[(D^2L_\psi^\dagger g)(L_\psi a)](w,v) = L_\psi^\dagger[D^2g(L_\psi u))](L_\psi w, L_\psi v). \tag{5.54}$$

Using the properties of L_ψ, and (5.53) we estimate

$$||[D^2g(L_\psi a)](L_\psi w, L_\psi v)||^2 \leq 12||a||^2||w||^2||v||^2. \tag{5.55}$$

Using (5.53)-(5.55) we have that for $u \in \mathcal{B}(\rho)$, $\phi \in S^1$, $w, v \in X$

$$||[D_1^2 \tilde{V}_S^1(u,\phi)](w,v)|| \leq \tag{5.56}$$

$$\leq \frac{2|\gamma|}{\Omega} \left\{ \int_0^{\phi} ||L_{\psi}^{\dagger}[D^2 g(L_{\psi}u)](w,v)|| d\psi + \int_0^{\phi} \left(\frac{1}{2\pi} \int_0^{2\pi} ||L_{\theta}^{\dagger}[D^2 g(L_{\theta}u)](w,v)|| d\theta \right) \right\} \leq$$

$$\leq 8\pi\sqrt{12}\frac{|\gamma|}{\Omega}||u||||w||||v|| \leq \frac{8\sqrt{12}\pi|\gamma|\rho}{\Omega}||w||||v||.$$

■

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8. Figure Captions

Fig. 1: Conjectured range of validity of the one-peak breather continuation argument.

Fig. 2(a): Spectrum of (nontrivial modified) Floquet map for single peak breather. The unit eigenvalue is double. $\overline{D} = 0$, $\frac{\beta}{\Omega} = \frac{0.4}{13}$

Fig. 2(b): Single peak breather, averaged system. $\overline{D} = 0$, $\frac{\beta}{\Omega} = \frac{0.4}{13}$

Fig. 3(a): Spectrum of (nontrivial modified) Floquet map for single peak breather. The unit eigenvalue is double. $\overline{D} = 0$, $\frac{\beta}{\Omega} = \frac{0.8}{13}$

Fig. 3(b): Single peak breather, averaged system. $\overline{D} = 0$, $\frac{\beta}{\Omega} = \frac{0.8}{13}$

Fig. 4(a): Spectrum of (nontrivial modified) Floquet map for single peak breather. The unit eigenvalue is double. $\overline{D} = 0$, $\frac{\beta}{\Omega} = \frac{2}{13}$

Fig. 4(b): Single peak breather, averaged system. $\overline{D} = 0$, $\frac{\beta}{\Omega} = \frac{2}{13}$

Fig. 5(a): Spectrum of (nontrivial modified) Floquet map for $2-$peak breather. The unit eigenvalue is double. $\overline{D} = 0$, $\frac{\beta}{\Omega} = \frac{0.1}{13}$

Fig. 5(b): $2-$peak breather, averaged system. $\overline{D} = 0$, $\frac{\beta}{\Omega} = \frac{0.1}{13}$

Fig. 6(a): Spectrum of (nontrivial modified) Floquet map for $3-$peak breather. The unit eigenvalue is double. $\overline{D} = 0$, $\frac{\beta}{\Omega} = \frac{0.1}{13}$

Fig. 6(b): $3-$peak breather, averaged system. $\overline{D} = 0$, $\frac{\beta}{\Omega} = \frac{0.1}{13}$

