Physics Letters A 383 (2019) 164–169

Contents lists available at ScienceDirect

Physics Letters A

www.elsevier.com/locate/pla

Classical breathers and quantum coherent states in discrete NLS systems

Ricardo Martínez-Galicia [a], Panayotis Panayotaros [b,*]

[a] Depto. Matemáticas, Facultad de Química – U.N.A.M., 04510 México D.F., Mexico
[b] Depto. Matemáticas y Mecánica, I.I.M.A.S.-U.N.A.M., Apdo. Postal 20–726, 01000 Cd. México, Mexico

A R T I C L E I N F O

Article history:
Received 8 August 2018
Received in revised form 16 October 2018
Accepted 17 October 2018
Available online 22 October 2018
Communicated by M.G.A. Paris

Keywords:
Discrete nonlinear Schrödinger equation
Breather solutions
Bosonic quantization
Coherent states
Localization

A B S T R A C T

We present a comparison of quantum and "semiclassical" trajectories of coherent states that correspond to classical breather solutions of finite discrete nonlinear Schrödinger (DNLS) lattices. The main goal is to explain earlier numerical observations of recurrent return to the vicinity of initial coherent states corresponding to stable breathers that are also spatially localized. This effect can be considered as a quantum manifestation of classical spatial localization. We show that these phenomena are encoded in a simple expression for the distance between the quantum and semiclassical states that involves the basic frequencies of the classical and quantum systems, as well as the breather amplitude and quantum spectral decomposition of the system. A corollary is that recurrence phenomena are robust under perturbation of the initial conditions for stable breathers.

1. Introduction

The concept of energy localization in lattices has been the subject of intense research during the last decades and is relevant to many physical systems, such as photonic systems, Bose–Einstein condensates in lattices, electrons in molecules, etc. [1–3]. It has been shown that nonlinearity enhances spatial localization and that spatially localized solutions can be stable and robust. One of the main models to study these phenomena is the discrete nonlinear Schrödinger equation (DNLS) where we can also have spatially localized solutions of breather type, i.e. relative equilibria of the global phase symmetry of DNLS systems [4–6]. The nonlinear dynamics of these solutions has been studied extensively, see e.g. [7–11] for works on small chains.

A natural question on these DNLS lattices is how the existence and properties of classical breathers are reflected in the corresponding quantized system. There is not a unique answer to this question since the quantum and classical descriptions deal with different objects, and the literature has considered various properties of quantized DNLS systems with localized breathers, see [12–21], and [22] for a brief discussion. The general approach in this paper is to consider the quantum dynamics of Glauber and $SU(f)$ coherent states [23,24], a subset of the states of the system that can be identified with points in the phase space of the classical system. We also want to compare the (exact) quantum evolution of coherent states to states obtained by evolving the coherent state parameters by the classical equations of motion. Such trajectories will be referred to as "classical" or "semiclassical" trajectories of coherent states. The semiclassical trajectories arise from a variational principle as conditions for minimizing the distance between the set of coherent states and the exact quantum state at each instant, see [25–27] for precise definitions. In general however the semiclassical evolution of coherent states is only expected to give a good approximation of the quantum states in the linear limit of the DNLS (and for a large number of quanta), see [28,29], where spatial localization is not pronounced.

In previous works [22,30] we compared the quantum and semiclassical trajectories for coherent states corresponding to breathers and points in their vicinity away from the linear limit, and saw numerically that their distance can exhibit recurrences to relatively small values, suggesting the return to the vicinity of quantum coherent states corresponding to spatially localized breathers. The present work is a first step towards a quantification of these recurrences. In particular, we consider the evolution from $SU(f)$ coherent states in invariant subspaces of a fixed number of quanta (i.e. normalized projected Glauber states) [31,27,28], and give an expression for the distance between the quantum and semiclassical trajectories that involves the amplitude and frequency of the classical breather and the spectral data of the quantum problem, see Proposition 3.1. This expression also implies that the recurrence phenomena persist under perturbation of the classical initial

* Corresponding author.
E-mail address: panos@mym.iimas.unam.mx (P. Panayotaros).

https://doi.org/10.1016/j.physleta.2018.10.020
0375-9601/

conditions in the vicinity of orbitally stable classical breathers. This observation is stated as Proposition 3.2, where we use a more relaxed (finite time) notion of orbital stability that seems applicable to more breather solutions. In [22,30] and more recent numerical computations we see that the recurrences are more pronounced in subspaces of a small number of quanta where the breather coherent states are also close to superpositions of only a few eigenfunctions of the Hamiltonian operator. In such cases the exact expressions for the distance can be approximated by simpler expressions that lead to predictions for the minimum distance and the time of recurrence to the minimum distance. The accuracy and limitations of these approximations have been studied numerically for 3- and 5-site lattices, and we summarize some observations at the end of section 3. Details will be presented elsewhere.

The proposed comparison between the quantum and semiclassical evolution from coherent states requires classical and quantum quantities that are computed numerically. We believe that these quantities can be approximated analytically in special perturbative regimes, e.g. the small intersite coupling limit, but this is not attempted here. Possible generalizations and further questions are discussed in section 4.

