Linear stability of breathers of the discrete NLS

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Abstract

We consider real breather solutions of the discrete cubic nonlinear Schrödinger equation near the limit of vanishing coupling between the lattice sites and present leading order asymptotics for the eigenvalues of the linearization around the breathers. The expansion is given in fractional powers of the intersite coupling parameter and determines the linear stability of the breathers. The method we use relies on normal form ideas and applies to one and higher dimensional lattices. We also present some examples.

1 Introduction

The discrete cubic nonlinear Schrödinger equation (DNLS) is a nonlinear lattice system that appears in many areas of physics, e.g. nonlinear optics [CLS], Bose-Einstein condensates [LFO], biomolecular chains [KAT]. One of the main problems in nonlinear lattices is to understand spatially localized structures and their role in the dynamics of the system. In this work we study the linear stability of real breather solutions of the DNLS in the regime where the coupling between the sites is weak. This is the first step in a systematic study of the stability of breathers.

Breathers are spatially localized time-periodic solutions of the DNLS of the general form $e^{-i\omega t}A$, where $\omega \in \mathbf{R}$ is the temporal frequency, and $A : \mathbf{Z}^d \to \mathbf{C}$ is independent of time and decays to zero at infinity. The existence of breathers has been shown by different methods in one and higher dimensional lattices. In the case of the weak intersite coupling the basic existence results are in [MA94], [PKF05a], [PKF05b], [PP08]. In the limit of vanishing intersite coupling, breathers are localized in a finite set of "active" sites where the modulus of A is of $O(1)$; outside this set $|A|$ is bounded by a quantity that is proportional to the intersite coupling parameter δ. In this work we consider breather solutions with A real valued, up to a global phase. Such breathers will be referred to as *real* breathers. They exist in any dimension, although for $d \geq 2$ not all breathers are real.

An important property of breathers is that they are relative equilibria, i.e. equilibria in a suitable "moving frame". Using the Hamiltonian structure of the DNLS, the corresponding linearization around these equilibria has the Hamiltonian form $J\mathcal{H}$, with J the symplectic operator, and $\mathcal{H}$ a symmetric operator. To understand the spectrum of $J\mathcal{H}$ we use the fact that, for $|\delta|$ sufficiently small, the breather amplitude A can be written explicitly as a convergent power series in δ, so that

$J\mathcal{H}$ is similarly expanded in powers of δ. Observing that the spectrum of $J\mathcal{H}$ for $\delta = 0$ is known, the idea is to develop a perturbation theory for small $|\delta|$ spectrum.

The paper describes and justifies an algorithmic procedure for obtaining expansions for the eigenvalues of $J\mathcal{H}$ in fractional powers of δ. More precisely, noting that the eigenvalues of $J\mathcal{H}$ vanish as $\delta \to 0$, we describe how to obtain the lowest order nontrivial term in an expansion in powers of $\sqrt{|\delta|}$ for each eigenvalue. Under certain nondegeneracy conditions this asymptotic determines the number of unstable directions in the linearized problem. The main result is stated in Theorem 3.3. In a preliminary step (Propositions 3.1, 3.2) we see that the number of eigenvalues of $J\mathcal{H}$ is finite (it is twice the number of "active sites" of the breather), and that $J\mathcal{H}$ also has continuous spectrum that lies on the imaginary axis.

In the case of real breathers the operator $\mathcal{H}$ has a block diagonal structure, with symmetric blocks L_-, L_+. The expansion of the eigenvalues in $\sqrt{|\delta|}$ follows from a perturbative analysis of the $O(\delta)$-size eigenvalues of L_-, in particular we use normal form ideas, see e.g. [M02], to decouple the point spectrum from the continuous spectrum and to diagonalize the finite dimensional point spectrum block. The remaining part of the argument connects the spectral analysis of L_- to that of $J\mathcal{H}$, showing how the sign of the eigenvalues of L_- determines linear stability.

The block diagonal structure of $\mathcal{H}$ is well known in the literature on nonlinear Schrödinger equations. The method here relies crucially on the discreteness of the problem, especially the presence of the "anti-continuous limit" where the sites are uncoupled, and the fact that the operator $J\mathcal{H}$ is bounded so that we can decouple the discrete spectrum from the continuous spectrum. Also, the present work is partially motivated by [PKF05a], where the sign of eigenvalues of L_- is determined by an argument that appears applicable only to one dimension. In contrast, we can here in principle calculate the leading order nonzero part of eigenvalues of L_- pertubatively in one or several dimensions, but do not have an a-priori criterion for their sign.

The paper is organized as follows. Section 2 contains definitions and the main result on the existence of real breathers that we use. Section 3 states and proves the main results on the spectrum of $J\mathcal{H}$. In Section 4 we present some examples.

2 Breather solutions of the discrete NLS

The cubic discrete NLS equation in the $d-$dimensional integer lattice $\mathbf{Z}^d$ is

$$\dot{u}_n = i\delta(\Delta u)_n - 2i|u_n|^2 u_n, \tag{2.1}$$

where f_n is the value of the complex function f at the site $n \in \mathbf{Z}^d$, and Δ is the discrete Laplacian, defined by

$$(\Delta u)_n = \sum_{j=1}^{d} (u_{n+\hat{e}_j} + u_{n-\hat{e}_j}) - 2du_n, \tag{2.2}$$

with $\hat{e}_j$ the unit vector in the $j-$th direction. The discrete NLS (2.1) can be also written as the Hamiltonian system

$$\dot{u}_n = -i\frac{\partial H}{\partial u_n^*}, \quad n \in \mathbf{Z}^d, \quad \text{with} \quad H = \delta \sum_{n\in\mathbf{Z}^d}\left(\sum_{j=1}^{d}|u_{n+\hat{e}_j} - u_n|^2\right) + \sum_{n\in\mathbf{Z}^d}|u_n|^4. \tag{2.3}$$

System (2.1) conserves the Hamiltonian H, and the quantity $P = \sum_{n\in\mathbf{Z}^d}|u_n|^2$ (the "power").

A *breather* is a solution of the discrete NLS (2.1) that has the form $u = e^{-i\omega t}A$, with ω real, and $A : \mathbf{Z}^d \to \mathbf{C}$ decaying to zero at infinity. By (2.1), A, ω must satisfy

$$-\omega A_n = \delta(\Delta A)_n - 2|A_n|^2 A_n, \quad n \in \mathbf{Z}^d. \tag{2.4}$$

Observe that if A satisfies (2.4) so does $e^{i\theta}A$, for arbitrary real θ (independent of n). A *real* breather is a breather with $A = e^{i\phi}\tilde{A}$, where $\tilde{A}$ is real-valued, and ϕ an arbitrary real (independent of n).

