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Gravity Waves on the Surface of the Sphere

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This paper is dedicated to the memory of Juan C. Simo

Summary. We propose a Hamiltonian model for gravity waves on the surface of a fluid layer surrounding a gravitating sphere. The general equations of motion are nonlocal and can be used as a starting point for simpler models, which can be derived systematically by expanding the Hamiltonian in dimensionless parameters. In this paper, we focus on the small wave amplitude regime. The first-order nonlinear terms can be eliminated by a formal canonical transformation. Similarly, many of the second order terms can be eliminated. The resulting model has the feature that it leaves invariant several finite-dimensional subspaces on which the motion is integrable.

1. Introduction

The goal of this paper is to describe a Hamiltonian formulation for waves on the surface of a fluid layer surrounding a gravitating spherical body. The fluid satisfies hydrodynamic equations inside the layer, and the surface of the layer is moving consistently with the motions of the fluid. The resulting system is a free boundary problem in which the region where the hydrodynamic equations of motion hold is also part of the unknowns.

We will assume that the fluid in the layer is inviscid, incompressible and moving irrotationally. The assumption that the flow is irrotational (potential flow) is very restrictive and in many cases inappropriate. Our model is, however, useful for isolating and studying hydrodynamic wave phenomena where the restoring force is gravitational, such as, for instance, sea waves and atmospheric tides.

In the case where the potential gravity waves take place over a plane, a very elegant Hamiltonian formulation was first introduced by Zakharov [Z] (see also [M]). More recently it has been shown that the formulation leads to a very systematic discussion of

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approximate gravity wave equations (see [CG]) and to efficient numerical methods [CS]. Using the formalism presented here, similar algorithms could be derived for the sphere. The theory of Zakharov has also been extended to general inviscid, incompressible free boundary flows in [LMMR]. In that work the Hamiltonian formulation of the potential flow case is derived from the canonical theory of the Euler equations (see [MG], [MW]).

In Sections 2, 3, and 4 we show how to use the potential flow formalism on the sphere and discuss some physical applications and limitations of our model. The main result of these sections is that, indeed, the free boundary problem can be written in a Hamiltonian form. The Poisson bracket does not have the standard form (such brackets are often referred to as noncanonical), but we also find a change of variables that reduces it to standard form. Note that there are already several well-known examples of noncanonical Poisson brackets in hydrodynamics, plasma, magnetism and other areas. The canonical variables in the Hamiltonian formulation of free surface potential flow are the wave amplitude (a function giving the shape of the surface) and the hydrodynamic potential at the surface. These two functions, defined on the sphere, completely determine the velocity inside the layer. This reduction in the dimension of the problem is one of the benefits of the potential flow assumption. On the other hand, the equations of motion for wave amplitude and surface potential are nonlocal. We describe a method that allows us to write the Hamiltonian and the equations of motion in terms of Fourier multiplier operators in Section 5.

The Hamiltonian formalism is particularly convenient for deriving approximate gravity wave equations, for it suffices to consider approximations of the Hamiltonian. These approximations are based on dimensional analysis and are discussed in Section 6, where we identify the dimensionless parameters of the problem and indicate interesting asymptotic regimes. In the present work we will be concerned with a small amplitude regime and we focus on "intermediate depth" waves.

The theory of small amplitude water waves is analogous to the theory of motions of a Hamiltonian system near an elliptic fixed point. The completely quiescent state is the fixed point, and the linear plane waves correspond to the normal modes of the linearised system. The motion of the waves in the linear approximation is governed by the quadratic terms in the Hamiltonian, while the nonlinear evolution arises from cubic and higher order terms of the Hamiltonian. The cubic and higher order terms of the Hamiltonian can be also interpreted as describing the interaction between three waves (cubic terms), four waves (quartic terms) and so on. One of the tools used in the study of motion around an elliptic fixed point is the Poincaré-Birkhoff method of successive canonical changes of coordinates. In each canonical transformation of the method one tries to eliminate the lowest order nonlinear terms of the Hamiltonian. After a number of such transformations the Hamiltonian is reduced to the so-called Birkhoff normal form containing only the resonant nonlinear terms. Analysis of normal form systems can yield extra information on the behavior of the system. In Section 7 we show that for water waves on the surface of the sphere: (a) the cubic (first-order in the nonlinearity) terms of the Hamiltonian can be eliminated, and (b) the only quartic (second-order in the nonlinearity) terms that cannot be eliminated are those of a very reduced class. The canonical transformations are given explicitly as formal time-1 maps of suitable Hamiltonian flows. We will remark, however, that these normal form calculations are more useful for intermediate and large depth water waves. As an application of these calculations we find some finite dimensional manifolds that are invariant under the evolution determined by the quartic normal form Hamiltonian. On these finite dimensional manifolds we can use the standard methods of dynamical systems to, in particular, identify periodic orbits of the second-order normal form system. These approximate solutions of the full water wave system are traveling and standing waves with amplitude dependent frequency.

