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Averaging and Benjamin–Feir instabilities in a parametrically forced nonlinear Schrödinger equation

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Abstract

We study small amplitude solutions of a parametrically forced nonlinear Schrödinger equation, and give exact expressions for the quartic resonant mode interactions by analyzing the corresponding Diophantine equations. We use this information to find some classes of periodic orbits of the averaged equation and characterize the linearly unstable directions of Stokes wave solutions.

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1. Introduction

In this Letter we study small amplitude solutions of the parametrically forced cubic nonlinear Schrödinger equation $u_t = id(t)u_{xx} - 2i\gamma|u|^2u$, with $d(t)$ a periodic real valued function and γ a real constant. The equation has been used to model the propagation of signals in optical fibers with dispersion management, and there is extensive literature on solutions of amplitude $O(1)$ in the parameter range where the average δ and period T of the function $d(t)$, and the parameter γ are all comparable and small in absolute value (see, e.g., [1,4–7,9,12] for further references). Here we consider the equation in a periodic domain, and study small amplitude solutions, or equivalently solutions of $O(1)$ amplitude with $|\gamma| \ll |\delta|$.

In the parameter range of interest we have a weakly nonlinear system that is expected to exhibit nontrivial dynamics over a long time. An unusual feature of the system is that the resonance conditions determining the lowest order normal form (averaged) equation can be analyzed completely. In particular, we see that the sets of solutions of the Diophantine equations for the quartic resonances admit explicit parameterizations. More generally, we can here parameterize the

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level sets of the functions appearing in the resonance conditions. We remark that determining resonant interactions and small divisors is one of the main obstacles in normal form calculations for nonlinear dispersive systems. An understanding of the resonances can have many applications, for instance in finding adiabatic constants of motion (see, e.g., [2,10]).

The averaged equation is seen to have several classes of periodic orbits and we use the parameterization of the resonances to obtain linear stability information on periodic orbits that describe traveling near-monochromatic waves ("Stokes waves"). Owing to the special structure of the cubic NLS nonlinearity the linear stability analysis of these orbits is exact and we can determine all linearly unstable directions of their Floquet maps. A similar analysis applies to other dispersive systems of interest, and we also briefly discuss an alternative approach to the linear stability analysis of Stokes waves in forced NLS equations (see [3]). We also present numerical simulations showing the growth of the unstable modes indicated by the linear stability analysis. The simulations suggest that the growth of the unstable modes eventually saturates at small amplitudes, and it is likely that the Stokes waves may be nonlinearly stable over long intervals. These phenomena will be examined elsewhere.

The Letter is organized as follows. In Section 2 we discuss the Hamiltonian structure of the system, and formally compute a normal form by analyzing the relevant resonance conditions. The main application is in Section 3 where we describe periodic orbits and the linear stability analysis of Stokes waves. In Section 4 we state an averaging theorem on the distance between solutions of the averaged and full systems, discuss its application to the results of Section 3, and compare the linear stability analysis of the Stokes waves to some numerical simulations of the full system.

2. Hamiltonian structure and normal forms

We consider the initial value problem for the non-autonomous equation

$$u_t = i d(t) u_{xx} - 2i\gamma |u|^2 u, \tag{2.1}$$

with $u(x,t)$ a complex valued function satisfying periodic boundary conditions $u(x,t) = u(x+2\pi,t)$. The function $d(t)$ and the parameter γ are real. We assume that $d(t)$ is T-periodic, we decompose it as

$$d(t) = \delta + \tilde{d}(t), \quad \text{with} \quad \delta = \frac{1}{T}\int_0^T d(s)\,ds. \tag{2.2}$$

Letting $\Omega = \frac{2\pi}{T}$, we will be interested in the parameter range where $|\Omega| \geqslant O(1)$ (and possibly $\gg 1$), $|\delta|$ is of $O(1)$, and $|\gamma| \ll 1$.

It is easy to see that (2.1) has the structure of a nonautonomous Hamiltonian system. To perform normal form calculations it will be convenient to make the system autonomous by introducing an additional angle variable. In particular, let $u_k(t)$ be the Fourier transform of $u(x,t)$ and define the variables $a_k(t)$, $k \in \mathbf{Z}$ by

$$a_k(t) = u_k(t) e^{i\omega_k \tilde{\Lambda}(t)}, \quad \text{with}$$
$$\omega_k = k^2, \qquad \tilde{\Lambda}(t) = \int_0^t \tilde{d}(s)\,ds. \tag{2.3}$$

Eq. (2.1) is then

$$\begin{aligned}\dot{a}_k = {} & -i\delta\omega_k a_k \\ & - 2i\gamma \sum_{k_1,k_2,k_3 \in \mathbf{Z}} a_{k_1} a_{k_2} a^*_{k_3} \delta_{k_1+k_2-k_3-k} \\ & \times e^{-i(\omega_{k_1}+\omega_{k_2}-\omega_{k_3}-\omega_k)\tilde{\Lambda}(t)},\end{aligned} \tag{2.4}$$

$k \in \mathbf{Z}$, where $\delta_r = 1$ if $r = 0$, 0 otherwise. The initial condition is $a_k(0) = u_k(0)$, $k \in \mathbf{Z}$. The right-hand side of (2.4) is T-periodic. Also, we consider an angle $\phi \in [0, 2\pi)$, and add to (2.4) the equation $\dot{\phi} = \Omega$, with $\phi(0) = 0$. The function Λ defined by $\Lambda(\phi) = \tilde{\Lambda}(t(\phi)) = \tilde{\Lambda}(\frac{\phi}{\Omega})$ is 2π-periodic with zero average. Adding an "action" variable $J \in \mathbf{R}$, we further define the Poisson bracket $[\,,\,]$ on pairs of functions F, G of the variables a_k, a_k^*, $k \in \mathbf{Z}$, and ϕ, J by

$$\begin{aligned}[F,G] = {} & -i\sum_{k\in\mathbf{Z}}\left(\frac{\partial F}{\partial a_k}\frac{\partial G}{\partial a_k^*} - \frac{\partial F}{\partial a_k^*}\frac{\partial G}{\partial a_k}\right) \\ & + \frac{\partial F}{\partial J}\frac{\partial G}{\partial \phi} - \frac{\partial F}{\partial \phi}\frac{\partial G}{\partial J}.\end{aligned} \tag{2.5}$$

We then have the following:

Proposition 2.1. *The evolution equation for the variables a_k, $k \in \mathbf{Z}$, and ϕ, J above is the Hamiltonian* *system*

$$\dot{a}_k = [a_k, H], \quad k \in \mathbf{Z},$$
$$\dot{\phi} = [\phi, H], \qquad \dot{J} = [J, H], \tag{2.6}$$

where the Hamiltonian H given by

$$H = \delta \sum_{k\in\mathbf{Z}} \omega_k |a_k|^2 - \Omega J + \gamma \sum_{k_1,k_2,k_3,k_4,n\in\mathbf{Z}} e^{in\phi} a_{k_1} a_{k_2} a^*_{k_3} a^*_{k_4} \times I(k_1, k_2, k_3, k_4, n), \tag{2.7}$$

and the coefficients $I(k_1, k_2, k_3, k_4, n)$ are given by

$$I(k_1, k_2, k_3, k_4, n) = \hat{f}_m(n) \delta_{k_1+k_2-k_3-k_4}, \tag{2.8}$$

$$m = \omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4},$$
$$\hat{f}_m(n) = (2\pi)^{-1} \int_0^{2\pi} e^{-im\Lambda(\phi)} e^{-in\phi}\, d\phi. \tag{2.9}$$

This setup is quite general, and we can also consider other dispersion relations, γ time dependent, and quasi-periodic dispersion management functions. The structure of the coefficients $\hat{f}_m(n)$ is discussed below.