Breathers are also trajectories of the Hamiltonian vector field of the power P. They are thus group orbits (or relative equilibria), analogous for instance to circular orbits of the central force problem, or traveling waves in translation invariant nonlinear wave equations. Given a solution of (2.4) we define the "moving frame" coordinates $v(t)$ by $u(t) = e^{-i\omega t}v(t)$, so that (2.1) is equivalent to

$$\dot{v}_n = -i\frac{\partial H_\omega}{\partial v_n^*}, \quad n \in \mathbf{Z}^d, \quad \text{with} \quad H_\omega = H - \omega P, \tag{2.5}$$

and H as in (2.3). A solution A of (2.4) is a fixed point of (2.5) and belongs to the circle $e^{i\theta}A$, $\theta \in \mathbf{R}$, of fixed points of (2.5).

An alternative real notation for (2.5) is obtained by identifying complex functions with $\mathbf{R}^2$-valued functions. Let $z = [q,p]^T$, with $z_n = [q_n, p_n]^t$, $q_n = \mathrm{Re}v_n$, $p_n = \mathrm{Im}v_n$, $n \in \mathbf{Z}^d$, i.e. q, p are real functions on $\mathbf{Z}^d$. Then (2.5) can be written as

$$\dot{z} = J\nabla h_\omega, \quad \text{with} \quad h_\omega = \frac{1}{2}H_\omega, \tag{2.6}$$

and $(Jz)_n = -[p_n, q_n]^T$, i.e. J is the standard symplectic operator.

To study the linear (relative) stability for breather solutions we linearize (2.6) around a solution A of (2.4). Using the real notation, the linearization around the breather is

$$\dot{z} = J\mathcal{H}z, \quad \text{with} \quad \mathcal{H} = \nabla^2 h_\omega(A), \tag{2.7}$$

i.e. $\mathcal{H}$ is the Hessian of h_ω at A (the dependence of $\mathcal{H}$ on ω is suppressed from the notation). Let $a_n = \mathrm{Re}A_n$, $b_n = \mathrm{Im}A_n$, and $\langle f, g\rangle = \sum_{n\in\mathbf{Z}^d} f_n g_n$, with f, g real valued functions on $\mathbf{Z}^d$. Then (2.7) is equivalent to the Hamiltonian system

$$\dot{z} = J\nabla h, \quad \text{with} \quad h = \frac{1}{2}\langle p, L_+ p\rangle + \frac{1}{2}\langle q, L_- q\rangle + \langle q, \tilde{L}p\rangle, \tag{2.8}$$

where L_+, L_-, $\tilde{L}$ are infinite matrices with entries

$$L_+(n,n) = -\omega + 4|A_n|^2 + 2(a_n^2 - b_n^2) + 2d\delta, \quad L_+(n, n\pm\hat{e}_j) = -\delta, \quad j = 1,\ldots,d, \quad n \in \mathbf{Z}^d; \tag{2.9}$$

$$L_-(n,n) = -\omega + 4|A_n|^2 - 2(a_n^2 - b_n^2) + 2d\delta, \quad L_-(n, n \pm \hat{e}_j) = -\delta, \quad j = 1, \ldots, d, \quad n \in \mathbf{Z}^d; \tag{2.10}$$

$$\tilde{L}(n,n) = 4\gamma a_n b_n, \quad n \in \mathbf{Z}^d. \tag{2.11}$$

All entries corresponding to index pairs not specified by (2.9)-(2.11) are zero. By the above definitions $L_\pm$, $\tilde{L}$ are symmetric.

We also have

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} L_+ & \tilde{L} \\ \tilde{L} & L_- \end{bmatrix}, \tag{2.12}$$

where blocks correspond to real and imaginary parts of complex functions on $\mathbf{Z}^d$.

We are here interested in breathers obtained for $|\delta|$ small. Although there are theoretical existence results for a wider parameter regime, these appear to be nonconstructive in that they do not yield a systematic way to approximate the breather solution. On the other hand, for $|\delta|$ small breathers can be essentially obtained by expansions in δ. Our goal is to extend the use of these expansions to the stability question.

For $|\delta|$ small, solutions of (2.4) are obtained by continuation from solutions of (2.4) with $\delta = 0$ (the "anticontinuous limit"). These solutions have the form

$$A_n = e^{i\phi_n}\sqrt{\frac{\omega}{2}}, \quad \text{for} \quad n \in U; \quad A_n = 0, \quad \text{for} \quad n \in U^c, \tag{2.13}$$

with $\omega > 0$, U a finite subset of $\mathbf{Z}^d$, $U^c = \mathbf{Z}^d \setminus U$, and $\phi_n \in \mathbf{R}$ arbitrary.

The simplest continuation result for $\delta \neq 0$ concerns real solutions of (2.4). Let $Y = l_2(\mathbf{Z}^d, \mathbf{R})$ with the inner product $\langle f, g \rangle = \sum_{n \in \mathbf{Z}^d} f_n g_n$. Y is a real Hilbert space.

Proposition 2.1 *Consider a solution $A(0)$ of (2.4) with $\delta = 0$ that has the form (2.13) with $\phi_n = 0$ or π, $\forall n \in U$. Then there exist $\tilde{\delta}$ such that for $|\delta| < \tilde{\delta}$ equation (2.4) has a unique real solution $A(\delta) \in Y$ that satisfies $A(\delta) \to A(0)$ in Y. Moreover $A(\delta)$ is real analytic in δ, i.e. it is a real analytic function from $(-\tilde{\delta}, \tilde{\delta})$ to Y.*

Note that $\tilde{\delta}$ will in general depend on $A(0)$, U, and ω. Proofs (of slightly different versions) of Proposition 2.1 are in [MA94], [PKF05a], [PKF05b], [PP08]. Analyticity implies that small $|\delta|$ breathers can be obtained by starting with a power series Ansatz $A_n(\delta) = A_{n,0} + \delta A_{n,1} + \delta^2 A_{n,2} + \ldots$ and matching powers of δ. The term $A_{n,0}$ is given by (2.13). We refer to breathers obtained by continuation from the $\delta = 0$ solutions (2.13) as *$k-$peak breathers*, where k is the number of sites in U. (U is the set of "active sites".) The solutions of Proposition 2.1 are clearly examples of real breathers. A theory on the continuation of solutions of (2.13) with arbitrary ϕ_n is developed in [PKF05a], [PKF05b] (see also [PP08]).