Even if at this point these transformations are purely formal, we hope that they can be made rigorous and at least be used to prove good lower bounds for the time of existence of solutions with small initial data. It also seems possible that some of the periodic orbits and quasi-periodic orbits of the linear problem persist in the full nonlinear system. For finite dimensional systems, the persistence of families of periodic orbits in the full nonlinear system was proved first by Lyapunov. More general results of this type are in [Mo], [We]. The persistence of some of the quasi-periodic orbits is also proved in finite dimensions using KAM theory. Since this paper is concerned with developing the formalism, we will postpone these questions to future work.

We also point out that our formalism is well adapted to the development of numerical methods for the problem. For example, to develop finite-dimensional approximations we truncate the Hamiltonian. The truncated system will automatically be Hamiltonian. We plan to discuss the numerical implementations of that scheme in a forthcoming paper.

2. Equations of Motion

We consider a sphere of radius b and, on top of the sphere, a layer of fluid ("sea" or "atmosphere") of thickness ("depth") h. Using the standard spherical coordinates r = radius, ϑ = polar, φ = azimuth, the surface of the sea will be at $r(\vartheta, \varphi) = \rho + \eta(\vartheta, \varphi)$ with $\rho = b + h$. The amplitude of the water waves is described by the single valued function $\eta(\vartheta, \varphi)$. The dynamical problem we want to consider is that of free surface potential flow of the layer of water under the influence of gravity. For such a flow, since the spherical shell is simply connected, there exists a velocity potential ϕ and the velocity is given by $\vec{u} = \nabla\phi$. The conservation of mass for an incompressible fluid is $\nabla \cdot \vec{u} = 0$. Hence, we should have

$$\Delta\phi = 0 \tag{2.1}$$

in the region occupied by the fluid. On the surface we have

$$\eta_t = \frac{\partial\phi}{\partial r} - \frac{1}{r^2}\frac{\partial\phi}{\partial\vartheta}\frac{\partial\eta}{\partial\vartheta} - \frac{1}{r^2\sin^2\vartheta}\frac{\partial\phi}{\partial\varphi}\frac{\partial\eta}{\partial\varphi}, \tag{2.2}$$

and

$$\phi_t = -\frac{1}{2}|\nabla\phi|^2 + \frac{K}{\rho+\eta} - p, \tag{2.3}$$

and at the bottom $r = b$ we have

$$\frac{\partial\phi}{\partial r} = 0. \tag{2.4}$$

The equations of motion (2.1)–(2.4) are obtained from the Euclidean Euler equations (see [L]) by a change of variables: (2.1) is the conservation of mass for an incompressible fluid, (2.4) is the rigid wall boundary condition at the bottom of the sea and (2.2) is the condition, in polar coordinates, that the surface is transported by the flow, or

$$\left[\frac{\partial}{\partial t} + \vec{u} \cdot \nabla\right] F = 0, \tag{2.5}$$

where $F(\vec{r}, t) = r - \rho - \eta(\vartheta, \varphi) = 0$ is the implicit representation of the surface and $\vec{u}$ is the velocity at the surface. The dynamical boundary condition (2.3) follows from Euler's equation at the surface

$$\nabla\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}|\vec{u}|^2 + V(\eta) + p\right) = 0. \tag{2.6}$$

The term $V(\eta) = \frac{K}{\rho+\eta} = \frac{K}{\rho} - g\eta + \cdots$ is the gravitational potential due to the solid sphere of radius b. Physically, $K = GM$ with M the mass of the planet, G the gravitational constant and g the acceleration of gravity at $r = b$. Also we require that the pressure p be constant in space. ($\nabla p \neq 0$ would correspond to the presence of an additional external force.) Note that in (2.3) quantities with no spatial dependence (e.g. p, $\frac{K}{\rho}$) do not play any role and can be set to zero.

The equations of motion suggest that if at any instant t_0 we know the function $\eta(\vartheta, \varphi)$ and the potential ϕ on the surface, i.e., the function $\Phi(\varphi, \vartheta) = \phi(\varphi, \vartheta, \rho + \eta(\vartheta, \varphi))$, we can determine ϕ at t_0 in the whole region occupied by the fluid by solving the boundary value problem: $\Delta\phi = 0$ inside the fluid with $\phi = \Phi$ at the surface $r = \rho + \eta(\vartheta, \varphi)$ and $\frac{\partial \phi}{\partial r} = 0$ at $r = b$. Thus we are essentially interested in the evolution of η and Φ which is given by (2.2) and (2.3). Note that $\frac{\partial \Phi}{\partial t} = \frac{\partial \phi}{\partial t} + \frac{\partial \eta}{\partial t}\frac{\partial \phi}{\partial r}|_{r=\rho+\eta(\vartheta,\varphi)}$. Both equations are nonlocal since they contain the term $\frac{\partial \phi}{\partial r}$. To evaluate $\frac{\partial \phi}{\partial r}$ we need information about the solution of the boundary value problem. A method for writing the equations of motion in terms of η and Φ alone is given in Section 3.