The Hamiltonian H of (2.7) has the form $H = H_2 + H_4$, where

$$H_2 = \delta \sum_{k\in\mathbf{Z}} \omega_k |a_k|^2 - \Omega J, \tag{2.10}$$

and H_4 contains the quartic terms and we see that the parameter regime of interest describes a "weakly" nonlinear system with H_2 describing the "fast" motions.

To eliminate the lowest order resonant terms of H_4, consider a function $\chi_1(a, a^*, \phi, J)$ and the canonical transformation generated by the time-1 map $\Phi^1_{\chi_1}$ of the Hamiltonian flow of χ_1. Formally, we have

$$H \circ \Phi^1_{\chi_1} = \exp \mathrm{Ad}_{\chi_1} H = H_2 + H_4 + [\chi_1, H_2] + Y_1, \tag{2.11}$$

with Y_1 the remainder (containing terms of order 6 and higher in a, a^*). By the definition of the Poisson bracket, the resonance condition is

$$n\Omega - \delta m = 0, \qquad k_1 + k_2 - k_3 - k_4 = 0,$$
$$\hat{f}_m(n) \neq 0, \quad k_1, \ldots, k_4, n \in \mathbf{Z} \tag{2.12}$$

with $m = \omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4}$. The sum of all monomials with indices satisfying (2.12) in H_4 (i.e., the resonant part of H_4) will be denoted by $\bar{H}_4$. For χ_1 satisfying $[\chi_1, H_2] + H_4 = \bar{H}_4$ we write the transformed Hamiltonian as

$$H \circ \Phi^1_{\chi_1}\big(a, a^*, \phi, J\big) = H_2 + \bar{H}_4 + Y_6. \tag{2.13}$$

The remainder Y_6 will be of $O(\gamma^2)$. To determine the quartic resonant terms we use the following:

Proposition 2.2. *Let $m \in \mathbf{Z}$, $\omega_k = k^2$, and let Λ_m the set of integers k_1, k_2, k_3, k_4 satisfying*

$$\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4} = m,$$
$$k_1 + k_2 - k_3 - k_4 = 0. \tag{2.14}$$

Then, (i) *if $m \in 2\mathbf{Z} + 1$ then Λ_m is empty, and* (ii) *if $m \in 2\mathbf{Z}$ then Λ_m is*

$$k_1 = \frac{1}{2}\left[\theta + \frac{1}{2}\left(s + \frac{2m}{s}\right)\right],$$
$$k_2 = \frac{1}{2}\left[\theta - \frac{1}{2}\left(s + \frac{2m}{s}\right)\right],$$
$$k_3 = \frac{1}{2}\left[\theta + \frac{1}{2}\left(s - \frac{2m}{s}\right)\right],$$
$$k_4 = \frac{1}{2}\left[\theta - \frac{1}{2}\left(s - \frac{2m}{s}\right)\right], \tag{2.15}$$

with

$$\theta \in 2\mathbf{Z}, \qquad s \in 2\mathbf{Z}^* \quad \text{s.t.} \quad s|m \quad \text{and}$$
$$\frac{1}{2}\left(s + \frac{2m}{s}\right) \in 2\mathbf{Z}, \tag{2.16}$$

or

$$\theta \in 2\mathbf{Z} + 1, \qquad s \in 2\mathbf{Z}^* \quad \text{s.t.} \quad s|m \quad \text{and}$$
$$\frac{1}{2}\left(s + \frac{2m}{s}\right) \in 2\mathbf{Z} + 1. \tag{2.17}$$

(*Notation*: $x|y$ means x divides y, and $\mathbf{Z}^* = \mathbf{Z} \setminus \{0\}$.)

Proof of Proposition 2.2. Let $k = [k_1, k_2, k_3, k_4] \in \mathbf{Z}^4$, and define new variables $a = [a_1, a_2, a_3, a_4]$ by

$$a_1 = k_1 + k_2, \qquad a_2 = k_1 - k_2,$$
$$a_3 = k_3 + k_4, \qquad a_4 = k_3 - k_4. \tag{2.18}$$

Let $\mathcal{P}(n) = +1, -1$ for n even, odd, respectively. The change of variables (2.18), written as $a = f(k)$ defines a function $f : \mathbf{Z}^4 \to \mathbf{Z}^4_{\mathcal{P}} = \{a \in \mathbf{Z}^4\colon \mathcal{P}(a_1) = \mathcal{P}(a_2), \mathcal{P}(a_3) = \mathcal{P}(a_4)\}$. We check that f is a bijection onto $\mathbf{Z}^4_{\mathcal{P}}$. Thus, by (2.14) and (2.18), it is sufficient to solve

$$a_2^2 - a_4^2 = 2m, \qquad a_1 = a_3,$$
$$m \in \mathbf{Z}, \qquad a \in \mathbf{Z}^4_{\mathcal{P}}. \tag{2.19}$$

Let $\mathbf{Z}^N_E$, $\mathbf{Z}^N_O$ denote the subsets of $\mathbf{Z}^N$ with even and odd coordinates, respectively. Eq. (2.19) reduces to

$$a_1 = a_3 = \theta \in 2\mathbf{Z}, \qquad a_2^2 - a_4^2 = 2m,$$
$$m \in \mathbf{Z}, \qquad a \in \mathbf{Z}^4_E, \tag{2.20}$$

and

$$a_1 = a_3 = \theta \in 2\mathbf{Z} + 1, \qquad a_2^2 - a_4^2 = 2m,$$
$$m \in \mathbf{Z}, \qquad a \in \mathbf{Z}^4_O. \tag{2.21}$$

In both cases we have $a_2^2 - a_4^3 = (a_2 + a_4)(a_2 - a_4) \in 4\mathbf{Z}$, and thus we can only have solutions for $m \in 2\mathbf{Z}$. To parameterize the solutions of (2.20), (2.21) we let

$$r = a_2 - a_4, \qquad s = a_2 + a_4. \tag{2.22}$$

We check that the new transformation defines functions $g_E : \mathbf{Z}^2_E \to \mathbf{Z}^2_{4x} = \{[r,s] \in \mathbf{Z}^2\colon r - s \in 4\mathbf{Z}\}$ and $g_O : \mathbf{Z}^2_O \to \mathbf{Z}^2_{4x+2} = \{[r,s] \in \mathbf{Z}^2\colon r - s \in 4\mathbf{Z} + 2\}$. Both g_E and g_O are bijections onto $\mathbf{Z}^2_{4x}$ and $\mathbf{Z}^2_{4x+2}$, respectively. Note that $\mathbf{Z}^2_E = \mathbf{Z}^2_{4x} \cup \mathbf{Z}^2_{4x+2}$ and that $\mathbf{Z}^2_{4x} \cap \mathbf{Z}^2_{4x+2} = \emptyset$. Then (2.20) is equivalent to

$$rs = 2m, \quad r, s \in 2\mathbf{Z}, \quad r - s \in 4\mathbf{Z}, \quad m \in 2\mathbf{Z}. \tag{2.23}$$

The solutions of (2.23) can be parameterized by s, and are given by

$$r = \frac{2m}{s}, \quad s \in 2\mathbf{Z}^*, \quad s|m, \quad s + \frac{2m}{s} \in 4\mathbf{Z}. \tag{2.24}$$

Returning to the variables k via g_E^{-1} and f^{-1}, (2.24) yields (2.15), (2.16). A similar parameterization for the solutions of (2.21) in the r, s variables yields (2.15), (2.17) via g_O^{-1}, f^{-1}. Note that in (2.16), (2.17) the number $\phi(s) = \frac{1}{2}(s \pm \frac{2m}{s}) \in \mathbf{Z}$, and we can parameterize Λ_m by $s \in 2\mathbf{Z}^*$, $s|m$ with θ even or odd depending on the parity of $\phi(s)$. □

Remark 2.3. In the special case of $m = 0$ (i.e., $s|m, \forall s \in 2\mathbf{Z}$ in (2.16), (2.17)), the solutions of (2.14) reduce to $k_1 = k_3$, $k_2 = k_4$ or $k_1 = k_4$, $k_2 = k_3$ with $k_3, k_4 \in \mathbf{Z}$.