The paper is organized as follows. In section 2 we review the basic definitions and facts about breather solutions of the DNLS and define the quantized problem using bosonic quantization. In section 3 we define the $SU(f)$ coherent states and describe the expressions used to compare their quantum and semiclassical evolutions. In section 4 we discuss our results.

2. Classical DNLS system and its bosonic quantization

We consider a finite set of f anharmonic oscillators evolving under the cubic discrete nonlinear Schrödinger equation (DNLS)

$$\frac{du_j}{dt} = -i\delta\,(\Delta u)_j - 2i|u_j|^2 u_j, \tag{2.1}$$

where $u_j \in \mathbb{C}$ is the complex amplitude of the oscillator at the lattice site j, $j \in \{1,\dots,f\}$, and δ is a real number. The discrete Laplacian Δ is defined by

$$\begin{aligned}(\Delta u)_j &= u_{j+1} + u_{j-1} - 2u_j, \quad j = 2,\dots,f-1,\\ (\Delta u)_1 &= u_2 - 2u_1, \quad (\Delta u)_f = u_{f-1} - 2u_f.\end{aligned} \tag{2.2}$$

The definitions at $j = 1, f$ amount to a discrete analogue of Dirichlet boundary conditions. System (2.1) can be written as a Hamiltonian system

$$\frac{du_j}{dt} = -i\frac{\partial H}{\partial u_j^*}, \quad j = 1,\dots,f, \tag{2.3}$$

where the Hamiltonian H is given by

$$H = -\delta\left(\sum_{j=1}^{f-1}|u_{j+1}-u_j|^2 + |u_1|^2 + |u_f|^2\right) + \sum_{j=1}^{f}|u_j|^4. \tag{2.4}$$

Another conserved quantity is the power $P = \sum_{j=1}^{f}|u_j|^2$; its conservation follows from the invariance of H under global phase change $u_j \mapsto e^{i\theta}u_j$, $j = 1,\dots f$.

A *breather* solution of (2.1) is a time-periodic solution of the form

$$u_j = e^{-i\omega t}A_j, \quad j = 1,\dots,f, \tag{2.5}$$

with ω real, and $A_j \in \mathbb{C}$, $j = 1,\dots,f$, independent of time t. Breather solutions are relative equilibria with respect to the S^1-action of global phase change, and equilibria of the corresponding S^1-reduced phase space at each value of P [11]. Breathers are in that sense the simplest nontrivial solutions of Hamiltonian DNLS systems. Equivalently, breather orbits consist of critical points of the Hamiltonian H for fixed power P [5]. This implies the existence of at least some breather solutions for any power, moreover breathers that correspond to isolated local extrema of the energy are linearly and nonlinearly (orbitally) stable [11]. Note that in the linear case breathers are the normal modes of the system. Breathers and their relation to the nontrivial global dynamics of the DNLS have been studied extensively for the $f = 3$ ("trimer") system [7–11]. The case $f = 2$ is integrable.

Also, breathers are examples of solutions that can exhibit spatial localization, e.g. for $|\delta| \to 0$ we can have breathers with $|A_j|$ of $O(1)$, and $O(\delta)$ respectively in complementary sets of sites, see [4,6,32]. Solutions can be continued to the large coupling region (i.e. $|\delta| \to \infty$ is the linear limit, after suitable rescaling), see e.g. [9,33]. Normal modes of the linear DNLS are also breathers.

To compute breathers we use (2.5), (2.1) and solve numerically the system

$$-\omega A_j = -\delta\,(\Delta A)_j - 2|A_j|^2 A_j, \quad \sum_{j=1}^{f}|A_j|^2 = C, \quad j = 1,\dots,f \tag{2.6}$$

with C (or ω) fixed. The linearization around breathers in the frame moving with the breather is an autonomous linear system and the linear stability of breathers can be determined numerically in a straightforward way, see e.g. [33,11].

We define a quantum version of the DNLS equations using the bosonic quantization rules, see e.g. [12,34] for the Dirac notation. Specifically, let V be the complex span of the *occupation number* basis elements $|n_1, n_2, \dots, n_f\rangle$, where $n_1,\dots,n_f \geq 0$. The $|n_1,\dots,n_f\rangle$ are also assumed to form an orthonormal basis, satisfying

$$\langle m_f,\dots,m_1|n_1,\dots,n_f\rangle = \delta_{m_1n_1}\dots\delta_{m_fn_f}, \tag{2.7}$$

with $\delta_{m_in_j}$ the Kronecker delta. We also let V_n be the complex subspace of states spanned by all $|n_1,\dots,n_f\rangle$ satisfying $n_1 + \dots n_f = n$. V_n is referred to as the subspace with n quanta. The complex dimension p_n of V_n is given by $p_n = \frac{(n+f-1)!}{(f-1)!n!}$.