3. Hamiltonian Formulation

We will show that the equations of motion (2.2), (2.3) for η and Φ form a Hamiltonian system. More precisely they can be written as

$$\eta_t = [\eta, H], \qquad \Phi_t = [\Phi, H] \tag{3.1}$$

where [,] is an appropriate Poisson bracket and H is the Hamiltonian of the system.

To express the equations of motion in Hamiltonian form, it is natural to guess that the Hamiltonian should be the physical energy and then try to find the Poisson bracket. In particular, here the Hamiltonian H will be the total energy $K + U$ (kinetic + potential) of the water mass

$$H = \frac{1}{2}\int_{S^2}\int_{r=b}^{r=\rho+\eta(\vartheta,\varphi)} |\nabla\phi|^2 dV + \int_{S^2}\int_{r=b}^{r=\rho+\eta(\vartheta,\varphi)} \left(-\frac{K}{r}\right) dV. \tag{3.2}$$

Integrating by parts the first term,

$$H = \frac{1}{2}\int_{S^2} \Phi R \frac{\partial \phi}{\partial n}\bigg|_{r=\rho+\eta(\vartheta,\varphi)} dA_r + \frac{1}{2}\int_{S^2}(-K)dA_r, \tag{3.3}$$

where $R = [1 + (\frac{1}{r}\frac{\partial\eta}{\partial\vartheta})^2 + (\frac{1}{r\sin\vartheta}\frac{\partial\eta}{\partial\varphi})^2]^{\frac{1}{2}}$ and $\frac{\partial\phi}{\partial n}|_{r=\rho+\eta(\vartheta,\varphi)}$ is the normal derivative at the surface. We also use the notation $dA_r = r^2\sin\theta d\theta d\varphi$ with $r = \rho + \eta(\vartheta,\varphi)$. The quantity RdA_r is the area element of the water surface.

We introduce the Dirichlet-Neumann ("flux") operator $G(\eta)$ by

$$G(\eta)\Phi = R\left.\frac{\partial\phi}{\partial n}\right|_{r=\rho+\eta(\vartheta,\varphi)} \tag{3.4}$$

and we can write the Hamiltonian as

$$H = \frac{1}{2}\int_{S^2}\Phi G(\eta)\Phi dA_r + \frac{1}{2}\int_{S^2}(-K)dA_r. \tag{3.5}$$

The Poisson bracket [,] of (3.1) is defined by

$$[F, G] = \int_{S^2}\left(\frac{\rho+\eta}{\rho}\right)^{-2}\left(\frac{\delta F}{\delta\eta}\frac{\delta G}{\delta\Phi} - \frac{\delta G}{\delta\eta}\frac{\delta F}{\delta\Phi}\right)\rho^2 dA_1. \tag{3.6}$$

The variational derivative $\frac{\delta F}{\delta x}$ is defined by $\langle\frac{\delta F}{\delta x}, \delta x\rangle = F'$, where F' is the Frechet derivative of F with respect to x, and $\langle\, ,\,\rangle$ is the L^2 inner product on S^2 (with radius ρ). From (3.6) the equations of motion (3.1) become

$$\eta_t = \left(\frac{\rho+\eta}{\rho}\right)^{-2}\frac{\delta H}{\delta\Phi}, \qquad \Phi_t = -\left(\frac{\rho+\eta}{\rho}\right)^{-2}\frac{\delta H}{\delta\eta}. \tag{3.7}$$

It is easy to see that the above bracket indeed satisfies the axioms of Poisson brackets (for the formalism of Poisson brackets for fields see [D], ch. 1). The only axiom that is not immediate is the Jacobi identity, which can be verified by a direct computation.

Proposition 3.1. *Hamilton's equations (3.1) and the original equations of motion* (2.2), *(2.3) are equivalent.*

Proof. We note that $\frac{\delta U}{\delta\Phi} = 0$ and

$$K(\Phi + \delta\Phi) = \frac{1}{2}\int_{S^2}\Phi G(\eta)\delta\Phi\, dA_r + \frac{1}{2}\int_{S^2}\delta\Phi G(\eta)\Phi\, dA_r + K(\Phi) + o(\delta\Phi)$$

so that $\frac{\delta K}{\delta\Phi} = (1 + \frac{\eta}{\rho})^2 G(\eta)\Phi$, and, therefore, Hamilton's equation for η_t is

$$\eta_t = G(\eta)\Phi = R\frac{\partial\phi}{\partial n}|_{r=\rho+\eta(x,y)} = \nabla\phi\cdot\left[1, -\frac{1}{r}\frac{\partial\eta}{\partial\vartheta}, -\frac{1}{r\sin\vartheta}\frac{\partial\eta}{\partial\varphi}\right], \tag{3.8}$$

which is (2.2). For (2.3),

$$\frac{\delta U}{\delta\eta} = -\left(1 + \frac{\eta}{\rho}\right)^2\frac{K}{\rho+\eta}$$

and

$$K(\eta+\delta\eta) = \frac{1}{2}\int_{S^2}|\nabla\phi|^2\delta\eta\, dA + \frac{1}{2}\int_{S^2}\int_{r=b}^{r=\rho+\eta(\vartheta,\varphi)}|\nabla\phi(\eta+\delta\eta)|^2\, dV + o(\delta\eta). \tag{3.9}$$