To use the above proposition in the resonance equation (2.12) we consider two cases: $\frac{\Omega}{\delta}$ rational and irrational, respectively. In the rational case we let $\frac{\Omega}{\delta} = \frac{p}{q}$ in prime terms, with $p, q \in \mathbf{Z}^*$ (with $|p| \gg |q|$ if $|\frac{\Omega}{\delta}|$ is large). Comparing with (2.12), the resonant terms have indices $n = kq$, and $[k_1, k_2, k_3, k_4] \in \Lambda_{kp}$, where $k \in \mathbf{Z}$ for p even, and $k \in 2\mathbf{Z}$ for p odd. Then,

$$\bar{H}_4 = \gamma \sum_{k \in \mathbf{Z}} \hat{f}_{kp}(kq) e^{ikq\phi} \times \sum_{[k_1,k_2,k_3,k_4] \in \Lambda_{kp}} a_{k_1} a_{k_2} a^*_{k_3} a^*_{k_4}, \tag{2.25}$$

with Λ_{kp} as in Proposition 2.2, and can immediately write an analogous expression for χ_1. Note that there are no small divisors. By Remark 2.3, the resonant quartic Hamiltonian in (2.25) can be further decomposed as $\bar{H}_4 = \bar{H}_{4,I} + \bar{H}_{4,NI}$, with

$$\bar{H}_{4,I} = \gamma \hat{f}_0(0) \sum_{k_1,k_2 \in \mathbf{Z}} |a_{k_1}|^2 |a_{k_2}|^2, \tag{2.26}$$

$$\bar{H}_{4,NI} = \gamma \sum_{k \in \mathbf{Z}^*} \hat{f}_{kp}(kq) e^{ikq\phi} \times \sum_{[k_1,k_2,k_3,k_4] \in \Lambda_{kp}} a_{k_1} a_{k_2} a^*_{k_3} a^*_{k_4}. \tag{2.27}$$

The part $\bar{H}_{4,I}$ is integrable in the sense that it depends only on the "actions" $|a_i|^2$, $i \in \mathbf{Z}$. The part $\bar{H}_{4,NI}$ contains the remaining terms of $\bar{H}_4$, and from Proposition 2.2 we see that $\bar{H}_{4,NI}$ cannot include any monomial $|a_i|^2|a_j|^2$, i.e., $\bar{H}_{4,NI}$ is the "nonintegrable" part of $\bar{H}_4$.

In the case where $\frac{\Omega}{\delta}$ is irrational, the resonance condition (2.12) is satisfied only for $n = m = 0$. Then $\bar{H}_4 = \bar{H}_{4,I}$. However, Proposition 2.2 and the coefficients $\hat{f}_m(n)$ computed in two examples below show that there can be an infinite number of quintets $n, k_1, \ldots, k_4$ coming arbitrarily close to satisfying the resonance condition. We thus have a small divisor problem and we would need additional assumptions on $\frac{\Omega}{\delta}$ to carry out the transformation. We will here concentrate on the case where $\frac{\Omega}{\delta}$ is rational.

We now examine the coefficients $\hat{f}_m(n)$ in $\bar{H}_4$. First, we note that the integrable part $\bar{H}_{4,I}$ never vanishes since $\hat{f}_0(0) = 1$, regardless of the choice of $\tilde{d}(t)$. To indicate the structure of the coefficients $\hat{f}_{kp}(kq)$ for $k \neq 0$ we consider some examples. The first is the piecewise constant dispersion management function

$$d(t) = \begin{cases} \delta + \tilde{A}, & \text{if } t \in [0, \tau), \\ \delta + \tilde{B}, & \text{if } t \in [\tau, T), \end{cases} \tag{2.28}$$

with $\tilde{A}\tau + \tilde{B}(T - \tau) = 0$. To simplify the result we look at the special case where $\tau = \frac{T}{2}$, $\tilde{A} = h$ and $\tilde{B} = -h$, where we have

$$\hat{f}_{kp}(kq) = -\frac{i}{\pi}\left[e^{-i(h\delta^{-1}+1)kq\pi} - 1\right] \times \frac{h\delta^{-1}}{kq(1 - h^2\delta^{-2})}, \quad k \in \mathbf{Z}^*. \tag{2.29}$$

(The expression for $h\delta^{-1} = \pm 1$ are omitted.) We see that if $h\delta^{-1}$ is an odd integer then $\hat{f}_{kp}(kq) = 0$, for all $k \in \mathbf{Z}^*$. Similarly, the coefficients vanish if p is odd and $h\delta^{-1}$ is an even integer (since Λ_m is empty for m odd by Proposition 2.2). Also, as $h \to +\infty$, the $\hat{f}_{kp}(kq)$ decay as $\delta|kh|^{-1}$. Another example is the real analytic dispersion management function

$$d(t) = \delta + h \sin \Omega t, \tag{2.30}$$

where we have

$$\left|\hat{f}_{kp}(kq)\right| = 2\left|\mathcal{J}_{|kq|}\left(\left|h\delta^{-1}\right||kq|\right)\right|, \quad k \in \mathbf{Z}^*, \tag{2.31}$$

with $\mathcal{J}_N$ the Bessel function of order $N \in \mathbf{Z}$. As $|h| \to \infty$, the $|\hat{f}_{kp}(kq)|$ therefore decay as $|hkq|^{-1/2}$. By (2.9), the decay of the $\hat{f}_{kp}(kq)$ in the amplitude $|h|$ of $\tilde{d}(t)$ can be seen by a stationary phase argument, and we observe that the decay is faster for the more singular (integrable) dispersion management functions.

3. Invariant subspaces and some instabilities

We will now consider the quartic normal form Hamiltonian $H_2 + \bar{H}_4$ and find classes of periodic orbits of the corresponding system. In some cases we also obtain stability information. Our constructions concern the general case where the nonintegrable part $H_{4,NI}$ may not vanish.

The orbits we find will belong to finite-dimensional subspaces M that are invariant under the flow of $H_2 + \bar{H}_4$. The underlying idea is the following: given M, we identify all terms $e^{in\phi}a_{k_1}a_{k_2}a^*_{k_3}a^*_{k_4}$ whose absence from $\bar{H}_4$ would imply the invariance of M under the flow of $H_2 + \bar{H}_4$. If the offending quartic terms are nonresonant we then have that M is invariant under the flow of $H_2 + \bar{H}_4$. Note that the argument involves checking a small number of resonance conditions since the invariance of M follows from the elimination of only a subset of the nonresonant quartic terms. We will therefore need only a small part of the information contained in Proposition 2.2 (and we can also use "partial" normal forms to extend the existence of invariant subspaces for higher orders). On the other hand, the quartic normal form $\bar{H}_4$ of (2.25) contains additional stability information on the orbits contained in the invariant subspaces.