Under quantization, the amplitudes of the complex modes u_j^* and u_j of a model such as (2.1) are mapped to the bosonic creation and annihilation operators, $B_j^\dagger$ and B_j, $j = 1,\dots,f$, defined by their action on the basis vectors as

$$\begin{aligned}&B_j^\dagger|n_1,n_2,\dots,n_j,\dots,n_f\rangle = \sqrt{n_j+1}\,|n_1,n_2,\dots,n_j+1,\dots,n_f\rangle,\\ &B_j|n_1,n_2,\dots,n_j,\dots,n_f\rangle = \sqrt{n_j}\,|n_1,n_2,\dots,n_j-1,\dots,n_f\rangle,\\ &\quad \text{if } n_j > 0,\\ &B_j|n_1,n_2,\dots,0,\dots,n_f\rangle = 0|n_1,n_2,\dots,0,\dots,n_f\rangle = 0.\end{aligned} \tag{2.8}$$

We also define the quantized Hamiltonian operator $\hat{H}$ by

$$\begin{aligned}\hat{H} &= (1-2\delta)\sum_{j=1}^{f}B_j^\dagger B_j + \sum_{j=1}^{f}B_j^\dagger B_j B_j^\dagger B_j\\ &\quad + \delta\sum_{j=1}^{f}\left(B_j^\dagger B_{j+1} + B_j B_{j+1}^\dagger\right),\end{aligned} \tag{2.9}$$

i.e. compare to the classical Hamiltonian of (2.4), see also [34] for different quantizations. Similarly the power P is "quantized" to the "number" operator $\hat{N}$, defined by $\hat{N} = \sum_{j=1}^{f}B_j^\dagger B_j$.

The dynamics of the quantum system is described by the Schrödinger equation

$$i\frac{\partial|\Psi(t)\rangle}{\partial t} = \hat{H}|\Psi(t)\rangle, \tag{2.10}$$

whose formal solution is

$$|\Psi(t)\rangle = e^{-i\hat{H}t}|\Psi(0)\rangle, \tag{2.11}$$

with $|\Psi(0)\rangle$ the initial state.

Operators $\hat{N}$ and $\hat{H}$ commute and this implies that for any $n \geq 0$ the subspaces V_n are invariant under the evolution of Schrödinger's equation. Also, the matrix representation of $\hat{H}$ in the occupation number basis has a block diagonal form where each block, denoted by $\hat{H}_n$, has entries $\langle v_i|\hat{H}|v_j\rangle$, with v_i, v_j elements of the occupation number basis of V_n. We will denote the eigenvalues and eigenvectors of $\hat{H}_n$ by $E_{l,n}$, $|\Psi(l,n)\rangle$ respectively, i.e.

$$\hat{H}_n|\Psi(l,n)\rangle = E_{l,n}|\Psi(l,n)\rangle, \tag{2.12}$$

where $l \in \{1,\ldots,p_n\}$ is some enumerating index, e.g. $E_{q,n} \leq E_{s,n}$ if $q < s$. (Multiple eigenvalues are repeated.) Denoting the occupation number basis vectors in V_n by $|v(j,n)\rangle$, $j \in \{1,\ldots,p_n\}$ an enumerating index, e.g. lexicographic order, we also use the notation

$$|\Phi(l,n)\rangle = \sum_{j=1}^{p_n} a_{lj}|v(j,n)\rangle, \quad |v(j,n)\rangle = \sum_{l=1}^{p_n} b_{jl}|\Phi(l,n)\rangle \tag{2.13}$$

for the coefficients relating the two bases in each V_n.

Calculations of the spectral properties of the quantized DNLS have been reported by many authors, and there are several possible definitions of spatial localization at the quantum level [12–21], see [22] for a brief discussion. It also appears that an intrinsically quantum notion of localization is not as natural for systems with translation symmetry [18], e.g. the DNLS with periodic boundary conditions. In the next section we turn instead to coherent states, quantum states that correspond in a natural way to points of the classical phase space.

3. Quantum and classical evolution of breather coherent states

In this section we will consider the quantum evolution of coherent states that correspond to classical breathers and their vicinity.

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_f) \in \mathbb{C}^f$. We define a Glauber coherent state $|(\alpha)\rangle$, $(\alpha) \in \mathbb{C}^f$, to be a normalized state in V that satisfies

$$\hat{B}_j|(\alpha)\rangle = \alpha_j|(\alpha)\rangle, \tag{3.1}$$

for all $j = 1, 2, \ldots, f$.

A normalized Glauber coherent state $|(\alpha)\rangle$ can be expressed as a linear combination of occupation number basis states, namely

$$|(\alpha)\rangle = e^{-\frac{1}{2}(|\alpha_1|^2+|\alpha_2|^2+\cdots+|\alpha_f|^2)} \times \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_f=0}^{\infty} \frac{\alpha_1^{n_1}\alpha_2^{n_2}\ldots\alpha_f^{n_f}}{\sqrt{n_1!n_2!\ldots n_f!}}|n_1, n_2, \ldots, n_f\rangle, \tag{3.2}$$

see e.g. [23,24]. Let P_n denote the orthogonal projection to the subspace V_n. Then

$$P_n|(\alpha)\rangle = e^{-\frac{1}{2}\sum_{k=1}^{f}|\alpha_k|^2} \sum_{j=1}^{p_n} \frac{\alpha_1^{n_{1j}}\alpha_2^{n_{2j}}\ldots\alpha_f^{n_{fj}}}{\sqrt{n_{1j}!n_{2j}!\ldots n_{fj}!}}|v(j,n)\rangle, \tag{3.3}$$

where $n_{1j} + n_{2j} + \ldots n_{fj} = n$.