3 Linear stability of real breathers

For the case of real breathers we may assume that the breather solution A_n is real, $\forall n \in \mathbf{Z}^d$. Then $\tilde{L}$ vanishes and L_+, L_- simplify to

$$L_+(n,n) = -\omega + 6\gamma A_n^2 + 2d\delta, \quad L_+(n, n \pm \hat{e}_j) = -\delta, \quad j = 1, \ldots, d, \quad n \in \mathbf{Z}^d; \tag{3.1}$$

$$L_-(n,n) = -\omega + 2\gamma A_n^2 + 2d\delta, \quad L_-(n, n \pm \hat{e}_j) = -\delta, \quad j = 1, \dots, d, \quad n \in \mathbf{Z}^d. \tag{3.2}$$

Moreover $\mathcal{H}$ takes the block diagonal form

$$\mathcal{H} = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix}. \tag{3.3}$$

For $|\delta|$ sufficiently small the convergent power series expansion of A in δ leads to formal expansions

$$L_+ = \sum_{j=0}^{\infty} \delta^j L_{+,j}, \quad L_- = \sum_{j=0}^{\infty} \delta^j L_{-,j}, \quad \mathcal{H} = \sum_{j=0}^{\infty} \delta^j \mathcal{H}_j. \tag{3.4}$$

The $L_{\pm,j}$ are computed readily from (3.1), (3.2), and the expressions for the A_n.

For any real $k-$peak breather continued from a $\delta = 0$ breather solution of (2.13), $k = |U|$, we have

$$L_-(n,n) = 0 + O(\delta), \quad \forall n \in U, \quad L_-(n,n) = -\omega + O(\delta), \quad \forall n \in U^c, \tag{3.5}$$

$$L_+(n,n) = 2\omega + O(\delta), \quad \forall n \in U, \quad L_+(n,n) = -\omega + O(\delta), \quad \forall n \in U^c. \tag{3.6}$$

Thus for $\delta = 0$, the spectrum of $\mathcal{H}$ consists of the eigenvalues 0, and 2ω, both of multiplicity k, and of the eigenvalue $-\omega$, of infinite multiplicity. Furthermore the spectrum of $J\mathcal{H}$ consists of an eigenvalue 0 of multiplicity $2k$, and the eigenvalues $\pm i\omega$, both of infinite multiplicity.

A heuristic analogy. In studying the spectrum of $J\mathcal{H}$ near $\delta = 0$ it is useful to draw an analogy with a model problem of mechanics. By (2.8), the Hamiltonian for $\dot{w} = J\mathcal{H}w$ is

$$h(q,p) = \frac{1}{2}\langle p, L_+ p\rangle + \frac{1}{2}\langle q, L_- q\rangle. \tag{3.7}$$

For q, $p \in \mathbf{R}^n$, with L_+, L_- symmetric real $n \times n$ matrices, L_+, L_- can be diagonalized by orthogonal transformations U_+, U_- respectively. If in addition $U_+ = U_- = U$ we have a canonical transformation to variables $\tilde{q} = Uq$, $\tilde{p} = Up$ for which

$$h(q,p) = h(U^{-1}\tilde{q}, U^{-1}\tilde{p}) = \frac{1}{2}\langle \tilde{p}, \Lambda_+ \tilde{p}\rangle + \frac{1}{2}\langle \tilde{q}, \Lambda_- \tilde{q}\rangle, \tag{3.8}$$

with Λ_+, Λ_- diagonal and real. The stability of the linear system is then immediately understood by examining the pairs of diagonal entries $\Lambda_+(j,j)$, $\Lambda_-(j,j)$, $j = 1, \dots, n$.

For the $L_\pm$ of (3.1), (3.2), the assumption $U_+ = U_-$ should fail since L_+, L_- do not commute for $\delta \neq 0$. Ignoring this fact for the moment, we note that for $\delta = 0$ the diagonal entries $L_+(n,n)$, $L_-(n,n)$, $n \in U^c$, are both negative and bounded away from zero. We expect that they will correspond to (a suitable analogue of) negative eigenvalues of the perturbed L_+, L_-, and pairs of imaginary eigenvalues for $J\mathcal{H}$. We similarly expect that the positive eigenvalues corresponding to the entries $L_{+,0}(n,n)$, $n \in U$, will be continued to positive eigenvalues of the perturbed L_+. Linear stability will then be determined by the sign of the continuation the zero eigenvalues of $L_{-,0}$, expected to be of $O(\delta)$. We will show that despite the differences, the model problem captures the qualitative features of the present case.

Let $Y = l_2(\mathbf{Z}^d, \mathbf{R})$ with the inner product $\langle u, v\rangle_Y = \sum_{n \in \mathbf{Z}^d} u_n v_n$. Let Y_c be the complexification of Y. Let $X = l_2(\mathbf{Z}^d, \mathbf{C})$ with the inner product $\langle u, v\rangle_X = \sum_{\mathbf{Z}^d}[(\mathrm{Re} u_n)(\mathrm{Re} v_n) + (\mathrm{Im} u_n)(\mathrm{Im} v_n)]$, i.e.

X has a real Hilbert space structure. Let $X_c = l_2(\mathbf{Z}^d, \mathbf{C}^2)$ be the complexification of X. Note that $X = Y \oplus Y$, and $X_c = Y_c \oplus Y_c$. Also, let $(\hat{e}(n))_m = 1$ if $n = m$; 0 otherwise. The set $\{\hat{e}(n)\}_{n\in\mathbf{Z}^d}$ is the standard basis of Y. For L a linear operator in Y we let $L(m,n) = L(\hat{e}(n))_m$.

Let M be a bounded operator in Y_c, or X_c. We denote the spectrum of M by $\sigma(M)$, and use the standard notation $\sigma_p(M)$, $\sigma_c(M)$, $\sigma_r(M)$ to denote the point, continuous, and residual spectra of M respectively. We use the definition of [K76], p.243, for the essential spectrum $\sigma_e(M)$ of M. Algebraic and geometric multiplicities are defined in the standard way for bounded operators in complex Banach spaces (unless otherwise specified multiplicity below refers to algebraic multiplicity).

In what follows we fix U, ω, and a solution $A(0)$ of the form (2.13). Let $|\delta| < \tilde{\delta}$. Consider $A(\delta)$ as in Proposition 2.1, and the corresponding $L_\pm$, $J\mathcal{H}$. $L_\pm$ are bounded operators in Y (and Y_c), and $J\mathcal{H}$ is a bounded operator in X (and X_c). By (2.9) we can choose $0 < \delta_1 \leq \tilde{\delta}$ such that L_+ is invertible $\forall |\delta| < \delta_1$. Let $\delta_0 = \min\{\delta_1, \tilde{\delta}\}$. Also let Y_S denote the span of $\{\hat{e}(n)\}_S$, with $S \subset \mathbf{Z}^d$. Let P_Z denote the orthogonal projection to a subspace Z of Y. For L a linear operator in Y, the matrix representations $P_{Y_U} L|_{Y_U}$, $P_{Y_{U^c}} L|_{Y_{U^c}}$ in the standard basis are referred to as the U, U^c blocks of L respectively.