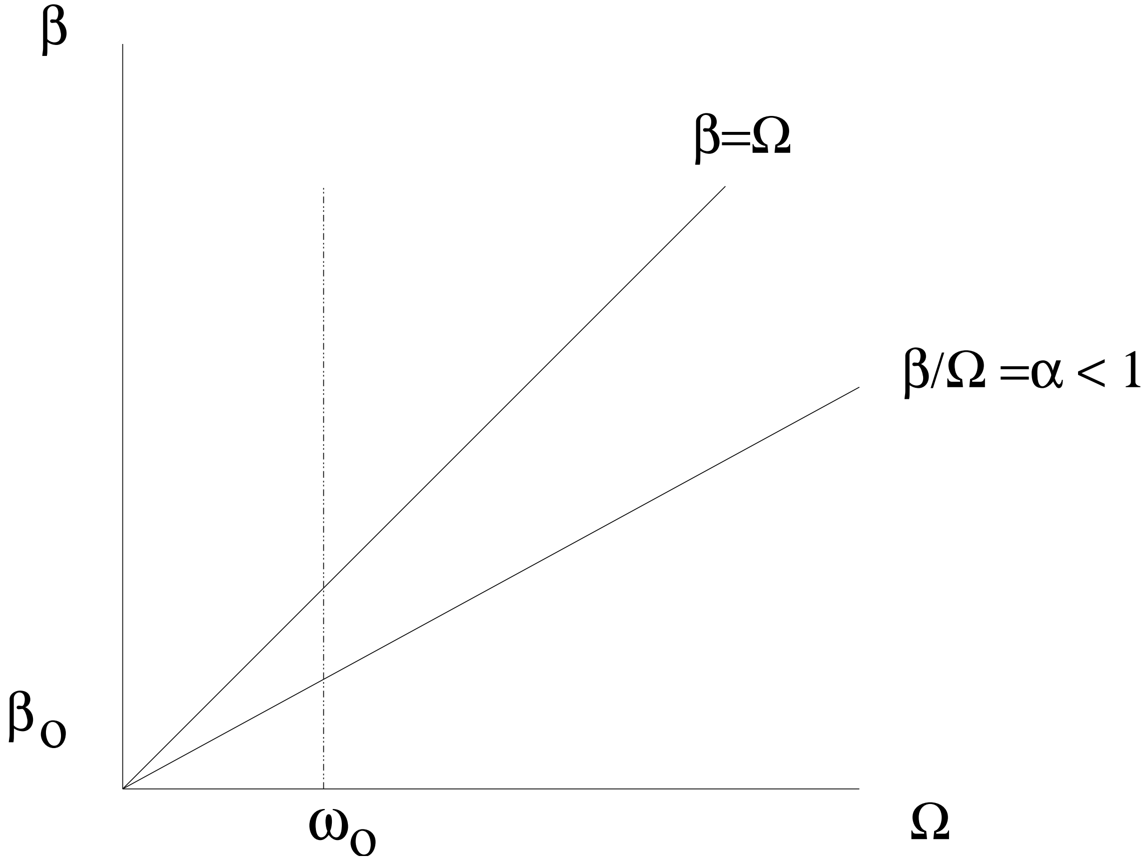


Figure 1