Let $\alpha(0) = (\alpha_1(0), \ldots, \alpha_f(0)) \in \mathbb{C}^f$ and consider the coherent state $|(\alpha(0))\rangle$. The quantum evolution of the initial state $P_n|(\alpha(0))\rangle$ is then given by

$$|\Psi_n(t)\rangle = e^{-i\hat{H}_n t}P_n|(\alpha(0))\rangle. \tag{3.4}$$

Using (2.13) in (3.3) we have

$$P_n|(\alpha)\rangle = \sum_{j=1}^{p_n} d_{j,n}|\Phi(j,n)\rangle,$$

$$\text{with} \quad d_{j,n} = e^{-\frac{1}{2}\sum_{k=1}^{f}|\alpha_k|^2} \sum_{l=1}^{p_n} \frac{\alpha_1^{n_{1l}}\alpha_2^{n_{2l}}\ldots\alpha_f^{n_{fl}}}{\sqrt{n_{1l}!n_{2l}!\ldots n_{fl}!}} b_{lj}. \tag{3.5}$$

Then (3.4) can be also written as

$$|\Psi_n(t)\rangle = \sum_{j=1}^{p_n} d_{j,n}e^{-iE_{j,n}t}|\Phi(j,n)\rangle. \tag{3.6}$$

The above also apply to initial conditions $c^{-1/2}P_n|(\alpha(0))\rangle$, with $c^{1/2}$ a normalizing factor. Normalized projected Glauber states are known as $SU(f)$ coherent states [24,31,27,28]. By (3.6) the quantum evolution of $SU(f)$ coherent states is given by

$$|\Psi_n(t)\rangle = \frac{1}{\sqrt{c}}\sum_{j=1}^{p_n} d_{j,n}e^{-iE_{j,n}t}|\Psi_j\rangle, \tag{3.7}$$

with $c = |d_{1,n}|^2 + \cdots + |d_{f,n}|^2$, and $d_{j,n}$ as in (3.5). As a particular case we use an initial condition $|(\alpha(0))\rangle$ with $\alpha(0) = A$, a breather amplitude satisfying (2.6), then

$$d_{j,n} = e^{-\frac{1}{2}\sum_{r=1}^{f}|A_r|^2} \sum_{l=1}^{p_n} \frac{A_1^{n_{1l}}A_2^{n_{2l}}\ldots A_f^{n_{fl}}}{\sqrt{n_{1l}!n_{2l}!\ldots n_{fl}!}} b_{lj}. \tag{3.8}$$

We also consider an alternative non-exact, and possibly approximate, evolution rule for quantum states defined in the following way. Let $\alpha(t)$ be solution of Hamilton's equations (2.3) with initial condition $\alpha(0)$ (as above). Also let $|(\alpha(t))\rangle$ be the corresponding coherent state at any time t. Then define $|\Psi^C(t)\rangle$, and $|\Psi_n^C(t)\rangle$ by $|\Psi^C(t)\rangle = |(\alpha(t))\rangle$, and

$$|\Psi_n^C(t)\rangle = P_n|(\alpha(t))\rangle, \tag{3.9}$$

respectively. State (3.9) gives a "semi-classical" approximation to the evolution of the projection of a coherent state.

Letting $\alpha_j = \alpha_j(t)$, $\alpha_j(t)$, $j = 1, \ldots, f$, a solution of the classical DNLS (2.3), and substituting into (3.3) we have

$$|\Psi_n^C(t)\rangle = \sum_{j=1}^{p_n} d_{j,n}^C(t)|\Phi(j,n)\rangle, \tag{3.10}$$

with

$$d_{j,n}^C(t) = e^{-\frac{1}{2}\sum_{k=1}^{f}|\alpha_k(t)|^2} \sum_{l=1}^{p_n} \frac{[\alpha_1(t)]^{n_{1l}}[\alpha_2(t)]^{n_{2l}}\ldots[\alpha_f(t)]^{n_{fl}}}{\sqrt{n_{1l}!n_{2l}!\ldots n_{fl}!}} b_{lj}, \tag{3.11}$$

using also (2.13).

For $\alpha(t)$ a breather solution $Ae^{-i\omega t}$ of the classical DNLS (2.3), we then have

$$d_{j,n}^C(t) = e^{-\frac{1}{2}\sum_{r=1}^{f}|A_r|^2} \times \sum_{l=1}^{p_n} \frac{[A_1e^{-i\omega t}]^{n_{1l}}[A_2e^{-i\omega t}]^{n_{2l}}\ldots[A_fe^{-i\omega t}]^{n_{fl}}}{\sqrt{n_{1l}!n_{2l}!\ldots n_{fl}!}} b_{lj}, \tag{3.12}$$

therefore

$$d_{j,n}^C(t) = e^{-\frac{1}{2}\sum_{r=1}^{f}|A_r|^2} \sum_{l=1}^{p} \frac{A_1^{n_{1l}} A_2^{n_{2l}} \dots A_f^{n_{fl}}}{\sqrt{n_{1l}! n_{2l}! \dots n_{fl}!}} e^{-i\omega N t} b_{lj}$$
$$= d_{j,n} e^{-i\omega n t}, \tag{3.13}$$

using $n_{1j} + n_{2j} + \dots n_{fj} = n$. Clearly $d_{j,n}^C(0) = d_{j,n}$, with $d_{j,n}$ as in (3.8), the coefficient appearing in the initial breather coherent state.