Proposition 3.1 *Consider the operators L_-, L_+ above, with $0 < |\delta| < \delta_0$. We have $\sigma_c(L_-) = \sigma_c(L_+) = [-\omega, -\omega + 4d\delta]$ for $\delta > 0$, and $\sigma_c(L_-) = \sigma_c(L_+) = [-\omega + 4d\delta, -\omega]$ for $\delta < 0$. The point spectrum of L_- consists of k eigenvalues (counting multiplicity) that are analytic in δ, with $0 \in \sigma_p(L_-)$, $\forall \delta \in (-\delta_0, \delta_0)$. The point spectrum of L_- consists of k eigenvalues (counting multiplicity) that are $O(\delta)$ close to 2ω.*

Proof. By $A \in l_2(\mathbf{Z}^d, \mathbf{R})$ we see that that the operator $L_- - (-\omega - \Delta)$ is Hilbert-Schmidt and therefore compact. (ω means ω times the identity operator.) Then $\sigma_e(L_-) = \sigma_e(-\omega - \Delta)$. Let $\delta > 0$. The set of bounded $u : \mathbf{Z}^d \to \mathbf{R}$, $\lambda \in \mathbf{C}$, that satisfy the eigenvalue equation $(-\omega - \Delta)u = \lambda u$ consists of $u_\kappa = e^{i\kappa\cdot n}$, $\lambda_\kappa = -\omega - 4\delta d \sum_{j=1}^d \sin^2 \frac{\kappa_j}{2}$, with $\kappa = [\kappa_1, \ldots, \kappa_d] \in [-\pi, \pi)^d$. The λ corresponding to unbounded solutions are easily seen not to be in $\sigma(-\omega - \Delta)$. We deduce that $\sigma_e(-\omega - \Delta) = \sigma_c(-\omega - \Delta) = [-\omega, -\omega + 4d\delta] = \sigma_e(L_-)$. Comparing the behavior at infinity of solutions of the eigenvalue equations for L_-, and $-\omega - \Delta$, we see that the solutions of the equation for L_- can not decay at infinity. Therefore $\sigma_e(L_-) = \sigma_c(L_-)$. The case $\delta < 0$, and the result for $\sigma_c(L_+)$ follow similarly.

Regarding the eigenvalues of $L_\pm$, we note that the families $L_\pm = L_\pm(\delta)$ of (3.4) are real analytic in δ, i.e. the series are convergent in the operator norm in Y. Since the $L_\pm$ are symmetric, and the zero eigenvalue of $L_\pm(0)$ has geometric multiplicity k, $L_\pm(0)$ has k eigenvectors that are real analytic in δ. Hence the k corresponding eigenvalues are real analytic in δ (for delta sufficiently small). Also, by (2.4), (2.10), $L_-(\delta)A(\delta) = 0$, $\forall |\delta| < \delta_0$. □

Proposition 3.2 *Consider the operator $J\mathcal{H}$, with $0 < |\delta| < \delta_0$. Then $\sigma_c(J\mathcal{H})$ consists of $z \in \mathbf{C}$ with $Re z = 0$, $Im z \in [-\omega, -\omega + 4d\delta] \cup [\omega - 4d\delta, \omega]$, for $\delta > 0$, and $Re z = 0$, $Im z \in [-\omega + 4d\delta, -\omega] \cup [\omega, \omega - 4d\delta]$, if $\delta < 0$. Furthermore, $\sigma_p(J\mathcal{H})$ consists of $2k$ eigenvalues (counting algebraic multiplicity) that belong to a disk of radius $o(1)$ around the origin. Moreover $0 \in \sigma_p(J\mathcal{H})$, $\forall \delta$ with $|\delta| < \delta_0$.*

Proof. The arguments for $\sigma_c(J\mathcal{H})$ are similar to the ones in the proof of Proposition 3.1. In particular, $\sigma_e(J\mathcal{H}) = \sigma_e(J\mathcal{H}_0)$, where

$$\mathcal{H}_0 = \begin{bmatrix} -\omega - \Delta & 0 \\ 0 & -\omega - \Delta \end{bmatrix}.$$

This follows from the observation that $J\mathcal{H} - J\mathcal{H}_0$ is Hilbert-Schmidt. Examining the behavior at infinity of the solutions of $J\mathcal{H}u = \lambda u$, and solving $J\mathcal{H}_0 u = \lambda u$ explicitly, we see that $\sigma_c(J\mathcal{H}) = \sigma_e(J\mathcal{H}) = \sigma_e(J\mathcal{H}_0) = \sigma_c(J\mathcal{H}_0)$. The result follows from the calculation of $\sigma_c(J\mathcal{H}_0)$.

The statement on the multiplicity of eigenvalues of $J\mathcal{H}$ follows from the fact that, for $\delta = 0$, $J\mathcal{H}$ has a zero eigenvalue of algebraic multiplicity $2k$, and from the continuity of finite rank spectral projections. The perturbed eigenvalues are of $o(1)$ by the upper semicontinuity of finite algebraic multiplicity eigenvalues under bounded perturbations (see [K], p. 208). Also, by Lemma 3.9, and $L_-(\delta)A(\delta) = 0$, we have the $0 \in \sigma_p(J\mathcal{H})$, $\forall |\delta| < \delta_0$. □

To examine the eigenvalues of $J\mathcal{H}$ we consider the following assumptions.

A I There exists a smallest positive integer r for which the near-zero eigenvalues $\rho_1, \ldots, \rho_k$ of L_- can be written as $\rho_1 = 0$, $\rho_2 = c_2\delta^{r_2} + O(\delta^{r_2+1})$, ..., $\rho_k = c_k\delta^{r_k} + O(\delta^{r_k+1})$, with $c_j \neq 0$, $r_j \leq r$, $\forall j = 2, \ldots, k$ (i.e. multiple eigenvalues, if any, are repeated). We say that the zero eigenvalue of $L_-(0)$ *unfolds* at order r.

A II Assume *AI*. Let $r' \in 1, \ldots, r$ and consider the set $I(r')$ of all $j \in \{2, \ldots, k\}$ for which $\rho_j = c_{j'}\delta^{r'} + O(\delta^{r'+1})$, with $c_{j'} \neq 0$. Suppose further that for all nonempty $I(r')$ we have $c_a \neq c_b$, for any a, $b \in I(r')$. Then we say that the zero eigenvalue of $L_-(0)$ *unfolds nondegenerately* at order r.

Conditions AI, AII can be verified in an algorithmic way. The procedure we use is described in the proof of Lemma 3.5..