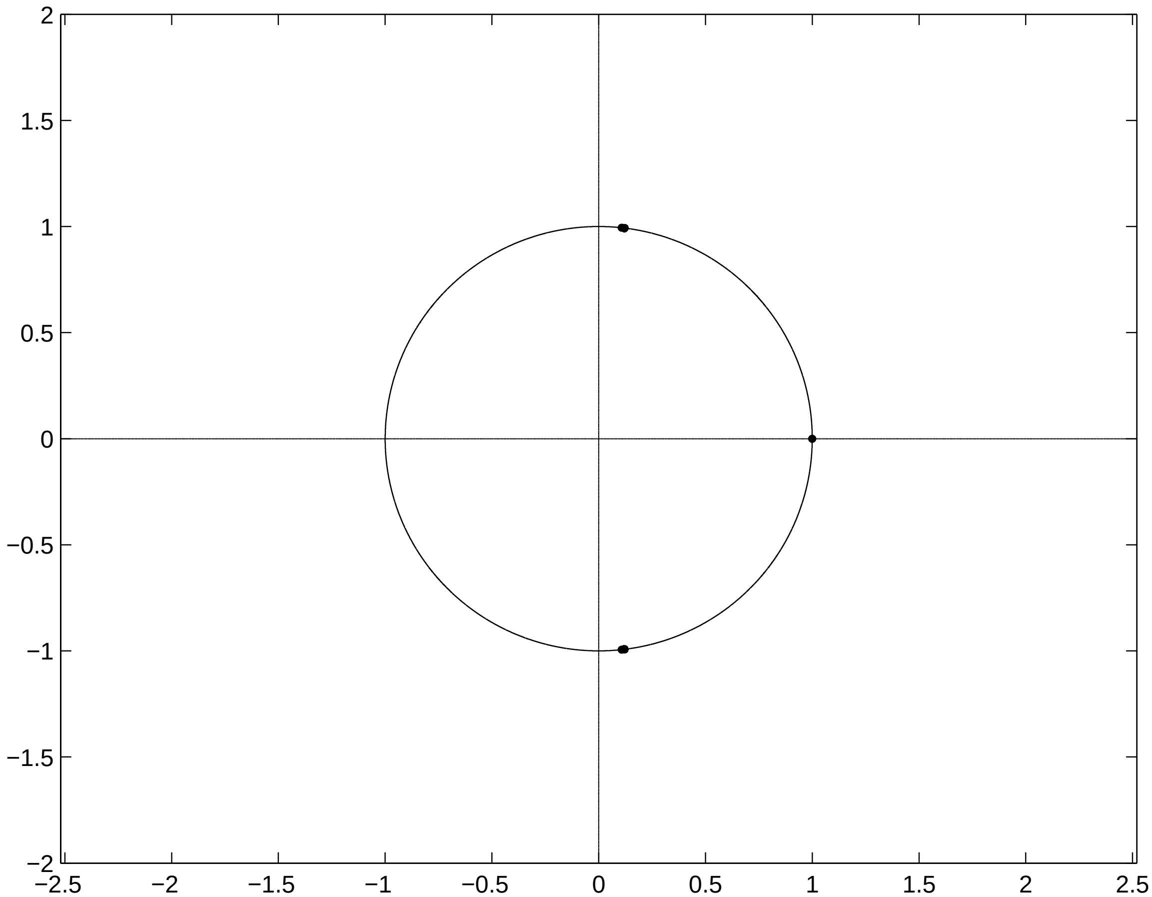
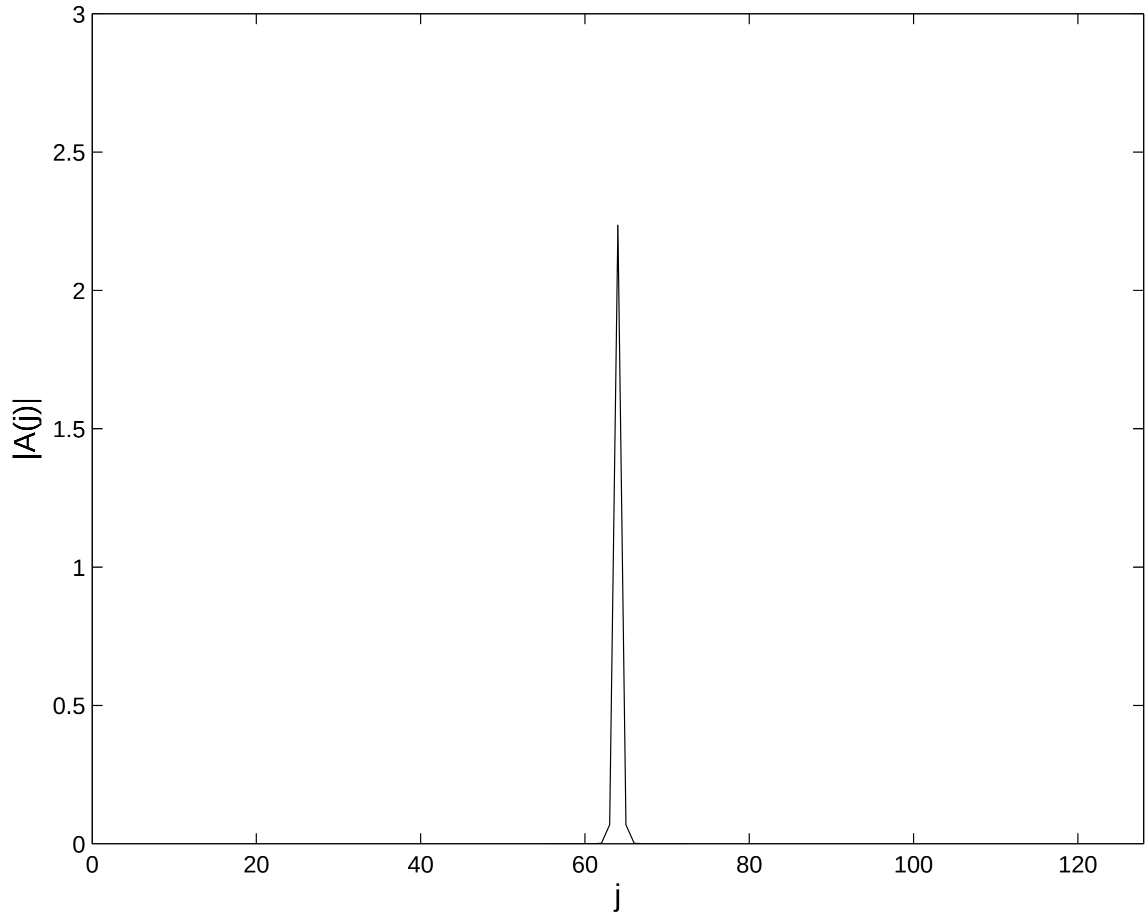