The quantum and "semi-classical" states from normalized breather coherent states in are therefore given respectively by

$$|\Psi_n(t)\rangle = \frac{1}{\sqrt{c}} \sum_{j=1}^{p_n} d_{j,n} e^{-iE_{j,n}t} |\Phi_{j,n}\rangle, \tag{3.14}$$

and

$$|\Psi_n^C(t)\rangle = \frac{1}{\sqrt{c}} \sum_{j=1}^{p_n} d_{j,n} e^{-i\omega N t} |\Phi_{j,n}\rangle, \tag{3.15}$$

with $d_{j,n}$ as in (3.8), and $c = |d_{1,n}|^2 + \dots + |d_{f,n}|^2$ the normalization constant. The squared modulus of each coefficient $d_{j,n}/\sqrt{c}$ can be interpreted as the probability that the system is in the state $|\Phi_{j,n}\rangle$ $j = 1, 2, \dots, p_n$.

By (3.15) state $|\Psi_n^C(t)\rangle$ has the time-dependence of an eigenstate. Thus breather coherent states mimic stationary states. For $|\delta|$ small the number of breathers is larger that f [32]. For $|\delta|$ sufficiently large, the breathers approach the normal modes of the linear system, and we have f breathers, see [33].

The difference between the quantum and classical evolutions in V_n can be measured by

$$D_n(t) = \inf_{\phi \in S^1} \| e^{i\phi} |\Psi_n(t)\rangle - |\Psi_n^C(t)\rangle \|. \tag{3.16}$$

In the case of breather coherent state initial conditions we use (3.14), (3.15) to obtain

$$D_n^2(t) = \inf_{\phi \in S^1} \left(2 - 2 \sum_{j=1}^{p_n} \frac{|d_{j,n}|^2}{c} \cos\left((E_j - \omega n) t + \phi \right) \right), \tag{3.17}$$

where the $d_{j,n}$ are given by (3.8). The value of ϕ that attains the infimum depends on t.

Proposition 3.1. *The functions D_n, $n \geq 1$, of (3.17) are given by*

$$D_n^2(t) = 2 - \frac{2}{c} \sqrt{ \sum_{j=1}^{p_n} |d_{j,n}|^4 + 2 \sum_{i<j}^{p_n} |d_{i,n}|^2 |d_{j,n}|^2 \cos(E_{j,n} - E_{i,n})t }. \tag{3.18}$$

Proof. Fix n, t and write D_n^2 as the function

$$G(\phi) = 2 - 2 \sum_{j=1}^{p_n} \frac{|d_{j,n}|^2}{c} \cos(\lambda_j t + \phi), \quad \lambda_j = E_{j,n} - \omega n. \tag{3.19}$$

G is a smooth 2π-periodic real function and has at least two critical points in $[-\pi, \pi)$, corresponding to its maximum and the minimum. By

$$G(\phi) = 2 - 2A\cos\phi + 2B\sin\phi, \tag{3.20}$$

with

$$A = \sum_{j=1}^{p_n} \frac{|d_{j,n}|^2}{c} \cos\lambda_j t, \quad B = \sum_{j=1}^{p_n} \frac{|d_{j,n}|^2}{c} \sin\lambda_j t, \tag{3.21}$$

and

$$G'(\phi) = 2A\sin\phi + 2B\cos\phi, \tag{3.22}$$

the critical points satisfy

$$\tan\phi = -\frac{B}{A}. \tag{3.23}$$

There are then exactly two critical points in $[-\pi, \pi)$, the minimum and the maximum. Moreover the two ϕ satisfying (3.23) are in the intervals $(-\pi/2, \pi/2)$, and $(-\pi, -\pi/2) \cup (\pi/2, \pi)$. We claim that for $A > 0$ the minimum is in $(-\pi/2, \pi/2)$, and that for $A < 0$ the minimum is in $(-\pi, -\pi/2) \cup (\pi/2, \pi)$. We evaluate

$$G''(\phi) = 2A\cos\phi - 2B\sin\phi = 2\cos\phi(A - B\tan\phi) \tag{3.24}$$

at the critical points. ϕ a solution of (3.23) and $\cos\phi = \pm(1 + B^2/A^2)^{-1/2}$ imply

$$G''(\phi) = \pm 2A(1 + \tan^2\phi)^{-1/2}. \tag{3.25}$$

Therefore $G''(\phi) > 0$ requires that A and $\cos\phi$ have the same sign, implying the claim. Using this fact we then check that at the minimum, $\cos\phi = \pm(1 + B^2/A^2)^{-1/2}$ implies

$$G(\phi) = 2 \mp 2\frac{A}{|A|}\sqrt{A^2 + B^2} = 2 - 2\sqrt{A^2 + B^2} \tag{3.26}$$

by $A = \pm|A|$. Therefore

$$D_n^2(t) = 2 - 2\sqrt{ \left(\sum_{j=1}^{p_n} \frac{|d_{j,n}|^2}{c} \cos\left(E_{j,n} - \omega N\right) t \right)^2 + \left(\sum_{j=1}^{p} \frac{|d_{j,n}|^2}{c} \sin\left(E_{j,n} - \omega n\right) t \right)^2 }, \tag{3.27}$$

and we immediately obtain (3.18). □

We can use expression (3.18) to approximate $D_n(t)$. For example if we only consider the two coefficients $|d_{1,n}|$, $|d_{2,n}|$ in the sum (3.18), assumed to be the largest ones, we have the simplified expression

$$D_n^2(t) \approx 2 - \frac{2}{c}\sqrt{|d_{1,n}|^4 + |d_{2,n}|^4 + 2|d_{1,n}|^2|d_{2,n}|^2\cos(E_{2,n} - E_{1,n})t}. \tag{3.28}$$

We can then estimate the frequency of recurrence to the minima of D_n^2 as $(E_{2,n} - E_{1,n})/2\pi$.