To give an example, fix N integers $0 \leqslant \lambda_1 < \cdots < \lambda_N$, let $\mathcal{I} = \{\pm\lambda_1, \ldots, \pm\lambda_N\}$, and denote the (complex) span of the modes a_i, $i \in \mathcal{I}$ by $M_{\mathcal{I}}$. Also assume that $\frac{\Omega}{\delta} = \frac{p}{q} \in \mathbf{Q}$.

Proposition 3.1. *Consider a subspace $M_{\mathcal{I}}$ as above, and suppose that $8(\lambda_N)^2 < |p|$. Then $M_{\mathcal{I}}$ is invariant under the Hamiltonian flow of $H_2 + \bar{H}_4$. Moreover, the Hamiltonian flow of $H_2 + \bar{H}_4$ on $M_{\mathcal{I}}$ is integrable.*

Proof. To show that $M_{\mathcal{I}}$ is invariant it is enough to show that all terms $e^{in\phi}a_{k_1}a_{k_2}a^*_{k_3}a^*_{k}$ with $n \in \mathbf{Z}$, k_1, k_2, $k_3 \in \mathcal{I}$, and $k \in \mathbf{Z} \setminus \{\mathcal{I}\}$ of $\bar{H}_4$ are nonresonant. Suppose that some term of this type is resonant. Then we have some triple $\lambda = [k_1, k_2, k_3] \in \mathcal{I}^3$, $n \in \mathbf{Z}$ and $k \in \mathbf{Z} \setminus \{\mathcal{I}\}$ satisfying the resonance condition (2.12). Hence we must have

$$k = k(\lambda) = k_1 + k_2 - k_3, \tag{3.1}$$

which implies

$$\left|m(\lambda)\right| = |\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k(\lambda)}| \leqslant 8(\lambda_N)^2. \tag{3.2}$$

According to the resonance condition (2.12), $m(\lambda)$ and n must also satisfy

$$n = \frac{\delta}{\Omega}m(\lambda). \tag{3.3}$$

The solution $n = m(\lambda) = 0$ is not acceptable, for applying Proposition 2.2 to the case $m = 0$ we see that $m(\lambda) = 0$ can be satisfied only with $k \in \mathcal{I}$. Thus we must seek solutions of (3.3) with $n \in \mathbf{Z}^*$. By Proposition 2.2 we also need $n = rq$, with $r \in \mathbf{Z}^*$. On the other hand, (3.2), (3.3) and the hypothesis on $\frac{\Omega}{\delta}$ imply that

$$|r| < \frac{8(\lambda_N)^2}{|p|} < 1. \tag{3.4}$$

Hence (3.3) cannot be satisfied, and the term in question cannot be resonant, a contradiction. To see the integrability of the Hamiltonian flow of $H_2 + \bar{H}_4$ on $M_{\mathcal{I}}$ we note that if $[k_1, k_2, k_3, k_4] \in \mathcal{I}^4$ then

$$|m| = |\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_{k_4}| \leqslant 4(\lambda_N)^2. \tag{3.5}$$

By the hypothesis on $\frac{p}{q}$ we then have that

$$|m| \leqslant 4(\lambda_N)^2 < \frac{1}{2}|p|, \tag{3.6}$$

hence $m = kp$, $k \in \mathbf{Z}$ is only satisfied for $k = m = 0$. Using Proposition 2.2 for $m = 0$ we see that $\bar{H}_4$ restricted to $M_{\mathcal{I}}$ can only consist of terms from $\bar{H}_{4,I}$. Clearly, $\bar{H}_{4,I}$ does not vanish on $M_{\mathcal{I}}$. $\square$

We can thus fix $\frac{\Omega}{\delta} = \frac{p}{q}$ and find invariant subspaces of complex dimension up to $[2\sqrt{|p|/8} + 1]$ that are foliated by invariant tori (note that by (2.26) the tori are foliated by periodic orbits). Alternatively, we can seek invariant subspaces and tori of arbitrary (finite) dimension by increasing $\frac{\Omega}{\delta}$. Since $\frac{\Omega}{\delta}$ is assumed large we also have an immediate application to numerical studies using spectral methods:

Corollary 3.2. *The trajectories of any Galerkin projection of the Hamiltonian system of $H_2 + \bar{H}_4$ to modes a_k, $k \in \mathbf{Z}$ with $8k^2 < |p|$ are exact solutions of the full (i.e., infinite-dimensional) Hamiltonian system of $H_2 + \bar{H}_4$. Moreover, all such Galerkin systems are integrable.*

Expression (2.25) for the quartic normal form Hamiltonian also gives us information on the stability of some of the orbits described in Proposition 3.1. Consider invariant subspaces $M_{\mathcal{I}}$ with $\mathcal{I} = \{\pm k_0\}$, and $k_0 \neq 0$, $8|k_0|^2 < |p|$ (what follows also applies with minor modifications to the case $k_0 = 0$). By Proposition 3.1 the $M_{\mathcal{I}}$ are foliated by invariant 2-tori. Also, $M_{\mathcal{I}}$ has an invariant plane foliated by periodic solutions for which $a_k(t) \equiv 0$, if $k \neq k_0$, and $a_{k_0}(t) = a_0 e^{-i\Omega_{k_0} t}$, with $\Omega_{k_0} = \delta\omega_{k_0} + 2\gamma \hat{f}_0(0)|a_0|^2$. We refer to these orbits as Stokes waves, and denote the corresponding invariant circles by μ_{k_0}. Note that a variation on the arguments of Proposition 3.1 shows that these waves exist for all $k_0 \in \mathbf{Z}$, without any restrictions on p, q. The Floquet maps for these periodic orbits can have unstable directions. We can see this by the following argument.

Consider all quartic monomials of $\bar{H}_{4,NI}$ of the form $a_{k_1} a_{k_2} a^*_{k_0} a^*_{k_0}$ and $a_{k_0} a_{k_0} a^*_{k_1} a^*_{k_2}$ with $k_1, k_2 \neq k_0$. These monomials will be referred to as Benjamin–Feir terms (for k_0), and we denote the sum of all Benjamin–Feir terms in $\bar{H}_4$ by $\bar{H}_{4,\mathrm{BF}}(\mu_{k_0})$. Each Benjamin–Feir term corresponds to pair of indices $\{k_1, k_2\} \subset \mathbf{Z}$, and index pairs corresponding to different Benjamin–Feir terms are disjoint. The set of all Benjamin–Feir pairs is listed in Proposition 3.3, and we see that the union of all indices belonging to Benjamin–Feir pairs is a proper subset of $\mathbf{Z}$. Now consider the variational equation along the orbit μ_{k_0}. The Fourier coefficients of the perturbation will be denoted by b_k. Introducing the amplitude variables

$$B_k(t) = e^{i\Omega_k t} b_k(t), \quad \text{with}$$
$$\Omega_k = \delta\omega_k + 2\gamma \hat{f}_0(0)\big|a_0(0)\big|^2, \tag{3.7}$$

we see that the variational equation becomes autonomous and block-diagonal. In particular, if $\{k_1, k_2\}$ is a Benjamin–Feir index pair we have

$$\dot{B}_{k_1} = \rho(k_1, k_2) B^*_{k_2}, \qquad \dot{B}_{k_2} = \rho(k_1, k_2) B^*_{k_1}, \tag{3.8}$$

where $\rho(k_1, k_2)$ is proportional to the coefficient of the corresponding Benjamin–Feir term in $\bar{H}_4$ (and to $|a_0|^2$, see Remark 3.4). From (3.8) we immediately see that each Benjamin–Feir block yields two stable and two unstable directions, corresponding to double eigenvalues $\pm|\rho(k_1, k_2)|$. If $k \neq k_0$ does not belong to any Benjamin–Feir index pair, then $\dot{B}_k = 0$, moreover $\dot{B}_{k_0} = -2i\gamma \hat{f}_0(0)(|a_0|^2 B_{k_0} + a_0^2 B^*)$, i.e., we have a double zero eigenvalue and B_{k_0} is affine in time.