Integrating by parts, the second integral is

$$-\int_{S^2}\int_{r=h}^{r=\rho+\eta(\vartheta,\varphi)} \nabla\phi\nabla\left(\frac{\partial\phi}{\partial r}\delta\eta\right) dV + K(\eta) + o(\delta^2\eta)$$

$$= -\int_{S^2} \frac{\partial\phi}{\partial r}\delta\eta\frac{\partial\phi}{\partial n} R\,dA_r + K(\eta) + o(\delta^2\eta)$$

$$= -\int_{S^2} \frac{\partial\phi}{\partial r}\eta_t\delta\eta\,dA_r + K(\eta) + o(\delta\eta) \tag{3.10}$$

using Green's identity and the equation for η_t. From (3.6), (3.7), (3.10) we obtain

$$-\left(\frac{\rho+\eta}{\rho}\right)^{-2}\frac{\delta H}{\delta\eta} = -\frac{1}{2}|\nabla\phi|^2 - g\eta + \eta_t\frac{\partial\phi}{\partial r}. \tag{3.11}$$

But we also have $\frac{\partial\Phi}{\partial t} = \frac{\partial\phi}{\partial t} + \frac{\partial\eta}{\partial t}\frac{\partial\phi}{\partial r}|_{r=\rho+\eta(\vartheta,\varphi)}$, so that by (3.9), $\frac{\partial\Phi}{\partial t} = -\left(\frac{\rho+\eta}{\rho}\right)^{-2}\frac{\delta H}{\delta\eta}$ is equivalent to (2.3). □

Remark 3.1. A comparison of the above derivation of the Hamiltonian structure of water wave equations on the sphere with the derivations of the analogous result in Euclidean space ([Z], also [BO]) will reveal that the adoption of the nonstandard bracket was needed because of the form of the surface element in polar coordinates. The departure of our bracket from the standard one is small—in a sense that will be made precise later—but not small enough to be discarded.

It is possible to make a change of variables in such a way that the bracket becomes the standard one. (We will refer to this process as "diagonalising" the bracket.) In particular, choosing a transformation $f(\eta,\Phi) = (\tilde\eta,\tilde\Phi)$ defined by

$$\tilde\eta = \eta\left(1+\frac{\eta}{2\rho}\right), \qquad \tilde\Phi = \Phi\left(1+\frac{\eta}{\rho}\right), \tag{3.12}$$

the bracket is diagonalised: $Df\cdot A\cdot Df^T = J$, where A, J are the respective cosymplectic forms of our bracket and the standard bracket. The transformation of (3.12) is invertible in the range of interest $\sup|\eta| < h$.

4. Applicability of the Model to Physical Problems

The main physical limitation of our model is the potential flow assumption. The absence of vorticity and angular momentum in the rest frame has the consequence that the model can not describe phenomena such as Rossby waves or the bulging of the equator observed in rotating liquids. Thus our model isolates the effects of gravitation. We can also add to our formalism the apparent motions due to the rotation of the observer's reference frame, as well as another gravitational effect, namely the tides. Also, the model can be useful for studying phenomena involving short waves, in particular energy transport by wind-generated ocean waves.

First we consider the water wave equations in a frame rotating with angular velocity Ω around the z-axis ($\vartheta = 0$). We introduce the rotating frame canonical variables $u_\Omega = (\eta_\Omega, \Phi_\Omega)$ related to the rest frame variables $u = (\eta, \Phi)$ by $u_\Omega = A_\Omega^{-1}(t)u$, where $A_\Omega(t)$ acts on functions defined on the sphere by

$$A_\Omega(t)f(\vartheta, \varphi) = f(\vartheta, \varphi + \Omega t).$$

The flow $A_\Omega(t)$ is generated by

$$f_t = [f, \Omega L_z],$$

where [,] is the Poisson bracket introduced previously and L_z, the angular momentum, is

$$L_z = \int_{S^2}\int_{r=b}^{r=\rho+\eta(\vartheta,\varphi)} (\vec{r} \times \nabla\phi) \cdot \hat{z}\, dV = \int_{S^2} \Phi \frac{\partial \eta}{\partial \varphi} dA_r. \tag{4.1}$$