Expression (3.17) also implies that the distance $D_n(t)$ depends only on the breather orbit: by (3.8), changing the breather initial condition from A to $e^{i\theta}A$ multiplies all the $d_{j,n}$ by a common phase $e^{in\theta}$ leaving $D_n(t)$ invariant.

We can also see that the values of $D_n(t)$ for classical initial conditions in the neighborhood of a stable breather are those obtained for the breather plus a small error. To make this more precise we consider a breather solution $Ae^{-i\omega t}$, and a solution $\tilde{u}$ that satisfies

$$\tilde{u}(t) = Ae^{-i(\omega t + \theta(t))} + \delta(t),$$
$$\text{with} \quad \phi(0) = 0, \quad ||\delta(t)|| < \epsilon, \quad \forall t \in [0, T], \tag{3.29}$$

for some functions $\theta : [0, T] \to \mathbb{R}$, $\delta : [0, T] \to \mathbb{C}^f$. ($|| \cdot ||$ is the Euclidean norm in $\mathbb{C}^f \approx \mathbb{R}^{2f}$.) (3.29) states that the orbit $\tilde{u}$ remains ϵ-close to the breather orbit in $[0, T]$. ($\theta(t)$ can grow so that $\tilde{u}(t)$ is not necessarily close to $Ae^{-i\omega t}$.) We can define finite time orbital stability for the breather $Ae^{-i\omega t}$ as the property that given any $0 < \epsilon < \epsilon_m$ we can choose δ_0 such that $||\tilde{u}(0) - A|| < \delta_0$ implies (3.29) for suitable functions θ, δ. For $T = \infty$ we have the usual definition of orbital stability.

Orbital stability is known only for breather orbits corresponding to local extrema of the energy at fixed power, and there are many linearly stable breathers that are not local extrema. We are not aware of theoretical results of nonlinear stability analysis for such breathers, but numerically we see trajectories that remain indefinitely in the neighborhood of the breather, as well as cases of eventual escape from the neighborhood of linearly stable breather [33,11,35]. These considerations should make finite time stability a reasonable assumption. From a more practical point of view we are interested in trajectories $\tilde{u}$ observed to satisfy (3.29) for ϵ "small".

Proposition 3.2. *Consider a breather solution $Ae^{-i\omega t}$ and trajectory $\tilde{u}$ satisfying (3.29) for some $T > 0$ and suitable functions θ, δ. Let $\tilde{A} = \tilde{u}(0) = A + \delta(0)$, $|\delta(0)| < \epsilon$, and consider quantum and semiclassical states $|\tilde{\Psi}_n(t)\rangle$, $|\tilde{\Psi}^C_n(t)\rangle$ as above that evolve from the initial condition $|(\tilde{A})\rangle$. Then, for any $t \in [0, T]$, the difference $\tilde{D}_n(t)$ between $|\tilde{\Psi}_n(t)\rangle$, $|\tilde{\Psi}^C_n(t)\rangle$, defined as in (3.16), satisfies*

$$\tilde{D}^2_n(t) = 2 - \frac{2}{c}\sqrt{\sum_{j=1}^{p_n} |d_{j,n}|^4 + 2\sum_{i<j}^{p_n} |d_{i,n}|^2 |d_{j,n}|^2 \cos(E_{j,n} - E_{i,n})t} + O(\epsilon) \tag{3.30}$$

as $\epsilon \to 0$, with $d_{j,n}$ as in (3.8).

Proof. By the assumption, $\alpha_j(t) = A_j e^{-i(\omega t + \phi(t))} + \delta_j(t)$, with $|\delta_j(t)| < \epsilon$ for all $j = 1, \ldots, f$, $t \in [0, T]$. Then the semiclassical coefficients $d^C_{j,n}(t)$ of (3.11) with $\alpha(t)$ the trajectory $\tilde{u}$ are

$$d^C_{j,n}(t) = e^{-\frac{1}{2}\sum_{r=1}^{f} |A_r|^2} \sum_{l=1}^{p} \frac{A_1^{n_{1l}} A_2^{n_{2l}} \ldots A_f^{n_{fl}}}{\sqrt{n_{1l}! n_{2l}! \ldots n_{fl}!}} e^{-iN(\omega t + \theta(t))} b_{lj} + O(\epsilon), \tag{3.31}$$

and we have that the semiclassical state corresponding to $\tilde{u}$ satisfies

$$|\tilde{\Psi}^C_n(t)\rangle = \frac{1}{\sqrt{c}} \sum_{j=1}^{p_n} d_{j,n} e^{-iN(\omega t + \theta(t))} |\Phi_{j,n}\rangle + O(\epsilon) \tag{3.32}$$