To see $\bar{H}_{4,\mathrm{BF}}(\mu_{k_0})$ let p be a positive integer, and $p_1, \ldots, p_n$ the prime factors of p that are different from 2 (i.e., when p is even). Also, define the non-negative integers $\alpha(2)$, $\alpha(p_1), \ldots, \alpha(p_n)$ by $p = 2^{\alpha(2)} p_1^{\alpha(p_1)} \cdots p_n^{\alpha(p_n)}$. We then have:

Proposition 3.3. *Let $\frac{\Omega}{\delta} = \frac{p}{q}$, with $p > 0$. Then, the sum of Benjamin–Feir quartic terms for the invariant* *circle μ_{k_0} above is*

$$\bar{H}_{4,\mathrm{BF}}(\mu_{k_0}) = 2\gamma \sum_{\nu\in\mathbf{Z}^*} \hat{f}_{m(\nu)}(n(\nu)) e^{in(\nu)\phi} \times a_{k_0+\nu\lambda_p r_p} a_{k_0-\nu\lambda_p r_p} a^*_{k_0} a^*_{k_0} + \text{c.c.}, \tag{3.9}$$

with

$$r_p = p_1\cdots p_n, \qquad \lambda_p = c_p p_1^{\beta(p_1)}\cdots p_n^{\beta(p_n)}, \tag{3.10}$$

$$m(\nu) = M_p \nu^2 \lambda_p^2 r_p^2, \qquad n(\nu) = M_p \nu^2 \lambda_p^2 r_p^2 \frac{q}{p}. \tag{3.11}$$

The $\beta(p_i)$ in (3.10) *are*: (i) $\beta(p_i) = 0$, *if* $\alpha(p_i) \leqslant 2$; (ii) $\beta(p_i)$ *is the smallest positive integer satisfying* $\beta(p_i) \geqslant \frac{1}{2}\alpha(p_i) - 1$, *if* $\alpha(p_i) > 2$. *The* c_p, M_p *in* (3.11) *are*: (i) $c_p = 1$, $M_p = 2$ *if* $\alpha(2) \leqslant 1$; *and* (ii) $c_p = 2^{\beta(2)+1}$, *with* $\beta(2)$ *the smallest positive integer satisfying* $\beta(2) \geqslant \frac{1}{2}\alpha(2) - \frac{3}{2}$, *and* $M_p = 8$, *if* $\alpha(2) > 1$.

Proof. Given $k_0 \in \mathbf{Z}^*$ we want a list of all resonant quartic monomials of the forms $a_{k_1}a_{k_2}a^*_{k_3}a^*_{k_4}$ with $k_3 = k_4 = k_0$, and $a_{k_1}a_{k_2}a^*_{k_3}a^*_{k_4}$ with $k_1 = k_2 = k_0$. It will be sufficient to list the monomials of the first type only since the ones of the second type are their complex conjugates. Using (2.25), we fix $m = kp$ and look for the Benjamin–Feir terms (of the first type) in each level set Λ_{kp}, $k \in \mathbf{Z}^*$. By (2.15), the requirement $k_3 = k_4 = k_0$ with $m = kp$ leads to the equation

$$2kp = s^2, \quad s|kp, \quad k, s \in \mathbf{Z}^*, \tag{3.12}$$

(and to $2\theta = k_0$, i.e., consistently with (2.16)). If k, s satisfy (3.12), then by (2.15) we have a resonant monomial $a_{k_1}a_{k_2}a^*_{k_3}a^*_{k_4}$ with

$$k_1 = k_0 + \frac{kp}{s}, \qquad k_2 = k_0 - \frac{kp}{s},$$
$$k_3 = k_0, \qquad k_4 = k_0. \tag{3.13}$$

It thus remains to solve (3.12). Noting that $2kp = s^2$ implies $s|kp$, the first equation in (3.12) can be solved by equating the prime exponents of the two sides, obtaining linear relations between the unknown prime exponents of k, s and the prime exponents of p (we omit the details). The solutions of (3.12) can be parameterized by $\nu \in \mathbf{Z}^*$: $k(\nu)p = m(\nu)$ is shown in (3.11), while $s(\nu) = m(\nu)(\nu\lambda_p r_p)^{-1}$. The statement follows by combining the solution of (3.12) with (2.25) and (3.13), and adding complex conjugates. □

Remark 3.4. From Proposition 3.3 and the proceeding discussion we have that, given k_0, its Benjamin–Feir pairs are $\{k_1, k_2\} = \{k_1(\nu), k_2(\nu)\} = \{k_0 + \nu\lambda_p r_p, k_0 - \nu\lambda_p r_p\}$ with $\nu \in \mathbf{Z}$ and λ_p, r_p as in (3.11). The corresponding coefficients $\rho(k_1, k_2)$ in (3.8) are given by $\rho(k_1, k_2) = \rho(k_1(\nu), k_2(\nu)) = 2\hat{f}_{m(\nu)}(n(\nu))|a_0|^2$, with $m(\nu)$, $n(\nu)$ as in (3.11).

Remark 3.5. The passage from the variational equation to an autonomous system through (3.7) is exact due to the special structure the nonlinearity of the NLS. A similar simplification in the linear stability analysis of Stokes waves also occurs in some autonomous dispersive systems. For instance, consider a system $\dot{a}_k = -i\frac{\partial H}{\partial a^*_k}$, $k \in \mathbf{Z}\setminus\{0\}$, with the quadratic part of the Hamiltonian H given by $\sum_{k\in\mathbf{Z}\setminus\{0\}} \omega(k)|a_k|^2$. Also assume that the dispersion $\omega(k)$, $k \in \mathbf{R}$ is even, satisfies $\omega(0) = 0$, and is increasing and concave for $k > 0$. Then we can see that cubic resonances and quartic resonant monomials of the forms $a_{k_1}a_{k_2}a_{k_3}a^*_{k_4}$ and $a^*_{k_1}a^*_{k_2}a^*_{k_3}a_{k_4}$ are absent, Stokes waves exist, and the variational equation around them has the block diagonal structure above. An example is the system describing 2D gravity water waves of finite depth (see [11]).

Also note that (2.1) admits exact Stokes wave solutions (see [3]). Using the variables a_k of (2.4), these are given by $a_k(t) \equiv 0$ if $k \neq k_0$, and $a_{k_0}(t) = a_0 e^{-i\Omega_0 t}$, with $\Omega_0 = \delta\omega_{k_0} + 2\gamma|a_0|^2$ (for arbitrary $k_0 \in \mathbf{Z}$). The solutions of the normal form equation have therefore the same form as the exact Stokes waves solutions of the full system. Also, the variational equation around the exact orbits is block diagonal in the Fourier basis (with real 2×2 time-periodic blocks), and unstable directions can only come in pairs of wavenumbers $k_0 \pm k$. The two approaches therefore describe the same qualitative picture of Stokes waves and the Benjamin–Feir instability, and will be compared in future work. Clearly, the approach starting from the exact solutions has several advantages (e.g., it is valid for arbitrary amplitude, and $d(t)$ quasi-periodic) and leads to several interesting problems on the modulational instability of Stokes waves. On the other hand, the normal form approach here is closer to the original argument for the Benjamin–Feir instability and, when valid, poses the problem of modulational instabilities in a more general setting.