Note that $[H, L_z] = 0$ so that if the rest frame variables evolve under $u_t = [u, H]$ the rotating frame observables evolve under

$$\frac{\partial \eta_\Omega}{\partial t} = [\eta_\Omega, H - \Omega L_z], \qquad \frac{\partial \Phi_\Omega}{\partial t} = [\Phi_\Omega, H - \Omega L_z]. \tag{4.2}$$

Also note that the rotating frame velocity u_Ω is given by $u_\Omega = \nabla\phi_\Omega + U_\Omega$ where ϕ_Ω is the solution of the Neumann problem (i.e., (2.1), (2.4) for ϕ_Ω) with Φ_Ω as the boundary value at the surface, and U_Ω is the velocity of rigid rotation with angular velocity Ω. A Hamiltonian formalism for general Euler free boundary flows is given in [LMMR]. There the authors use the unique decomposition of divergenceless velocity field u to $w + \nabla\phi$, with w divergenceless and tangent to the boundary, to write the canonical theory for w, the surface potential and the shape of the boundary. However, the description of the fluid motion by functions of two variables is now lost.

Tides result from the gravitational attraction of another body, in particular, the acceleration of points on the surface of the sphere relative to the acceleration of the center of the sphere. The effect of tides is described by a tidal potential, which can be added to the equation for the evolution of the surface hydrodynamic potential. The tidal potential W at the surface of the fluid layer has, in lowest order in the distance to the attracting body, the form (see [P])

$$W = AP_2(\zeta), \tag{4.3}$$

where $P_2(\zeta)$ is the second Legendre function and A is a physical constant. If we consider a point x on the surface of the layer, the point O at the center of the sphere and the point S at the center of the distant star, ζ is the angle between the lines Ox and OS. The tidal potential can be expressed as a function of the polar and azimuth angles ϑ and φ of the point x and of the polar and azimuth angles δ' and T' of the external body

$$W = A[P_2(\vartheta)P_2(\delta') + \frac{3}{4}\sin 2\varphi \sin 2\delta' \cos(T' + \varphi) + \frac{3}{4}\cos^2\varphi \cos^2\delta' \cos 2(T' + \varphi)]. \tag{4.4}$$

If the sphere rotates with angular velocity Ω, then $T' = \Omega t$, and, similarly, we can consider variations in time of δ'. We can then use the rotating frame formalism of equations (4.2) and the tidal potential to model the effects of the periodic variation of the tidal forces. Notice that in (4.3) we can write $P_2(\zeta) = Y_2^0(\zeta, \varpi)$ where ϖ is the angle around the axis OS. In the second form, (4.4), we are using the terrestrial system of axes and angles. Since the two systems are related by a time-dependent rotation, the tidal potential in the terrestrial system can be written as a time-dependent linear combination of the spherical harmonics Y_l^m with $l = 2$, $m = -2, -1, \ldots, 2$. Thus the potential tide can be added to the spectral form of the equations of motion in a straightforward way provided that we have the time dependence of the coefficient of each spherical harmonic. Since the harmonics are eigenfunctions of the linearised problem, a simple way to mimic the varying tidal forces is parametric excitation (variation of one of the parameters of the problem) at frequencies that are at resonance with the $l = 2$ harmonics. In fact, the phenomenon of parametric excitation of nonlinear waves (see, for instance, [FS] for water waves) is of independent interest.

One other area of possible application of our model is in the transport of energy by wind-driven sea waves in the ocean. The typical wavelength of these waves is very small compared to the global scales, and the effect of the Coriolis force is minimal. We can include the effect of wind by adding to the equations of motion a pressure (gradient) term computed from given velocity profiles for the motion of the air masses above the sea. Since the approximations used in the following chapters are uniform in spatial wave number of the quantities involved, we expect that the response of short waves will yield rather reliable information on the long-range transport of wind energy by the sea waves.

5. The Flux Operator

We now describe the Dirichlet-Neumann operator $G(\eta)$ as a function of η. It will be assumed that $G(\eta)$ can be expanded around $\eta = 0$ so that we can write $G(\eta) = \sum_{i=0}^{\infty} G_i(\eta)$, the $G_i(\eta)$ being homogeneous of order i in η. Our task will be to calculate the $G_i(\eta)$. Clearly, from (3.5), the expansion of $G(\eta)$ will give us an expansion of the Hamiltonian $H = H_0 + H_1 + \cdots$ with $H_0 = \frac{1}{2}\int \Phi G_0(\eta)\Phi + \frac{1}{2}\int g\eta^2$, $H_1 = \frac{1}{2}\int \Phi G_1(\eta)\Phi$ and so forth. As we will show, it is possible to compute recursively the $G_i(\eta)$ in a way very similar to that in [CG], [CS] and [GAS]. Once the $G_i(\eta)$ are computed explicitly, the Hamiltonian of (3.5) can be systematically expanded in powers of η. If we use as the Hamiltonian some truncation of this expansion, the resulting model can be considered as a small-amplitude approximation of the water waves problem. This formalism, therefore, provides us with a systematic way to produce increasingly more refined small-amplitude expansions that can be computed rather effectively. We remark, however, that the expansion here is formal. We leave the problem of convergence and the choice of appropriate spaces for η and Φ open. For waves on the line the question has been addressed (in a more general context) in [CM].