for all $t \in [0, T]$, with $d_{j,n}$ obtained from the unperturbed breather by (3.8). The assumption also implies that the exact quantum state obtained from the initial condition $|(\tilde{u})\rangle$ satisfies

$$|\Psi_n(t)\rangle = \frac{1}{\sqrt{c}} \sum_{j=1}^{p_n} d_{j,n} e^{-iE_{j,n}t} |\Phi_{j,n}\rangle + O(\epsilon) \tag{3.33}$$

for all real t, with $d_{j,n}$ as in (3.32). Defining the difference $\tilde{D}_n(t)$ between $|\tilde{\Psi}_n(t)\rangle$ and $|\tilde{\Psi}^C_n(t)\rangle$, as in (3.16) we have

$$\tilde{D}^2_n(t) = \inf_{\phi \in S^1} \left(2 - 2\sum_{j=1}^{p_n} \frac{|d_{j,n}|^2}{c} \cos\left(\left(E_j - \omega n\right) t + \theta(t) + \phi\right) \right) + O(\epsilon), \tag{3.34}$$

for all $t \in [0, T]$. The infimum over $\phi \in S^1$ for any given t is not affected by the addition of an angle $\theta(t)$, and we again obtain the expression of Proposition 3.1, up to an error $O(\epsilon)$. □

The lack of significant change in $D_n(t)$ for initial conditions in the vicinity of stable breathers was also observed in [22,30], where we also saw significant change in $D_n(t)$ as we perturb from initial conditions on unstable breather orbits.

Numerical calculations of $D_n(t)$ with $f = 3, 5$ sites, using as initial conditions coherent states that correspond to linearly stable breathers, suggest that recurrences to the vicinity of the initial coherent state, i.e. to relatively small local minima of $D_n(t)$, are observed mainly for small numbers of quanta (up to about 10 for $\delta \sim 0.3$). This behavior was already seen in [22,30] and we have now checked that in these cases (3.28) gives a good approximation of $D_n(t)$, and allows us to give accurate quantitative estimates of the recurrence times and minima of $D_n(t)$ using only two eigenvalues and eigenvectors computed numerically. We have also seen that the effectiveness of this approximation is correlated to a smaller number of eigenfunctions needed to capture the initial coherent state. Details on these numerical results will be presented elsewhere. Another observation is that the minimum of $D_n(t)$ is significantly smaller for the "one-peak" breathers, see [32], expected to be local minima of the energy H at fixed power P and therefore orbitally stable for $\delta \geq 0$ sufficiently small, see [11].

4. Discussion

We have compared quantum and semiclassical trajectories for $SU(f)$ coherent states corresponding to classical breathers. Our main motive has been to give a possible quantum analogue of classical localization by considering spatially localized classical breathers and associated quantum states. We saw that the distance D_n between the quantum and semiclassical trajectories is reduced to a relatively simple expression that can be evaluated numerically and can explain some features of recurrence to the vicinity of the initial state seen in earlier studies [22,30]. We were particularly interested in the regime of weak coupling between the lattice sites, where we expect to be able to use perturbation arguments to approximate analytically several of the quantities in the expression for the D_n in Proposition 3.2. In [22,30] we had seen that the phenomenon of recurrence to the vicinity of the initial quantum state is more pronounced for small numbers of quanta. In more recent numerical studies, to be reported elsewhere, we see that in such cases the initial states considered are a superposition of a small number of eigenstates, and D_n can be effectively approximated by the simplified expression (3.28). The behavior of D_n as we increase the number of quanta n, and the related observation of increased degeneracy of the eigenstates, should be analyzed in more detail.

The ideas of the paper can be applied to other systems with global phase symmetry, e.g. we may also consider the same DNLS system in linear normal mode variables, i.e. use the framework of sections 2, 3 with u_j the normal mode coordinates and the corresponding Hamiltonian. We also believe that analogous but more cumbersome expressions for the distance between quantum and Glauber semiclassical trajectories can be obtained for general periodic orbits and Hamiltonian systems. Such expressions require more involved computations, but may give a precise way to measure the accuracy of coherent state semiclassical approximations for a large class of orbits.

Acknowledgement

The authors acknowledge partial support from grant PAPIIT IN103916.

References

[1] J.C. Eilbeck, P.S. Lomdahl, A.C. Scott, The discrete self-trapping equation, Physica D 16 (1985) 318–338.
[2] D.N. Christodoulides, R.I. Joseph, Discrete self-focusing in nonlinear arrays of coupled waveguides, Opt. Lett. 18 (1988) 794.