4. Discussion of the asymptotics

The above results are formal and their validity must be supported by theoretical considerations or numerical simulations. A first step is Theorem 4.2 where we estimate the distance between solutions of the averaged and full systems. The estimate gives a satisfactory explanation of the meaning of Proposition 3.1 and its corollaries for the original equation. Note that the proof follows the arguments of the periodic averaging theorem of [8, Chapter 3], extended to the infinite-dimensional setting, and we here only state the result (we consider the case where $\frac{\Omega}{\delta}$ is rational only). The theory connecting the Benjamin–Feir instability calculations to the solutions of (2.1) is not complete, nevertheless we see that the heuristic predictions of Proposition 3.3 are consistent with the preliminary numerical results we present below.

Recall that in Section 2, Eq. (2.1) for $u(t)$ was transformed to (2.4) for $a(t) = V_t u(t)$, i.e., $a(t)$ is the Fourier series of the $a_k(t)$, and V_t is defined by (2.4). Consider the (amplitude) variables $A(t) = U_{-t} a(t)$, where U_t is the evolution operator solving $\dot{u} = i\delta u_{xx}$. Then $A(t)$ satisfies

$$\dot{A} = \gamma U_{-t} V_{-t} G(U_t V_t A), \qquad A(0) = A_0, \quad \text{with} \quad G(u) = i|u|^2 u. \tag{4.1}$$

(We have used the fact the U_t, V_t commute with their generators. Also note that $A_0 = u_0$.) We rewrite (4.1) as

$$\dot{x} = \gamma f(x,t), \qquad x(0) = x_0. \tag{4.2}$$

In the case where $\frac{\Omega}{\delta} = \frac{p}{q} \in \mathbf{Q}$, the solutions of (2.1) with $\gamma = 0$ are $\tilde{T}$-periodic, with $\tilde{T} = 2\pi q|\delta|^{-1}$, and we see that $f(x,t)$ is also $\tilde{T}$-periodic. We then consider the averaged equation

$$\dot{y} = \gamma g(y), \qquad y(0) = y_0 = x_0, \quad \text{with} \quad g(y) = \frac{1}{\tilde{T}} \int_0^{\tilde{T}} f(y,t)\,dt. \tag{4.3}$$

Eq. (4.3) is precisely Hamilton's equation for the quartic normal form Hamiltonian in amplitude variables: the evolution equation for $U_{-t} y(t)$ is Hamilton's equation for $H_0 + \bar{H}_4(t)$, where $\bar{H}_4(t)$ is the quartic normal form in (2.25), with $\phi = \Omega t$ (i.e., with the time dependence made explicit). To formulate the statement describing the relation between solutions of (4.2) and (4.3) we briefly mention the relevant facts from the local existence theory for the two equations. Considered in the Sobolev spaces H^s, $s \in \mathbf{R}$, of complex valued 2π-periodic functions with the norms $\| \|_s$ given by

$$\|u\|_s^2 = \sum_{k \in \mathbf{Z}} \bigl(1 + |k|^2\bigr)^s |u_k|^2 \tag{4.4}$$

(with u_k the Fourier coefficients of u). Note that $G(u) = i|u|^2 u$ satisfies the Lipschitz condition

$$\bigl\|G(u) - G(v)\bigr\|_s \leqslant L_G\bigl(\|u\|_s, \|v\|_s\bigr)\|u - v\|_s, \quad s > \frac{1}{2}, \tag{4.5}$$

with $L_G(\|u\|_s, \|v\|_s) = C_s^2(\|u\|_s^2 + \|u\|_s\|v\|_s + \|v\|_s^2)$, and C_s a constant satisfying $\|uv\|_s \leqslant C_s\|u\|_s\|v\|_s$ for $s > \frac{1}{2}$. A standard fixed point argument then implies the following:

Proposition 4.1. *Let $s > \frac{1}{2}$, $\alpha > 0$. Then there exists a time $t_1 \geqslant C(\alpha, \|x_0\|_s)|\gamma|^{-1}$ for which the initial value problems* (4.2), (4.3) *have unique solutions $x(t)$, $y(t) \in R_s(t_1, x_0, \alpha)$, where $R_s(t_1, x_0, \alpha) = \{u(t) \in C^0([0, t_1], H^s)$: $\|u(t) - x_0\|_s \leqslant \alpha\}$.*

We note that for $\|x_0\|_s$, $\alpha \sim O(1)$ we have $C(\alpha, \|x_0\|_s) \sim O(1)$, i.e., the local existence time t_1 is large for $|\gamma|$ small. Proposition 4.1 implies that $\|x(t) - y(t)\|_s \leqslant O(\alpha) \sim O(1)$ for $0 \leqslant t \leqslant t_1 \sim |\gamma|^{-1}$. The averaging theorem below leads to a significant improvement.

Theorem 4.2. *Let $s > \frac{1}{2}$, and assume that $\|y_0\|_s \leqslant C_0 \sim O(1)$ and that $\tilde{T}|\gamma|$ is sufficiently small. Then there exist constants $C_1 \leqslant O(1)$, $C_2 \geqslant O(1)$ for which*

$$\bigl\|x(t) - y(t)\bigr\|_s \leqslant C_1 \tilde{T}|\gamma|, \quad \forall t \in \bigl[0, C_2|\gamma|^{-1}\bigr]. \tag{4.6}$$

The constants C_1, C_2 depend on s, $\tilde{T}$, and C_0.

The periodic averaging theorem captures the main idea behind the formal calculation in that the error of the averaged equation is proportional to $|\gamma|$ for $|\gamma|$ sufficiently small. Clearly $\tilde{T}$ and hence the error also involves the size of q, i.e., the "complexity" of the rational $\frac{\Omega}{\delta}$. As an example, consider (2.1) with $\delta = \gamma = \Omega^{-1} = \epsilon > 0$ and replace u in (2.1) by ϵu. Also, assume $q < \epsilon^{-1}$, i.e., $\tilde{T} \sim O(1)$, and integrate (2.4) and the corresponding averaged system with an initial condition $a(0) = \sum_{k \in \mathbf{Z}} a_k(0) e^{-ikx}$ satisfying $\|a(0)\|_s \sim O(1)$ for some $s > \frac{1}{2}$. Then, for ϵ sufficiently small the H^s distance between the averaged and full systems will be at most of $O(\epsilon)$ up to a time of $O(\epsilon^{-3})$.

The averaging theorem also relates the special orbits of Proposition 3.1 to solutions of the original system. Specifically, by the averaging theorem, any initial condition on any set $\mathcal{S}$ that is invariant under the flow of the averaged equation, and whose points have H^s norm bounded by an $O(1)$ constant, will stay $O(|\gamma|)$ close to $\mathcal{S}$ in H^s under the evolution of the full system over a time of $O(|\gamma|^{-1})$ ($s > \frac{1}{2}$). By Proposition 3.1 these sets can be circles, tori, subspaces, etc. Note that this is not a stability statement since it does not guarantee that an initial condition near $\mathcal{S}$ will stay near $\mathcal{S}$. On the other hand if we consider a Galerkin projection of the averaged equation and take $\frac{\Omega}{\delta} = \frac{p}{q}$ with p sufficiently large, $\mathcal{S}$ can have small or zero codimension (see Corollary 3.2).