To calculate the $G_i(\eta)$ we first look for harmonic functions ϕ_γ of the form

$$\phi_\gamma(r, \vartheta, \varphi) = u_\gamma(r) Y_\gamma(\vartheta, \varphi).$$

The functions $Y_\gamma(\vartheta, \varphi)$ are the spherical harmonics, indexed by $\gamma = [l, m]$ with l a positive integer being the total angular momentum number, and $m = -l, -l+1, \ldots, l$

the azimuthal angular momentum number. The condition $\Delta\phi_\gamma = 0$ implies that

$$r^2 u_\gamma'' + 2r u_\gamma' - l(l+1)u_\gamma = 0. \tag{5.1}$$

The only solution of (5.1) (up to a multiplicative constant) that also satisfies $u_\gamma'(b) = 0$ is

$$u_\gamma(r) = (l+1)\left(\frac{r}{b}\right)^l + l\left(\frac{b}{r}\right)^{l+1}. \tag{5.2}$$

Since the $\phi_\gamma = u_\gamma Y_\gamma$, as above, are harmonic and satisfy $\frac{\partial \phi_\gamma}{\partial r} = 0$ at $r = b$, from the definition of $G(\eta)$, (3.4), we have that for every index γ

$$G(\eta)\phi_\gamma(\vartheta, \varphi, \rho + \eta) = \nabla\phi_\gamma(\vartheta, \varphi, \rho + \eta) \cdot \left[1, -\frac{1}{r}\frac{\partial \eta}{\partial \vartheta}, -\frac{1}{r \sin\vartheta}\frac{\partial \eta}{\partial \varphi}\right] \tag{5.3}$$

or

$$\begin{aligned}\sum_{i=0}^{\infty} G_i(\eta)u_\gamma(\rho+\eta)Y_\gamma &= u_\gamma'(\rho+\eta)Y_\gamma \\ &\quad - \frac{\partial \eta}{\partial \vartheta}\frac{u_\gamma(\rho+\eta)}{(\rho+\eta)^2}\frac{\partial Y_\gamma}{\partial \vartheta} - \frac{\partial \eta}{\partial \varphi}\frac{u_\gamma(\rho+\eta)}{(\rho+\eta)^2 \sin^2\vartheta}\frac{\partial Y_\gamma}{\partial \varphi}.\end{aligned} \tag{5.4}$$

Expanding $u_\gamma(r)$ and $\frac{1}{r^2}$ around $\eta = 0$ (i.e., $r = \rho$) and matching powers in η, we obtain, at order 0,

$$u_\gamma'(\rho)Y_\gamma = G_0(\eta)u_\gamma(\rho)Y_\gamma$$

and hence

$$G_0(\eta)Y_\gamma = \frac{1}{u_\gamma(\rho)}u_\gamma'(\rho)Y_\gamma. \tag{5.5}$$

At order 1 we have

$$\eta u_\gamma''(\rho)Y_\gamma - \frac{1}{\rho^2}u_\gamma(\rho)\left[\eta_\vartheta \frac{\partial Y_\gamma}{\partial \vartheta} + \frac{1}{\sin^2\vartheta}\eta_\varphi \frac{\partial Y_\gamma}{\partial \varphi}\right] = G_0(\eta)\eta u_\gamma'(\rho)Y_\gamma + G_1(\eta)u_\gamma(\rho)Y_\gamma$$

and hence

$$\begin{aligned}G_1(\eta)Y_\gamma &= \frac{1}{u_\gamma(\rho)}\left[\eta u_\gamma''(\rho)Y_\gamma - \frac{1}{\rho^2}u_\gamma(\rho)\right. \\ &\quad \left. \times\left[\eta_\vartheta \frac{\partial Y_\gamma}{\partial \vartheta} + \frac{1}{\sin^2\vartheta}\eta_\varphi \frac{\partial Y_\gamma}{\partial \varphi}\right] - G_0(\eta)\eta u_\gamma'(\rho)Y_\gamma\right].\end{aligned} \tag{5.6}$$

In general, at order k we have

$$\begin{aligned}&\frac{1}{k!}\eta^k u_\gamma^{(k+1)}(\rho)Y_\gamma - \frac{1}{(k-1)!}\eta^{k-1}\left(\frac{u_\gamma}{r^2}\right)^{(k-1)}(\rho)\left[\eta_\vartheta \frac{\partial Y_\gamma}{\partial \vartheta} + \frac{1}{\sin^2\vartheta}\eta_\varphi \frac{\partial Y_\gamma}{\partial \varphi}\right] \\ &\quad = \left[G_0(\eta)\frac{1}{k!}\eta^k u_\gamma^{(k)}(\rho) + G_1(\eta)\frac{1}{(k-1)!}\eta^{k-1}u_\gamma^{(k-1)}(\rho) + \cdots + G_k(\eta)u_\gamma(\rho)\right]Y_\gamma\end{aligned} \tag{5.7}$$