Figure 2(a)



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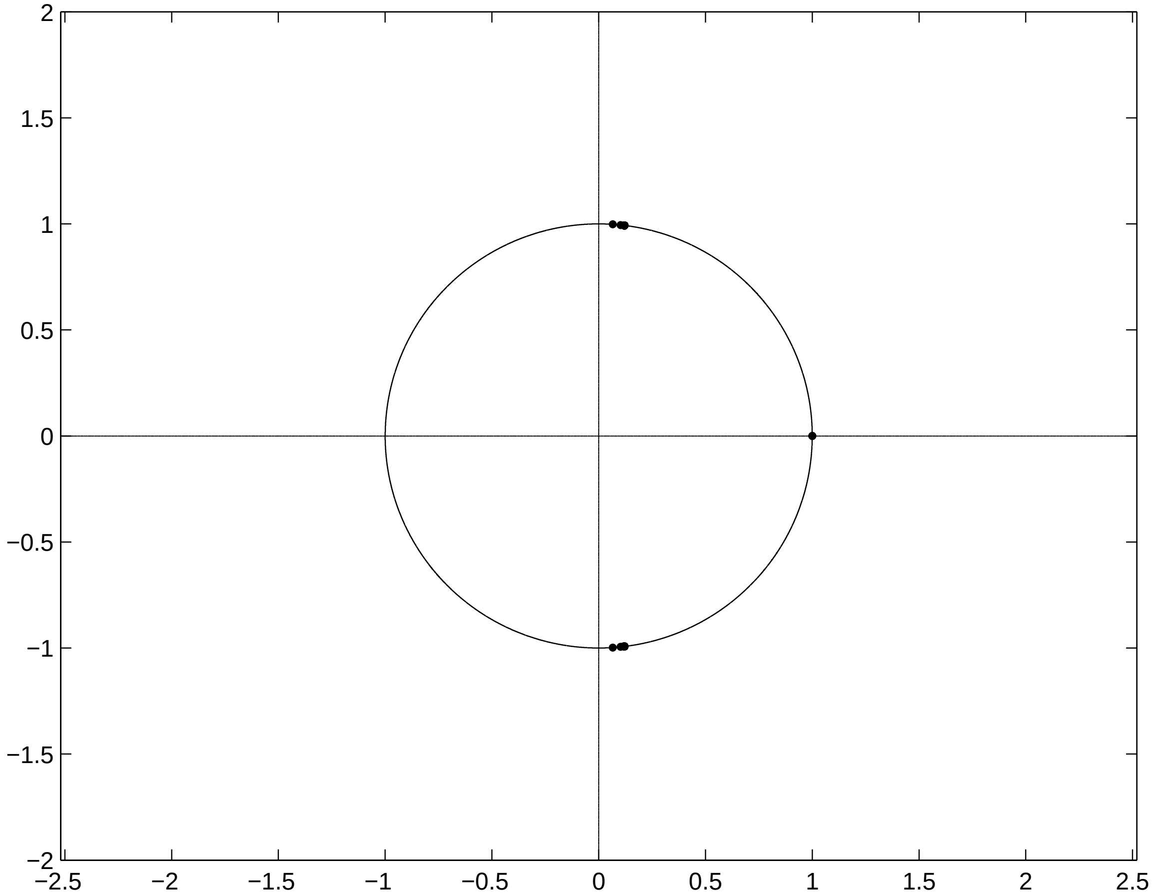
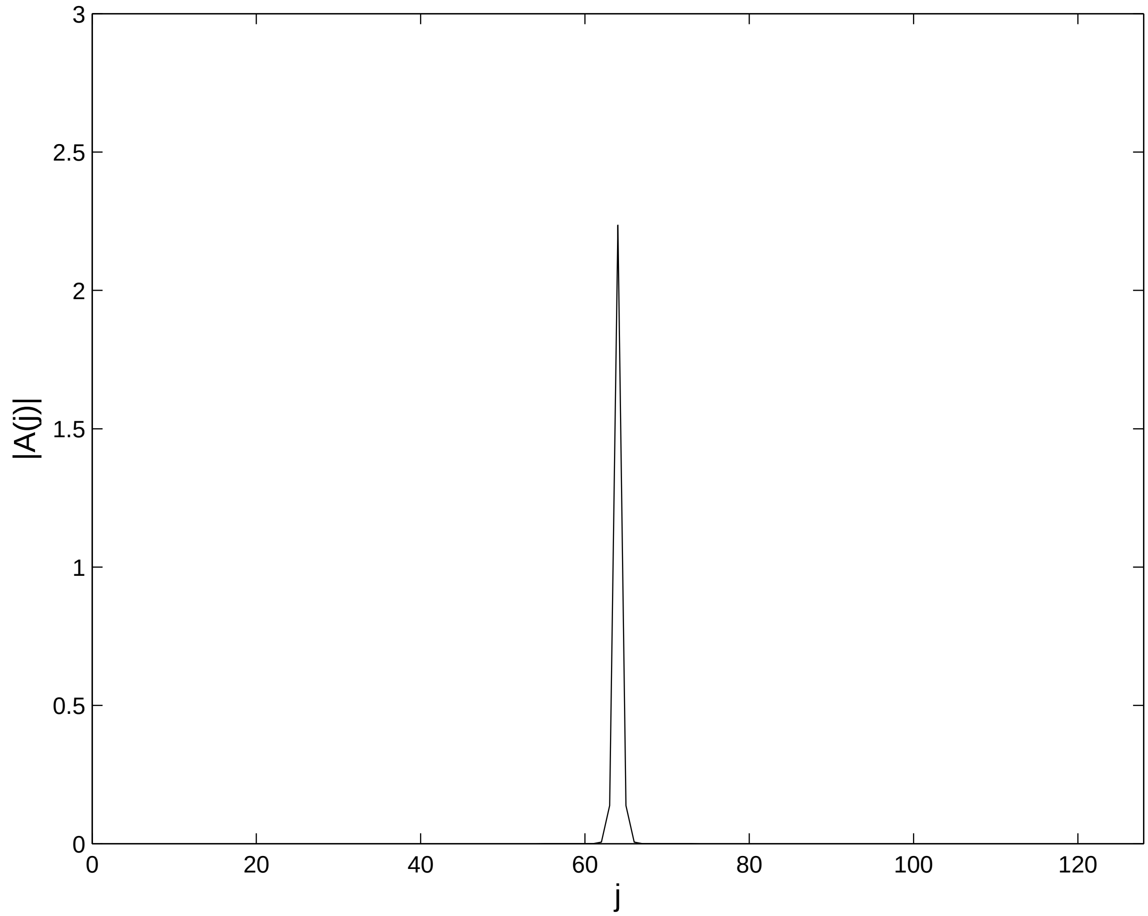


Figure 3(a)