The linear stability analysis of the Stokes waves solutions of the averaged equation is not theoretically expected give a picture of the dynamics near these solutions, since in addition to stable and unstable directions we generally have an infinite number of center directions corresponding to the modes that are not in Benjamin–Feir pairs. The Poincare map along the Stokes orbits is then not guaranteed to be locally conjugate (i.e., equivalent up to continuous or smoother changes of coordinates) to its linearization and the dynamics is very sensitive to the nonlinear terms (and possibly even numerical effects).

Heuristically, we may also expect that the existence of unstable directions in the Floquet map implies the growth of the corresponding Fourier modes since the unstable manifold is tangent to the unstable subspace of the Poincare map (i.e., assuming that the unstable manifold remains near the unstable subspaces for some distance away from the origin). To see what happens we have integrated numerically the full system (2.1) with initial conditions $a_{k_0}(0) = O(1)$ for some $k = k_0$, and $a_k(0)$, $k \neq k_0$, of smaller amplitude, e.g., 10^{-1}–10^{-4}. We also let $d(t) = 1 + \sin \Omega t$ (i.e., $\delta = h = 1$ in (2.30)), and consider γ between 10^{-2} and 10^{-4}. We discretize in space using 64 Fourier modes and integrate using a 4th order Runge–Kutta scheme. Varying k_0 and $\Omega = \frac{p}{q}$ we look for instabilities along the different directions suggested by Proposition 3.3. We monitor $|a_k(t)|$ and the "variation" $\Delta_I(k) = \max_{t \in I} |a_k(t)| - \min_{t \in I} |a_k(t)|$ of the moduli over sufficiently large (see below) time intervals I. The simplest formulas occur when p is prime; by (3.9)–(3.11) we should then have growth of the Fourier modes with indices $k_0 + \nu p$, with $\nu \in \mathbf{Z}$.

The first observation is the moduli of the expected unstable Fourier modes indeed vary significantly more that the moduli of the other modes. We also observe however that variation of the modulus of all modes is small compared to $|a_{k_0}(0)|$. In fact, choosing the integration interval I sufficiently large, e.g., a few multiples of γ^{-1}, we see that any increase of the moduli of the unstable modes saturates, and the unstable modes are eventually seen to perform small amplitude, slow oscillations with period comparable to γ^{-1}. For instance, in Figs. 1, 2 we choose $k_0 = 0$, $a_{k_0} = 1$ and consider $\Omega = \frac{3}{2}$, and $\frac{7}{3}$, respectively. In both cases we have $\gamma = 0.001$ and $a_k(0) = 0.0025$, for $|k| \leqslant 16$ ($k \neq 0$), and $a_k(0) = 0$ otherwise, and we show $\Delta_I(k)$ with $I = [0, 1300]$. In both cases we see that $\Delta_I(k)$ is largest for the Benjamin–Feir modes $k_0 + 3\nu$ and $k_0 + 7\nu$, respectively. In Figs. 3, 4 we see examples with $k_0 = 2$, $a_{k_0}(0) = 1$, and $\Omega = \frac{7}{3}$, and $\frac{5}{3}$, respectively. In Fig. 3 we have $a_k(0) = 0.0025$, for $-11 \leqslant k \leqslant 16$ ($k \neq 0$), and $a_k(0) = 0$ otherwise, while in Fig. 4 we have $a_k(0) = 0.0025i$, for $-9 \leqslant k \leqslant 3$, $a_k(0) = -0.0025i$, $3 \leqslant k \leqslant 9$, and $a_k = 0$ otherwise. The integration intervals are $[0, 750]$ and $[0, 3800]$, respectively. In both cases $\Delta_I(k)$ is largest for Benjamin–Feir modes $k_0 + 7\nu$ and $k_0 + 5\nu$, $\nu \in \mathbf{Z}$, respectively, although the picture is more complicated, e.g., in Fig. 3 the mode $k = 16$ exhibits very small variation.

The results above are typical for the perturbations we considered, and we can say that although the linear stability analysis allows us to make predictions, what we see numerically must be explained by nonlinear considerations. Perhaps the most important feature emerging from the simulations is that any significant

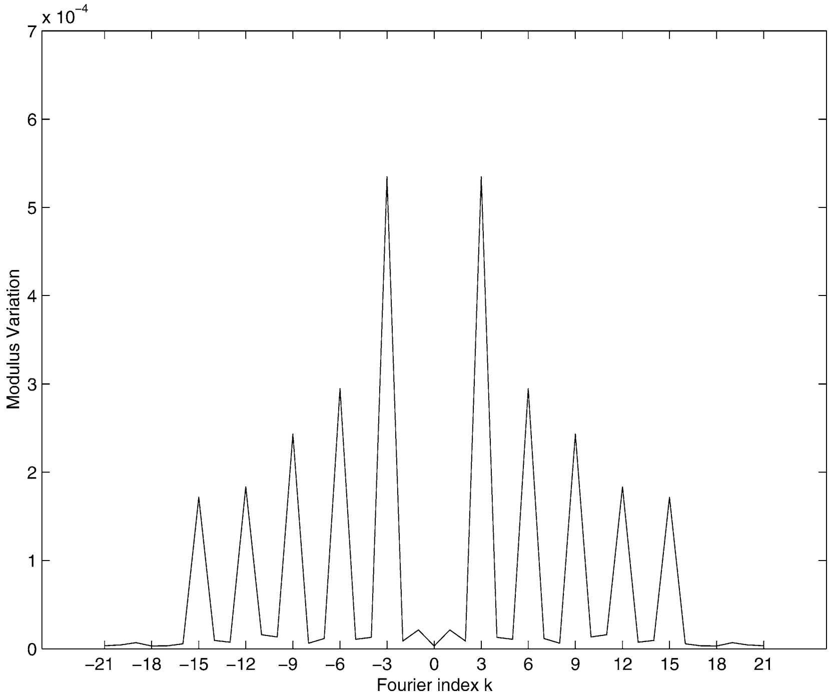

Fig. 1. $\Delta_I(k)$ for initial condition $a_{k_0} = 1$ plus small perturbation (see text): $k_0 = 0$, $\frac{\Omega}{\delta} = \frac{3}{2}$ (points in graph are connected).