($u_\gamma^{(k)} = k$-th derivative), and we can obtain $G_k(\eta)$ recursively in terms of the $G_i(\eta)$ with $i < k$. The above formulas determine the $G_i(\eta)$ as an operator in a, for the moment, unspecified space of functions in the sphere. Note that the $G_i(\eta)$ are given as Fourier multipliers, and it is clear that they are not local operators.

6. Scales and Dimensionless Variables

It will be advantageous to introduce several dimensionless quantities that take into account the scales of the problem. The relevant dimensionless quantities are (i) the ratio $\epsilon = \frac{A}{h}$ of the typical amplitude A of the waves to the depth h of the fluid layer, (ii) the ratio $\beta = \frac{h}{b}$ of the depth over the radius b of the planet, and in the rotating frame we also have (iii) $R = \frac{\Omega}{\omega_0}$ where Ω is the angular velocity of the frame and ω_0 is a typical frequency of linearised gravity waves in the rest frame. Here we take the reference frequency to be $\omega_0 = \frac{\sqrt{gh}}{b}$, which turns out to be the phase velocity of linearised shallow water waves.

The dimensionless variables η^*, Φ^* and t^* are introduced by

$$\eta^* = \frac{\eta}{A}, \qquad \Phi^* = \frac{\omega_0 \Phi}{gA}, \qquad t^* = \omega_0 t. \tag{6.1}$$

For the Dirichlet-Neumann operator, we observe from the formulas in the previous section that in each $G_k(\eta)$ we can factor out a term $\frac{1}{b^{k+1}}$ arising from the $u_Y{}^{(n)}$ so that

$$G_k(\eta) Y_\gamma = \frac{1}{b^{k+1}} G_k^*(\epsilon h \eta^*) Y_\gamma = \frac{1}{b} (\epsilon\beta)^k G_k^*(\eta^*) Y_\gamma. \tag{6.2}$$

The $G_k^*(\eta^*)$ defined by the first equality in (6.2) are dimensionless: they depend only on η^* and β. The Hamiltonian can be written as

$$H = (\epsilon\beta)^2 b^2 \frac{1}{2} \int_{S^2} \left[g\Phi^* \frac{1}{\beta} \sum_k (\epsilon\beta)^k G_k^*(\eta^*) \Phi^* + g(\eta^*)^2 \right] dA_\rho. \tag{6.3}$$

Note that g is the acceleration of gravity at $r = \rho$. Defining $\hat{H}$ by

$$\hat{H} = \frac{1}{2} \int_{S^2} \left[\Phi^* \frac{1}{\beta} \sum_k (\epsilon\beta)^k G_k^*(\eta^*) \Phi^* + (\eta^*)^2 \right] \rho^2 \, dA_1.$$

Hamilton's equations (3.7) become

$$\eta^*_{t^*} = (1 + \epsilon\beta\eta^*)^{-2} \frac{\delta \hat{H}}{\delta \Phi^*}, \tag{6.4}$$

$$\Phi^*_{t^*} = -(1 + \epsilon\beta\eta^*)^{-2} \frac{\delta \hat{H}}{\delta \eta^*}. \tag{6.5}$$

In the rotating frame we can use the time-scale $t^\dagger = Rt^*$ and Hamilton's equations become

$$\eta^*_{t^\dagger} = (1 + \epsilon\beta\eta^*)^{-2} \left[-\frac{\delta \hat{L}}{\delta \Phi^*} + \frac{1}{R} \frac{\delta \hat{H}}{\delta \Phi^*} \right] \tag{6.6}$$

and

$$\Phi^*_{t} = -(1+\epsilon\beta\eta^*)^{-2}\left[-\frac{\delta\hat{L}}{\delta\eta^*}+\frac{1}{R}\frac{\delta\hat{H}}{\delta\eta^*}\right] \tag{6.7}$$

with $\hat{L}_z = \int_{S^2}\Phi^*\frac{\partial\eta^*}{\partial\vartheta}dA_\rho$.

Note that the factor $(1+\epsilon\beta\eta^*)^{-2}$ which appears in the nonstandard bracket differs from unity in terms which are first order in the dimensionless variables ϵ and β. Hence, the difference between the standard bracket and the nonstandard one has to be considered in expansions in the dimensionless quantities of this order.