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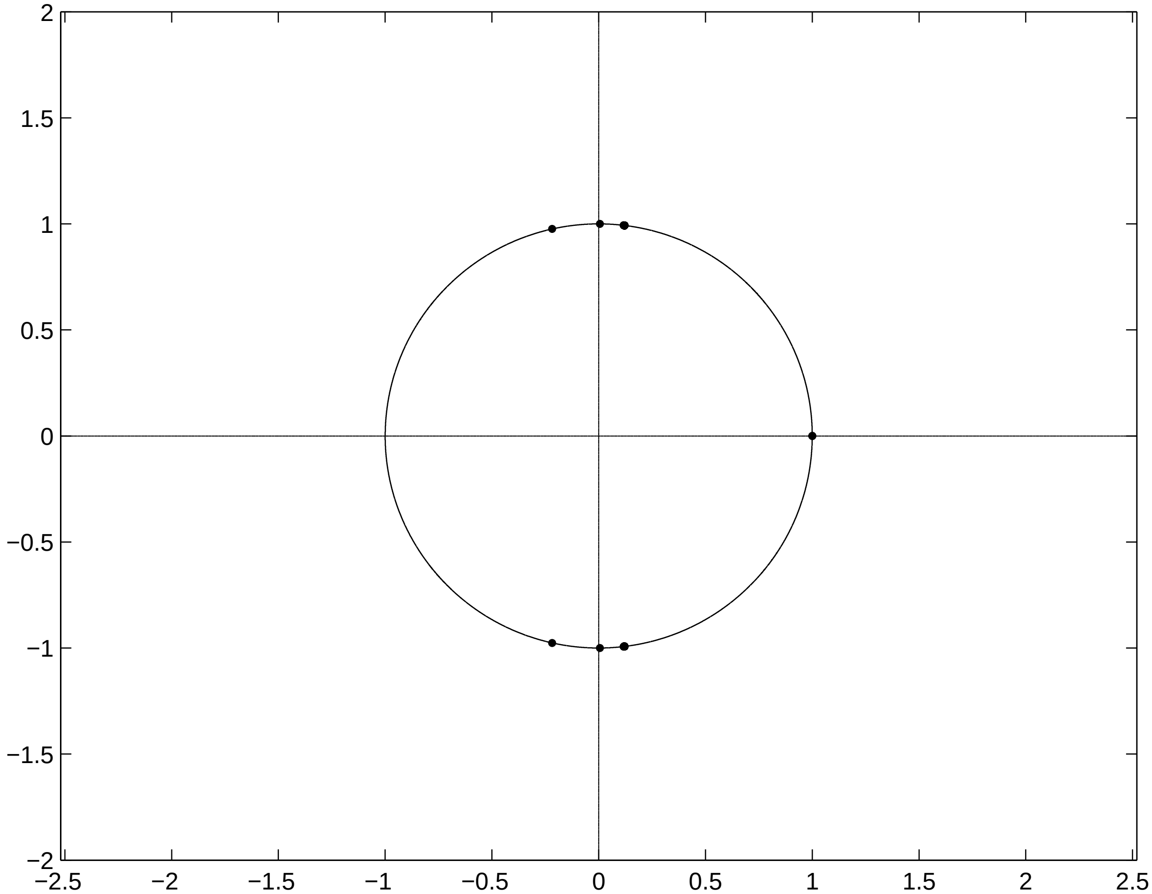

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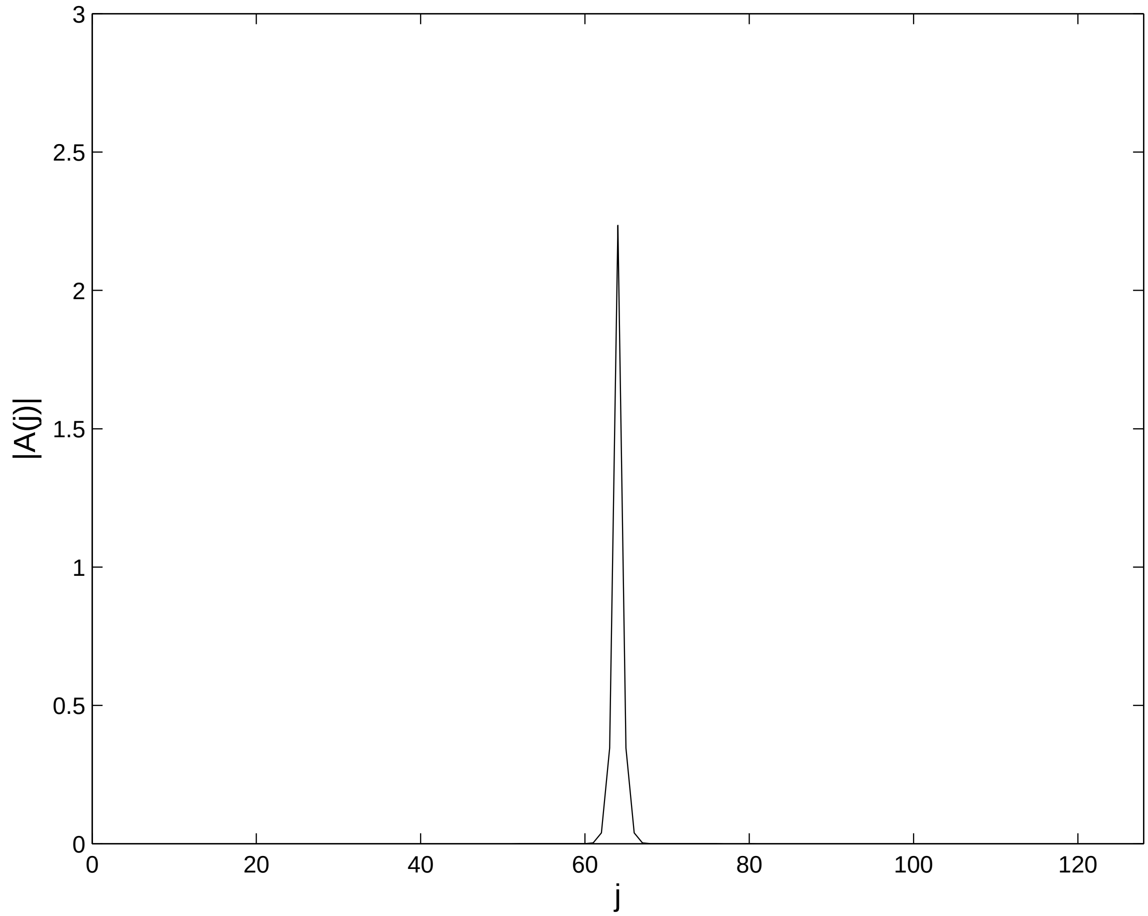