Theorem 3.3 *Let $0 \leq |\delta| < \delta_0$. Assume that the zero eigenvalue of $L_-(0)$ unfolds at order r. Then each c_j, $j = 2, \ldots, k$ above corresponds to a pair of eigenvalues $\lambda_{j,\pm}$ of $J\mathcal{H}$. Assume $\delta > 0$. If $c_j > 0$, then $\lambda_{j,\pm} = \pm i\sqrt{2\omega c_j}\delta^{\frac{r_j}{2}} + O(\delta^{\frac{r_j}{2}+\frac{1}{2}})$. If $c_j < 0$, then $\lambda_{j,\pm} = \pm\sqrt{2\omega|c_j|}\delta^{\frac{r_j}{2}} + O(\delta^{\frac{r_j}{2}+\frac{1}{2}})$. Also $J\mathcal{H}$ has a zero eigenvalue of algebraic multiplicity 2, $\forall \delta \in (-\delta_0, \delta_0)$. For $\delta < 0$, the saddles become centers and vice versa, with the formulas interchanged.*

Thus, up to an error $O(\delta^{\frac{r_j}{2}+\frac{1}{2}})$, the eigenvalues $\lambda_{j,\pm}$ belong to either the real or the imaginary axis. Assuming AII, we obtain a stronger statement.

Corollary 3.4 *Let $0 \leq |\delta| < \delta_0$. Assume that the zero eigenvalue of $L_-(0)$ unfolds nondegenerately at order r. Then each c_j, $j \in I(r')$, $r' \in \{1, \ldots, r\}$, corresponds to a pair of eigenvalues $\lambda_{j,\pm}$ of $J\mathcal{H}$. Assume $\delta > 0$. If $c_j > 0$, then $\lambda_{j,\pm} = \pm i\sqrt{2\omega c_j}\delta^{\frac{r_j}{2}} + O(\delta^{\frac{r_j}{2}+\frac{1}{2}}) \in i\mathbf{R}$. If $c_j < 0$ then $\lambda_{j,\pm} = \pm\sqrt{2\omega|c_j|}\delta^{\frac{r_j}{2}} + O(\delta^{\frac{r_j}{2}+\frac{1}{2}}) \in \mathbf{R}$. For $\delta < 0$, the saddles become centers and vice versa, with the formulas interchanged.*

To prove Theorem 3.3 we first use Lemmas 3.5, 3.7, 3.8 to calculate the eigenvalues of L_- to any desired order in δ. Lemma 3.9 relates the eigenvalues of $J\mathcal{H}$, and L_+L_-, while Lemma 3.11 shows how the expansions for the eigenvalues of L_- yield expansions for the eigenvalues of L_+L_-.

Lemma 3.5 *Let $|\delta| < \delta_0$. Let $r > 1$. There exists an orthogonal operator $\tilde{M} \in B(Y)$ such that $\tilde{L}_-^r = \tilde{M}^{-1} L_- \tilde{M}$ satisfies*

$$\tilde{L}_-^r(m,n) = O(\delta^{r+1}), \quad \forall (m,n) \quad \textit{with} \quad m \in U, n \in U^c, \quad \textit{or} \quad m \in U, n \in U^c. \tag{3.9}$$

Moreover, the U block of $\tilde{L}_-^r$ is diagonal, up to an error of $O(\delta^{r+1})$. The operator $\tilde{L}_+^r = \tilde{M}^{-1} L_- \tilde{M}$ has the form $\tilde{L}_+^r = L_{+,0} + O(\delta)$.

Proof. We can recursively define antisymmetric operators $\psi_1, \dots \psi_r$, such that for any $s = 1, \dots, r$, the operator

$$L_-^s = e^{-\delta^m \psi_s} \dots e^{-\delta \psi_1} L_- e^{\delta \psi_1} \dots e^{\delta^m \psi_s} \tag{3.10}$$

satisfies

$$L_-^s(m,n) = O(\delta^{s+1}), \quad \forall (m,n) \quad \text{with} \quad m \in U, n \in U^c, \quad \text{or} \quad m \in U, n \in U^c. \tag{3.11}$$

ψ_s is chosen to satisfy

$$(L_{-,0}(m,m) - L_{-,0}(n,n))\, \psi_s(m,n) = f(m,n), \tag{3.12}$$

$\forall (m,n)$ with $m \in U$, $n \in U^c$ or $m \in U$, $n \in U^c$, where f is the coefficient of δ^{s+1} in the expansion of L_-^{s-1}. For all other m, n we choose $\psi_s(m,n) = 0$.

The U block of L_-^r can be diagonalized recursively to $O(\delta^{r+1})$ by a sequence of similarity transformations by orthogonal block diagonal (with respect to the U, U^c subspaces) operators $T_1, \dots, T_r$. The U^c block of the T_j is the identity. The U block of the T_j is determined recursively so that it diagonalizes $T_1^{-1} \dots T_{j-1}^{-1} L_-^r T_1 \dots T_{j-1}$ to $O(\delta^{j+1})$. □

Remark 3.6 *The above proof is a sketch of a normal form calculation. To determine T_1 we must calculate the eigenvalues and eigenvectors of a $k \times k$ matrix, i.e. a nonperturbative step. If $L_-(0)$ does not unfold at order 1 we may need more calculations of this type, with matrices of size $k' \times k'$, $k' \leq k$. These calculations correspond to the metanormal form steps of [M02], ch. 2.*

Thus similarity by $\tilde{M}$ decouples and diagonalizes the U block of L_-, up to an error of $O(\delta^{r+1})$. In contrast, the U block of $\tilde{L}_+^r$ is neither decoupled from the U^c block nor diagonalized to $O(\delta^{r+1})$ by this transformation.

The U block of $\tilde{L}_-^r$ can be written as $\Lambda_-^r + O(\delta^{r+1})$, with Λ_-^r diagonal. The diagonal entries of Λ_-^r are polynomials of order r in δ. By Lemma 3.8 below, these are precisely the lowest order terms in the expansion for the eigenvalues in AI, AII. We first need the following analogue of Gershgorin's theorem.

Lemma 3.7 *Let A be a bounded operator in Y, and let $A(i,j)$ be the entries of its representation in the standard basis. Let $\lambda \in \mathbf{C}$ be an eigenvalue of A, with corresponding eigenvector $x \in Y_c$. Then*

$$|\lambda - A(i,i)||x_i| \leq \Big(\sum_{j \in \mathbf{Z}^d \setminus \{i\}} |A(i,j)|^2\Big)^{\frac{1}{2}} \Big(\sum_{j \in \mathbf{Z}^d} |x_j|^2\Big)^{\frac{1}{2}}, \quad \forall i \in \mathbf{Z}^d, \tag{3.13}$$

where $x = \sum_{j \in \mathbf{Z}^d} x_j \hat{e}(j)$.

Proof. Consider the diagonal operator D with matrix entries $D(i,i) = A(i,i)$, $i \in \mathbf{Z}^d$. From $Ax = \lambda x$ we have $(\lambda - A)x = (A - D)x$, or

$$\sum_{j \in \mathbf{Z}^d} (\lambda \delta_{ij} - D(i,j))\, x_j = \sum_{j \in \mathbf{Z}^d} (A(i,j) - D(i,j))\, x_j, \quad \forall i \in \mathbf{Z}^d. \tag{3.14}$$

Since A^T is bounded, and $x \in Y_c$, (3.13) follows from (3.14), and the Schwartz inequality. □

The following lemma identifies the diagonal entries of Λ_-^r with the expansions of AI, AII. (We give a sketch of the proof.)