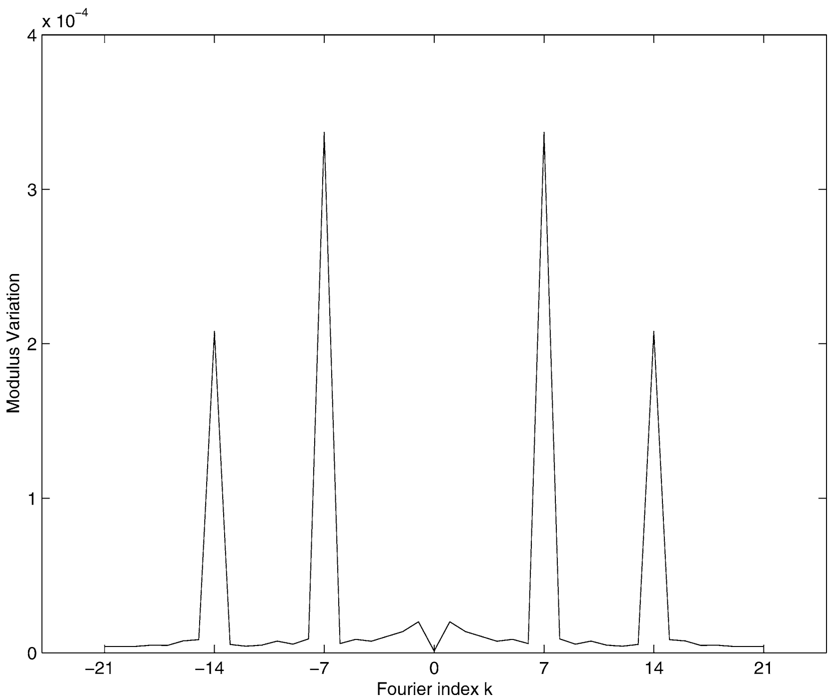

Fig. 2. $\Delta_I(k)$ for initial condition $a_{k_0} = 1$ plus small perturbation (see text): $k_0 = 0$, $\frac{\Omega}{\delta} = \frac{7}{3}$ (points in graph are connected).

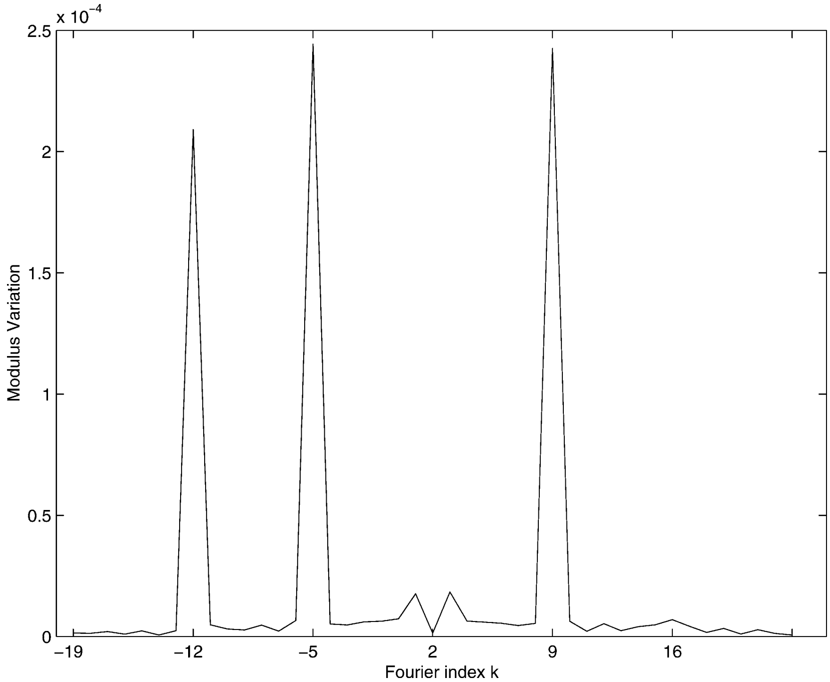

Fig. 3. $\Delta_I(k)$ for initial condition $a_{k_0} = 1$ plus small perturbation (see text): $k_0 = 2$, $\frac{\Omega}{\delta} = \frac{7}{3}$ (points in graph are connected).

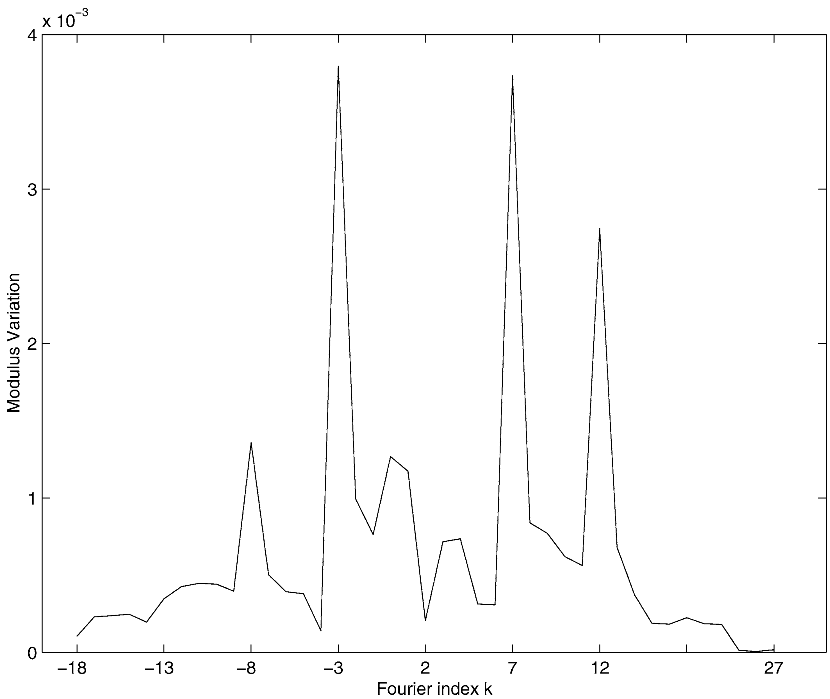

Fig. 4. $\Delta_I(k)$ for initial condition $a_{k_0} = 1$ plus small perturbation (see text): $k_0 = 2$, $\frac{\Omega}{\delta} = \frac{5}{3}$ (points in graph are connected).

excursion away form the Stokes solutions must be slow. Note that a small bound on $|b_{k_0}(t)|$ together the conservation of the L_2 norm would imply that perturbations of the Stokes waves must remain small, i.e., nonlinear stability is governed by the dynamics of the mode with index k_0. The dynamics of b_{k_0} is not however readily seen from the equations.

Further comparisons between theory and numerical experiments will be reported elsewhere. It could be useful to study numerically the averaged equation, and to make a closer comparison between the theory and the numerics. By changing the forcing frequency we also obtain systems that are interesting but more tractable. For instance, given a Galerkin projection of the averaged system we can choose $\frac{\Omega}{\delta}$ for which the Poincare map around the Stokes waves have two-dimensional stable and unstable manifolds. Another possibility would be to examine cases where we have a local center manifold of low dimension, e.g., for $\frac{\Omega}{\delta} = \frac{p}{q}$ with $p = 1$ the Poincare map around the Stokes waves has two center directions.

5. Summary

In summary we have used a detailed analysis of the lowest order resonant interactions to give an explicit expression for the averaged (or lowest order normal form) equation of a parametrically forced nonlinear Schrödinger equation. We see explicitly how the forcing frequency can be tuned to produce invariant tori foliated by periodic orbits. We have also used the structure of the equations and the analysis of the Diophantine equations to give a clear picture of the linearization around Stokes wave solutions. Analogous solutions are very common in nonlinear dispersive systems, and we indicate a class of physical systems, most notably gravity water waves, for which the linearization around Stokes waves has a similar structure. Solutions of the averaged system can be compared to the full system, at least for a time of $O(|\gamma|^{-1})$, and this allows us to connect the existence of special solutions of the averaged system to the dynamics of the full system. Numerical simulations with Stokes wave initial conditions show that the linear stability analysis predicts correctly the side-band modes that grow the most. However it also appears that the growth of the side-band modes saturates at small amplitudes, i.e., the plane wave structure is deformed only slightly, and any significant deformation (if any) should be a long term phenomenon. The present theory does not explain these nonlinear processes around the Stokes waves but points to some possible ways of analyzing them. We hope that some of these ideas will be fruitful in further work.

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