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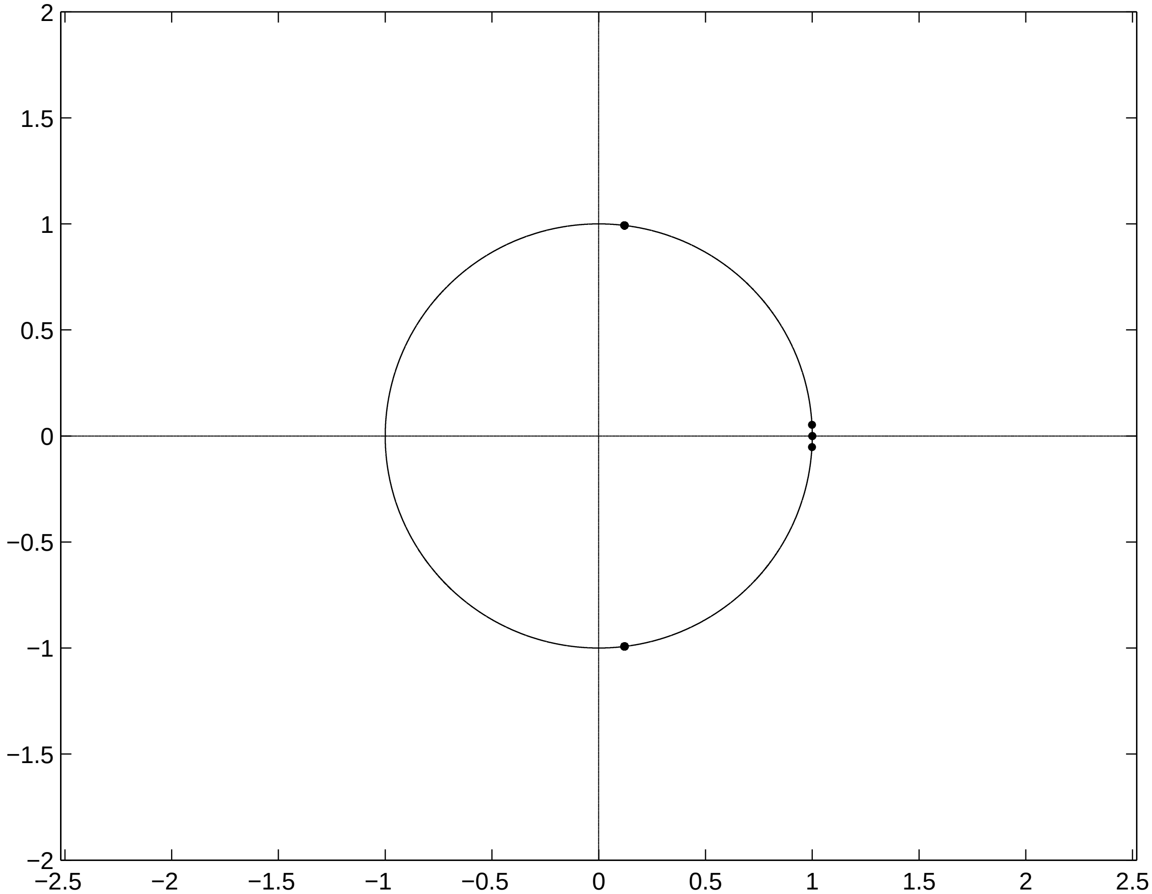
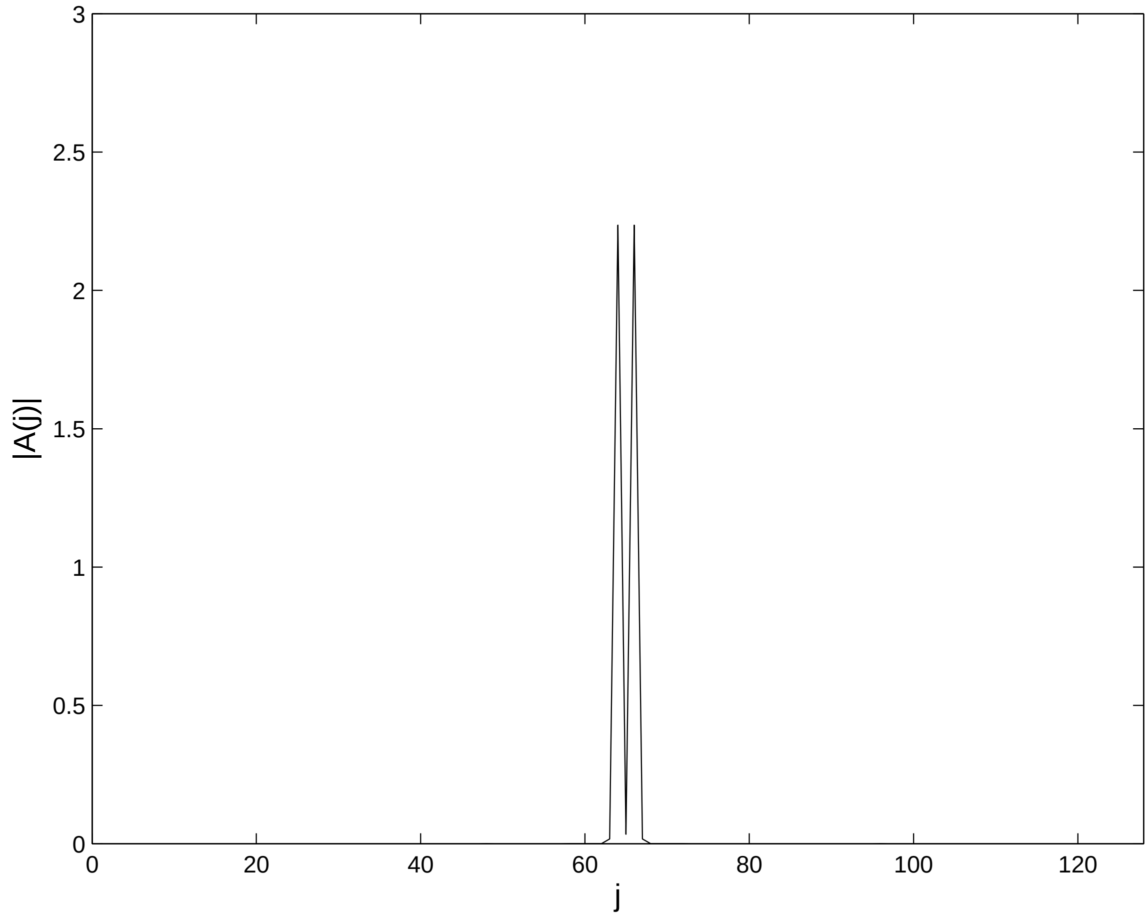