Lemma 3.8 *Let $|\delta| < \delta_0$. Let $r > 1$. Then the diagonal entries of the U block of $\tilde{L}_-^r$ coincide, up to an error of $O(\delta^{r+1})$, with the eigenvalues $\rho_1, \ldots, \rho_k$ of L_-.*

Proof. By Lemma 3.5 and the definition of Λ_-^r we can write

$$\tilde{L}_-^r = N + O(\delta^{r+1}), \quad \text{with} \quad N = \begin{bmatrix} \Lambda_-^r & 0 \\ 0 & D \end{bmatrix}, \tag{3.15}$$

D a bounded operator in Y_{U^c}, and $O(\delta^{r+1})$ standing for a bounded operator with norm of size $O(\delta^{r+1})$. For each $\Lambda_-^r(j,j)$, $j = 1, \ldots, k$, N has an eigenvector $[y^j, 0] \in Y_U \oplus Y_{U^c}$, with $y_j^j = 1$, and $y_i^j = 0$ for $i \neq j$. By the continuity of spectral projections $\tilde{L}_-^r$ has an eigenvector $\tilde{x}^j = [\tilde{y}^j, \tilde{z}]$ satisfying $\tilde{y}_j^j = 1 + o(1)$, and $||\tilde{x}^j||_{Y_c} = 1 + o(1)$. Applying Lemma 3.7 to $\tilde{L}_-^r$ and each $\tilde{x}^j$ we see that the corresponding eigenvalue is $\Lambda_-^r(j,j) + O(\delta^{r+1})$, i.e. $\rho_0(j) + \ldots + \rho_r(j)\delta^r + O(\delta^{r+1})$. These are also the eigenvalues of L_- by Lemma 3.5.

On the other hand, the analyticity of A in δ, and (3.2) implies that L_- has eigenvalues of $\sum_{m=1}^{\infty} a_m(j)\delta^m$ for $|\delta| < \delta_0$, with $j = 1, \ldots, k$. This follows by results on analyticity of eigenvalues for analytic families of symmetric operators (see e.g. [K76], p.385).

The statement follows by showing that the coefficients of the two expansions are the same, i.e. the ordered sets S_ρ of the $[\rho_0(j), \ldots, \rho_r(j)]$ and S_a of the $[a_0(j), \ldots, a_r(j)]$, $j = 1, \ldots, k$, are the same up to a permutation. Assuming that this is false we easily produce a contradiction with the claim above that the eigenvalues of L_- are $\Lambda_-^r(j,j) + O(\delta^{r+1})$. □

We now consider the operator $L_+ L_-$.

Lemma 3.9 *Let $|\delta| < \delta_0$. If $\lambda \in \mathbf{C}$ is an eigenvalue of $J\mathcal{H} \in B(X_c)$, then $-\lambda^2$ is an eigenvalue of $\tilde{L}_+^r \tilde{L}_-^r \in B(Y_c)$. Conversely, if $\tilde{\rho} \in \mathbf{C}$ is an eigenvalue of $\tilde{L}_+^r \tilde{L}_-^r \in B(Y_c)$, then any $\lambda \in \mathbf{C}$ satisfying $-\lambda^2 = \tilde{\rho}$ is an eigenvalue of $J\mathcal{H} \in B(X_c)$.*

Proof. We first note that $\sigma_p(\tilde{L}_+^r \tilde{L}_-^r) = \sigma_p(L_+ L_-)$ since the two operators are similar by Lemma 3.5. Let $v \in X_c$ satisfy $J\mathcal{H}v = \lambda v$. By (2.12), (3.3) this is equivalent to

$$L_- p = \lambda q, \quad -L_+ q = \lambda p, \tag{3.16}$$

where $v = [q, p]$, q, $p \in Y_c$. By the invertibility of L_+ we then have $L_+ L_- p = -\lambda^2 p$, i.e. $\tilde{\rho} = -\lambda^2$ is an eigenvalue of $L_+ L_-$. Conversely, suppose that $L_+ L_- p = \tilde{\rho} p$ for some $\tilde{\rho} \in \mathbf{C}$, $p \in Y_c$. Let λ satisfy $-\lambda^2 = \tilde{\rho}$ and let $q = -\lambda L_+^{-1} p$, $v = [q, p] \in X_c$. Then q, p satisfy (3.16) and therefore λ is an eigenvalue of $J\mathcal{H}$. □

Remark 3.10 *Note that $\tilde{L}_+\tilde{L}_-$ is not normal and it may have residual spectrum. We will use the weaker property that, assuming $|\delta| \leq \delta_0$, $\lambda \in \sigma_p(L_-L_+)$ implies $\overline{\lambda} \in \sigma_p(L_+L_-)$: first note that $\lambda \in \sigma_p(L_-L_+)$ implies $\overline{\lambda} \in \sigma_p(L_-L_+)$ by the reality of L_-L_+. On the other hand, $\lambda \in \sigma_p(L_-L_+)$ also implies that $\overline{\lambda} \in \sigma_p(L_+L_-) \cup \sigma_r(L_+L_-)$. Suppose that $\overline{\lambda} \in \sigma_r(L_+L_-)$. Then $\overline{\lambda} \notin \sigma_p(L_+L_-)$. Since L_+ is invertible for $|\delta| \leq \delta_0$ we have $\sigma_p(L_+L_-) = \sigma_p(L_-L_+)$ by a standard argument, hence $\overline{\lambda} \notin \sigma_p(L_-L_+)$, a contradiction.*

Lemma 3.11 *Let $|\delta| < \delta_0$. Assume that the zero eigenvalue of $L_-(0)$ unfolds at order r and consider the pairs of r_j, c_j, $j \in \{1, \dots, r\}$ of AI. Then $\tilde{L}_+^r\tilde{L}_-^r \in B(Y_c)$ has k eigenvalues $\tilde{\rho}_j$, $j = 1, \dots, k$, with $\tilde{\rho}_1 = 0$, and $\tilde{\rho}_j = 2\omega c_j \delta^{r_j} + O(\delta^{r_j+1})$, $j = 2, \dots, k$.*