[3] P.G. Kevrekidis, The Discrete Nonlinear Schrödinger Equation, Springer, New York, 2009.
[4] R.S. MacKay, S. Aubry, Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators, Nonlinearity 7 (1994) 1623–1643.
[5] M.I. Weinstein, Excitation thresholds for nonlinear localized modes on lattices, Nonlinearity 12 (1999) 673–691.
[6] D.E. Pelinovsky, P.G. Kevrekidis, D.J. Frantzeskakis, Stability of discrete solitons in nonlinear Schrödinger lattices, Physica D 212 (2005) 1–19.
[7] P. Buonsante, R. Franzosi, V. Penna, Dynamical instability in a trimeric chain of interacting Bose–Einstein condensates, Phys. Rev. Lett. 90 (2003) 050404.
[8] M. Johansson, Hamiltonian Hopf bifurcations in the discrete nonlinear Schrödinger trimer: oscillatory instabilities, quasi-periodic solutions and a 'new' type of self-trapping transition, J. Phys. A, Math. Gen. 37 (2004) 2201–2222.
[9] T. Kapitula, P.G. Kevrekidis, Z. Chen, Three is a crowd: solitary waves in photorefractive media with three potential wells, SIAM J. Appl. Dyn. Syst. 5 (2007) 598–633.
[10] R. Goodman, Hamiltonian Hopf bifurcations and dynamics of NLS/GP standing-wave modes, J. Phys. A, Math. Theor. 44 (2011) 425101.
[11] P. Panayotaros, Instabilities of breathers in a finite NLS lattice, Physica D 241 (2012) 847–856.
[12] L. Bernstein, J.C. Eilbeck, A.C. Scott, The quantum theory of local modes in a coupled system of nonlinear oscillators, Nonlinearity 3 (1990) 293–323.
[13] P.D. Miller, A.C. Scott, J. Carr, J.C. Eilbeck, Binding energies for discrete nonlinear Schrödinger equations, Phys. Scr. 44 (1991) 509–516.
[14] E. Wright, J.C. Eilbeck, M.H. Hays, P.D. Miller, A.C. Scott, The quantum discrete self-trapping equation in the Hartree approximation, Physica D 69 (1993) 18–32.
[15] A. Chefles, Nearest-neighbor level spacings for the non-periodic discrete nonlinear Schrödinger equation, J. Phys. A, Math. Gen. 29 (1996) 4515–4526.
[16] S. Flach, V. Fleurov, Tunneling in the nonintegrable trimer – a step towards quantum breathers, J. Phys. Condens. Matter 9 (1997) 7039–7061.
[17] S. Raghavan, A. Smerzi, V.M. Kenkre, Transitions in coherent oscillations between two trapped Bose–Einstein condensates, Phys. Rev. A 60 (1999) 1787.
[18] J.C. Eilbeck, F. Palmero, Trapping in quantum chains, Phys. Lett. A 331 (2004) 201–208.
[19] R.A. Pinto, M. Haque, S. Flach, Edge-localized states in quantum one-dimensional lattices, Phys. Rev. A 79 (2009) 052118.
[20] P. Buonsante, V. Penna, A. Vezzani, Quantum signatures of the self-trapping transition in attractive lattice bosons, Phys. Rev. A 82 (2010) 043615.
[21] R. Franzosi, S.M. Giampaolo, F. Iluminati, Quantum localization and bound-state formation in Bose–Einstein condensates, Phys. Rev. A 82 (2010) 063620.
[22] R. Martínez-Galicia, P. Panayotaros, Coherent states and localization in a quantized discrete NLS lattice, J. Nonlinear Opt. Phys. Mater. 25 (2016) 1650047.
[23] R.J. Glauber, Coherent and incoherent states of the radiation field, Phys. Rev. 131 (6) (1963) 2766–2788.
[24] J.C. Klauder, E.G.C. Sudarshan, Fundamentals of Quantum Optics, W.A. Benjamin, New York, 1968.
[25] L. Amico, V. Penna, Dynamical mean field theory of the Bose–Hubbard model, Phys. Rev. Lett. 80 (1998) 2189.
[26] A.G. Basile, V. Elser, Equations of motion for superfluids, Phys. Rev. E 51 (1995) 5688.
[27] P. Buonsante, V. Penna, Some remarks on the coherent state variational approach to nonlinear boson models, J. Phys. A, Math. Theor. 41 (2008) 175301.
[28] F. Trimborn, D. Witthaut, H.J. Korsch, Beyond mean-field dynamics of small Bose–Hubbard systems based on the number-conserving phase-space approach, Phys. Rev. A 79 (2009) 013608.
[29] M. Hiller, T. Kottos, T. Geisel, Wave-packet dynamics in energy space of a chaotic trimeric Bose–Hubbard system, Phys. Rev. A 79 (2009) 023621.
[30] R. Martínez-Galicia, P. Panayotaros, Localization and coherent states in a quantum DNLS trimer, Eur. Phys. J. Spec. Top. 225 (2016) 2717–2727.
[31] D.M. Gitman, A.L. Shelepin, Coherent states of the SU(N) groups, arXiv:hep-th/9208017v1, 1992.
[32] P. Panayotaros, Continuation and bifurcations of breathers in a finite discrete NLS equation, Discrete Contin. Dyn. Syst., Ser. S (2011) 1227–1245.
[33] P. Panayotaros, Continuation of normal modes in finite NLS lattices, Phys. Lett. A 374 (38) (2010) 3912–3919.
[34] D. Ellinas, M. Johansson, P.L. Christiansen, Quantum nonlinear lattices and coherent state vectors, Physica D 34 (1999) 126–143.
[35] P. Panayotaros, F. Rivero, Multi-peak breather stability in a dissipative discrete NLS equation, J. Nonlinear Opt. Phys. Mater. 23 (4) (2014) 1450044–1450057.