Figure 5(a)



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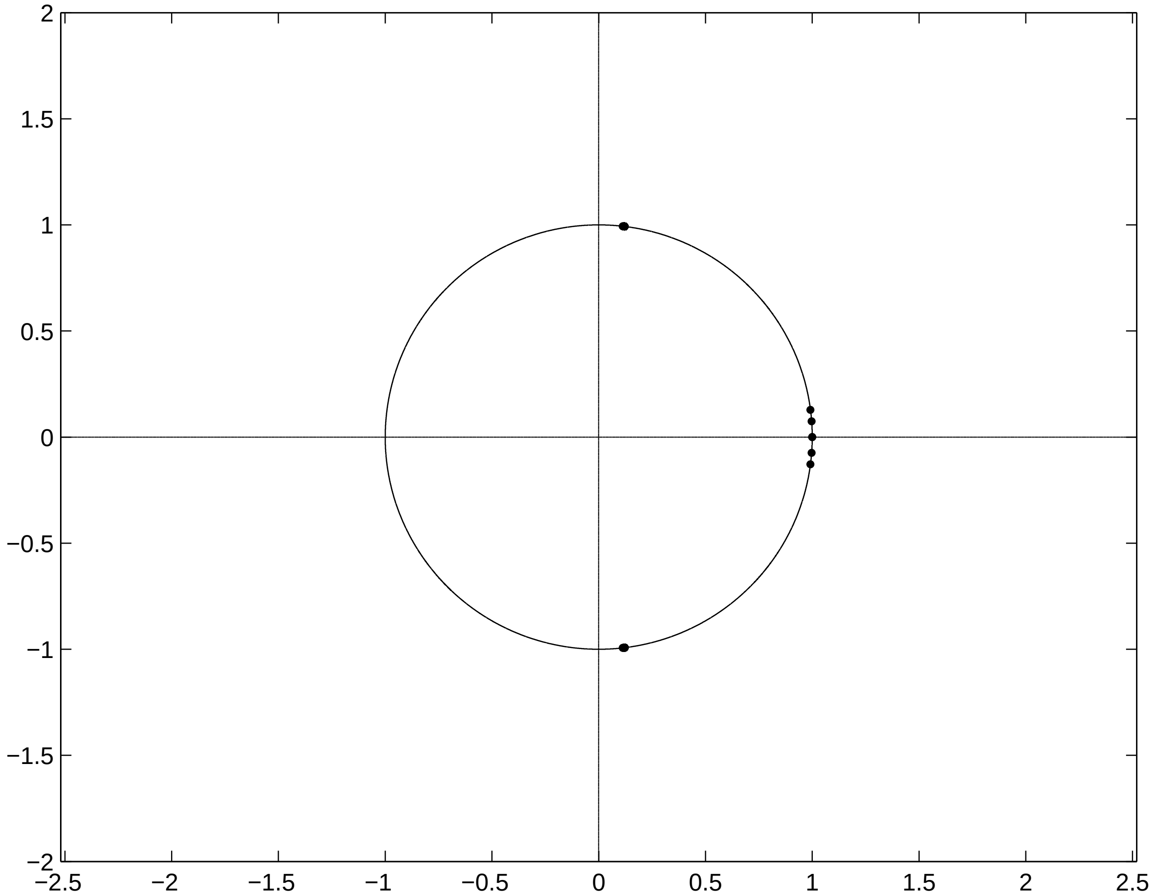