Proof. Let

$$\tilde{L}_+^r = \begin{bmatrix} A_+ & B_+ \\ C_+ & D_+ \end{bmatrix}, \quad \tilde{L}_-^r = \begin{bmatrix} A_- & B_- \\ C_- & D_- \end{bmatrix}, \tag{3.17}$$

with $A_\pm$, $D_\pm$ the U, U^c blocks respectively. By Lemma 3.5 we have

$$A_+ = \Lambda_1 + O(\delta), \quad B_+ = O(\delta), \quad C_+ = O(\delta), \quad D_+ = \Lambda_2 + O(\delta), \tag{3.18}$$

and

$$A_- = \Lambda_-^r + O(\delta^{r+1}), \quad B_- = O(\delta^{r+1}), \quad C_- = O(\delta^{r+1}), \quad D_+ = \Lambda_3 + O(\delta). \tag{3.19}$$

The operators Λ_1, Λ_2, Λ_3 are diagonal, with nonzero entries of $O(1)$. Then

$$\tilde{L}_+^r\tilde{L}_-^r = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}, \tag{3.20}$$

where by (3.17)-(3.19)

$$\tilde{A} = (\Lambda_1 + O(\delta))\Lambda_-^r + O(\delta^{r+1}), \quad \tilde{B} = O(\delta), \quad \tilde{C} = C_+\Lambda_-^r + O(\delta^{r+1}), \quad \tilde{D} = \Lambda_2\Lambda_3. \tag{3.21}$$

Let $j \in U$. The $j-$th column $\mathrm{col}_j(\tilde{A})$ of $\tilde{A}$ is

$$\mathrm{col}_j(\tilde{A}) = 2\omega c_j \delta^{r_j} \hat{e}_j + O(\delta^{r_j+1}), \tag{3.22}$$

where $\hat{e}_j$ is the $j-$th column of the $k \times k$ identity matrix. The $j-$th column $\mathrm{col}_j(\tilde{C})$ of $\tilde{C}$ is

$$\mathrm{col}_j(\tilde{C}) = c_j \delta^{r_j} C_+ \hat{e}_j = O(\delta^{r_j+1}), \tag{3.23}$$

since $C_+ = O(\delta)$. Also, we can write

$$(\tilde{L}_+^r\tilde{L}_-^r)^T = \begin{bmatrix} \tilde{A}^T & \tilde{C}^T \\ \tilde{B}^T & \tilde{D}^T \end{bmatrix} = M + O(\delta), \quad \text{with} \quad M = \begin{bmatrix} \Lambda_1\Lambda_-^r & 0 \\ 0 & \tilde{D}^T \end{bmatrix}. \tag{3.24}$$

$\Lambda_1\Lambda_-^r$ is diagonal and its eigenvectors are the standard basis vectors in $\mathbf{R}^k$. Then $[y, 0] \in Y_U \oplus Y_{U^c}$, with y one of the eigenvectors of $\Lambda_1\Lambda_-^r$ is an eigenvector of M. Also y satisfies $y_j = 1$, for some $j \in \{1, \dots, k\}$, and $y_i = 0$, for $i \neq j$. By the continuity of spectral projections $(\tilde{L}_+^r\tilde{L}_-^r)^T$ has an eigenvector $\tilde{x} = [\tilde{y}, \tilde{z}]$ satisfying $\tilde{y}_j = 1 + o(1)$, $||\tilde{x}||_{Y_c} = 1 + o(1)$. Let ρ be the corresponding eigenvalue. Then, by (3.22), (3.23), and Lemma 3.7

$$|\rho - \Lambda_1 \Lambda_-^r(j,j)| = O(\delta^{r+1}), \tag{3.25}$$

hence

$$|\overline{\rho} - \Lambda_1 \Lambda_-^r(j,j)| = O(\delta^{r+1}), \tag{3.26}$$

since $\Lambda_1 \Lambda_-^r(j,j) \in \mathbf{R}$, $\forall j \in \{2,\dots,k\}$. By Lemma 3.5 the $\tilde{L}_\pm^r$ are symmetric, hence $\overline{\rho} \in \sigma_p(\tilde{L}_-^r \tilde{L}_+^r) = \sigma_p(L_- L_+)$. Therefore $\tilde{\rho} = \overline{\rho} \in \sigma_p(L_+ L_-)$ by Remark 3.10. The estimates on the eigenvalues $\tilde{\rho}_j$ of $L_+ L_-$ then follow from (3.26), and (3.22). □

Proof of Theorem 3.3. The formulas for the eigenvalues of $J\mathcal{H}$ follow from the formulas for the $\tilde{\rho}_j$ in Lemma 3.11, and Lemma 3.9. □

Proof of Corollary 3.4. It is enough to show that if λ is an eigenvalue of $J\mathcal{H}$ the so are $-\lambda$, $\overline{\lambda}$, and $-\overline{\lambda}$, as in finite dimensions. For $-\lambda$ we argue using Lemma 3.9: by the lemma $\rho = -\lambda^2$ is an eigenvalue of $L_+ L_-$, hence $-\lambda$ is an eigenvalue of $J\mathcal{H}$, also by the lemma. For $\overline{\lambda}$ we use the reality of $J\mathcal{H}$, and $-\overline{\lambda}$ follows immediately. □

Remark 3.12 *For $k \geq 2$ we see that, even in the case of linear stability, the operator $\mathcal{H}$ is neither positive nor negative definite for we can find eigenvalues and points in the continuous spectrum with opposite signs.*

4 Examples

The above theory applies to any real breather. In practice the computations can be long. Also, the step of diagonalizing the U block or some of its sub-blocks may require numerical computation (see Remark 3.6). In contrast to a direct numerical computation of the spectrum of $J\mathcal{H}$, we only may require a numerical calculation for the coefficients the fractional powers of δ. This is an advantage in accuracy over the direct computation of the eigenvalues of $\mathcal{J}\mathcal{H}$.

Remark 4.1 *Two distinct sites n, $n' \in U \subset \mathbf{Z}^d$ are $g-$neighbors if $dist_d(n,n') = |n_1 - n'_1| + \ldots + |n_d - n'_d| = g$. The number $G(U)$ is the smallest g such that all sites of U have at least one $g'-$neighbor, $g' \leq g$. Observe that L_- can only unfold at some order $r \geq G(U)$.*

Below we present some simple examples where the calculations can be done by hand. We examine a $2-$peak breather in $\mathbf{Z}$, and a $3-$peak breather in $\mathbf{Z}^2$.

Example 1: Two consecutive peaks in $\mathbf{Z}$.