Figure 6(a)

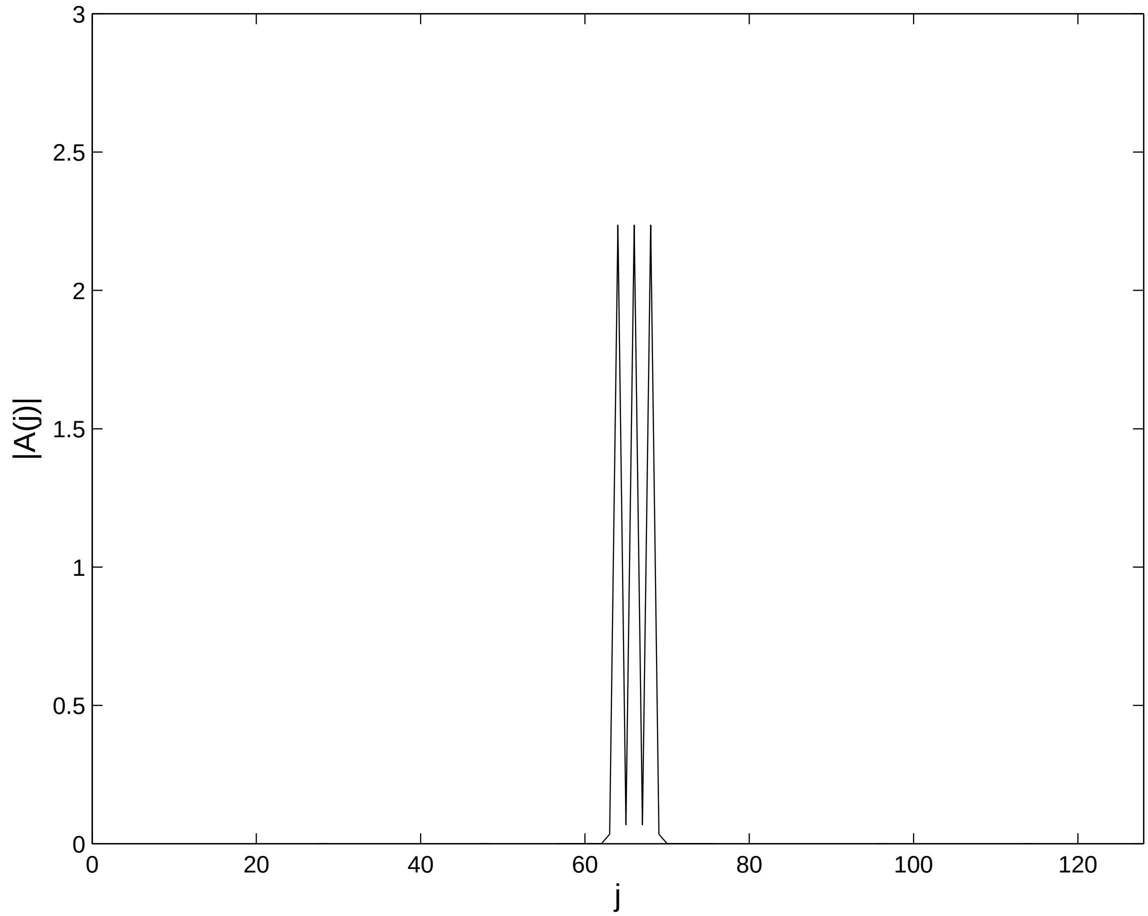

Figure 6(b)