Let $U = \{0,1\} \subset \mathbf{Z}$. We continue two breather solutions denoted $(+,+)$, $(+,-)$ that are respectively obtained by continuation from the $\delta = 0$ solutions

$$A_0(0) = \sqrt{\frac{\omega}{2}}, \quad A_1(0) = \pm\sqrt{\frac{\omega}{2}}, \quad A_n = 0, \quad \forall n \in U^c. \tag{4.1}$$

For the $(+,+)$ breather we compute

$$A_0(\delta) = \sqrt{\frac{\omega}{2}} - \sqrt{\frac{1}{8\omega}}\delta + O(\delta^2), \quad A_1(\delta) = \sqrt{\frac{\omega}{2}} - \sqrt{\frac{1}{8\omega}}\delta + O(\delta^2), \tag{4.2}$$

and $A_n(\delta) = O(\delta)$, $\forall n \in U^c$. Then

$$L_-(n,n) = 4A_{n,0}A_{n,1}\delta + 2\delta + O(\delta^2), \quad n \in U; \quad L_-(n,n) = -\omega + O(\delta^2), \quad n \in U^c. \tag{4.3}$$

The U block $L_-|_U$ of L_- is then

$$L_-|_U = \delta \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + O(\delta^2). \tag{4.4}$$

Choosing suitable ψ_1, the U, and U^c blocks of L_-^1 are decoupled up to an $O(\delta^2)$. The U block of L_-^1 is $L_-|_U$, up to an $O(\delta^2)$. The eigenvalues are 0, 2δ, up to $O(\delta^2)$. Therefore $J\mathcal{H}$ has a double zero eigenvalue, and the pair of eigenvalues $\pm i(2\sqrt{\omega}\delta^{\frac{1}{2}} + O(\delta^{\frac{3}{2}})) \in i\mathbf{R}$ for $\delta > 0$, or $\pm(2\sqrt{\omega}\delta^{\frac{1}{2}} + O(\delta^{\frac{3}{2}})) \in \mathbf{R}$ for $\delta < 0$.

For the $(+,-)$ breather we have

$$A_0(\delta) = \sqrt{\frac{\omega}{2}} - \sqrt{\frac{9}{8\omega}}\delta + O(\delta^2), \quad A_1(\delta) = -\sqrt{\frac{\omega}{2}} - \sqrt{\frac{9}{8\omega}}\delta + O(\delta^2), \tag{4.5}$$

and $A_n = O(\delta)$, $\forall n \in U^c$. Also L_- is given by (4.3), (4.5). The $U-$block $L_-|_U$ of L_- is then

$$L_-|_U = \delta \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} + O(\delta^2). \tag{4.6}$$

Similarly decoupling the U, U^c blocks to $O(\delta^2)$, the U block of L_- remains unchanged and we have the eigenvalues 0, -2δ, up to $O(\delta^2)$. Thus $J\mathcal{H}$ has a double zero eigenvalue, and the pair of eigenvalues $\pm(2\sqrt{\omega}\delta^{\frac{1}{2}} + O(\delta^{\frac{3}{2}})) \in \mathbf{R}$ for $\delta > 0$. The $(+,-)$ solution is therefore unstable for $\delta > 0$, and stable for $\delta < 0$.

Example 2: Three peaks in $\mathbf{Z}^2$.

Let $U = \{(0,1),(0,0),(1,0)\} \subset \mathbf{Z}^2$. We continue two breather solutions denoted $(+,+,+)$, $(-,+,+)$ that are respectively obtained by continuation from the $\delta = 0$ solutions

$$A_{(0,1)}(0) = \pm\sqrt{\frac{\omega}{2}}, \quad A_{(0,0)}(0) = A_{(1,0)}(0) = \sqrt{\frac{\omega}{2}}, \qquad A_n(\delta) = 0, \quad \forall n \in U^c. \tag{4.7}$$

For the $(+,+,+)$ breather we have

$$A_{(0,1)}(\delta) = \sqrt{\frac{\omega}{2}} - \frac{3}{2\sqrt{2\omega}}\delta + O(\delta^2), \quad A_{(0,0)}(\delta) = \sqrt{\frac{\omega}{2}} - \frac{1}{\sqrt{2\omega}}\delta + O(\delta^2), \tag{4.8}$$

$$A_{(1,0)}(\delta) = \sqrt{\frac{\omega}{2}} - \frac{3}{2\sqrt{2\omega}}\delta + O(\delta^2), \tag{4.9}$$

and $A_n(\delta) = O(\delta)$, $\forall n \in U^c$. The U block $L_-|_U$ of L_- is then

$$L_-|_U = \delta \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + O(\delta^2). \tag{4.10}$$

Column and row indices $1, 2, 3$ correspond to sites $(0, 1)$, $(0, 0)$, $(1, 0)$ respectively. Choosing suitable ψ_1, the $U-$, U^c blocks of L^1_- are decoupled, up to an $O(\delta^2)$ error. The U block of L^1_- is $L_-|_U$ up to an $O(\delta)$ error. The eigenvalues are 0, $\delta + O(\delta^2)$, $3\delta + O(\delta^2)$, therefore $J\mathcal{H}$ has a double zero eigenvalue, and two pairs of imaginary of eigenvalues for $\delta > 0$ (or two pairs of positive/negative eigenvalues for $\delta < 0$). We thus have stability for $\delta > 0$, and instability for $\delta < 0$.

For the $(-, +, +)$ breather we have

$$A_{(0,1)}(\delta) = -\sqrt{\frac{\omega}{2}} + \frac{5}{2\sqrt{2\omega}}\delta + O(\delta^2), \quad A_{(0,0)}(\delta) = \sqrt{\frac{\omega}{2}} - \frac{2}{\sqrt{2\omega}}\delta + O(\delta^2), \tag{4.11}$$

$$A_{(1,0)}(\delta) = \sqrt{\frac{\omega}{2}} - \frac{3}{2\sqrt{2\omega}}\delta + O(\delta^2), \tag{4.12}$$

and $A_n = O(\delta)$, $\forall n \in U^c$. The U block $L_-|_U$ of L_- is then

$$L_-|_U = \delta \begin{bmatrix} -1 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} + O(\delta^2). \tag{4.13}$$

Column and row numbering is as in (4.10). Choosing suitable ψ_1, the U, U^c blocks of L^1_- are decoupled, up to an $O(\delta^2)$ error. The U block of L^1_- is $L_-|_U$ up to an $O(\delta)$ error. The eigenvalues are 0, $\pm\sqrt{3}\delta$, up to $O(\delta^2)$. Thus the breather is unstable for $\delta \neq 0$.

5 Discussion

We have presented a method for analyzing the linear stability of real breathers of the DNLS near the limit of vanishing intersite coupling. Our results can be used directly to show the persistence of breathers in perturbed DNLS equations, e.g. in the DNLS with weak parametric forcing, see [P07], [PP08]. We also believe that the method we used here can be extended to the study of more general breathers.

A natural question arising from this study is that of the nonlinear stability of near-anticontinuous limit breathers. In the case of linearly stable breathers the linear stability analysis is not conclusive, see Remark 3.12. One scenario that should be checked is nonlinear stability due to the dispersive effects of the continuous spectrum. In the case of linearly unstable breathers it could be useful to study the evolution of the phases, expecting that for certain sets of active sites, stable breathers "attract" nearby unstable ones. Questions of this type will be considered in further work.

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