The parameter β is the analog of the depth to wavelength ratio in Euclidean space. The small β regime gives us an analog of the "shallow water" regime (e.g., see [Wi]), in which the $G_k^*(\eta^*)$ can be expanded in β. Note that the $G_k^*(\eta^*)$ are already multiplied by β^k in the Hamiltonian so that, for instance, in the $\beta\to 0$ limit (with ϵ nonzero) the evolution becomes linear. It is also possible to prescribe a relation between the parameters ϵ and β. For example, setting $\epsilon=\beta^2=\mu$ would lead to an analog of the Boussinesq regime in $\mathbf{R}^2$.

In what follows we will be concerned with the small ϵ regime with β arbitrary; in particular, we will use the $G_i(\eta)$ calculated previously to write the equations of motion to second order in ϵ. From the dimensional analysis we have to take into account terms arising from the nonstandard bracket. It is advantageous to present the Hamiltonian in the variables $\tilde{\eta}$, $\tilde{\Phi}$ in which the bracket is diagonal. Note that since ϵ is small the normalizations of the new variables can be taken to be the same as that of the original variables. The $O(\epsilon^2)$ Hamiltonian $H=H_0+H_1+H_2$ in $\tilde{\eta}$, $\tilde{\Phi}$ (unnormalized) is

$$H_0 = \frac{1}{2}\int_{S^2}\tilde{\Phi}G_0(\tilde{\eta})\tilde{\Phi}\rho^2\,dA_1 + \frac{1}{2}\int_{S^2}(\tilde{\eta})^2\rho^2\,dA_1, \tag{6.8}$$

$$H_1 = -\frac{1}{2}\int_{S^2}\tilde{\Phi}G_0(\tilde{\eta})(\tilde{\Phi}\tilde{\eta})\rho\,dA_1+\frac{1}{2}\int_{S^2}\tilde{\Phi}[G_0(\tilde{\eta})\tilde{\Phi}]\tilde{\eta}\rho\,dA_1+\frac{1}{2}\int_{S^2}\tilde{\Phi}G_1(\tilde{\eta})\tilde{\Phi}\rho^2\,dA_1 \tag{6.9}$$

and

$$\begin{aligned} H_2 =\ & \frac{3}{4}\int_{S^2}\tilde{\Phi}G_0(\tilde{\eta})(\tilde{\Phi}\tilde{\eta}^2)\,dA_1 - \frac{1}{2}\int_{S^2}\tilde{\Phi}[G_0(\tilde{\eta})(\Phi\tilde{\eta})]\tilde{\eta}\,dA_1 \\ & -\frac{1}{4}\int_{S^2}\tilde{\Phi}[G_0(\tilde{\eta})\tilde{\Phi}]\tilde{\eta}^2\,dA_1 - \frac{1}{2}\int_{S^2}\tilde{\Phi}G_1(\tilde{\eta})(\tilde{\Phi}\tilde{\eta})\rho\,dA_1 - \frac{1}{4}\int_{S^2}\tilde{\Phi}G_1(\tilde{\eta}^2)\tilde{\Phi}\rho\,dA_1 \\ & +\frac{1}{2}\int_{S^2}\tilde{\Phi}[G_1(\tilde{\eta})\tilde{\Phi}]\tilde{\eta}\rho\,dA_1 + \frac{1}{2}\int_{S^2}\tilde{\Phi}G_2(\tilde{\eta})\tilde{\Phi}\rho^2\,dA_1. \end{aligned} \tag{6.10}$$

If we use the spectral variables η_γ, Φ_γ (we drop the tilde from the notation) defined by $\eta=\sum_\gamma \eta_\gamma Y_\gamma$, $\Phi=\sum_\gamma \Phi_\gamma Y_\gamma$ with $\eta^*_{[l,m]}=\eta_{[l,-m]}$, $\Phi^*_{[l,m]}=\Phi_{[l,-m]}$, the Poisson bracket (which is now diagonalised) becomes

$$[f,g] = \sum_\gamma\left(\frac{\partial f}{\partial\eta_\gamma}\frac{\partial g}{\partial\Phi^*_\gamma} - \frac{\partial f}{\partial\Phi_\gamma}\frac{\partial g}{\partial\eta^*_\gamma}\right) \tag{6.11}$$

and Hamilton's equations are

$$\dot{\eta}_\gamma = \frac{\partial H}{\partial \Phi^*_\gamma}, \qquad \dot{\Phi}_\gamma = -\frac{\partial H}{\partial \eta^*_\gamma}. \tag{6.12}$$

Using the $G_i(\eta)$ from Section 5, the 4-wave Hamiltonian $H = H_0 + H_1 + H_2$ is

$$H_0 = \frac{1}{2}\rho^2 \sum_\gamma \frac{u_\gamma{}'(\rho)}{u_\gamma(\rho)} \Phi_\gamma \Phi^*_\gamma + \frac{1}{2}\rho^2 \sum_\gamma g \eta_\gamma \eta^*_\gamma, \tag{6.13}$$

$$\begin{aligned} H_1 &= \frac{1}{2}\rho^2 \sum_{\gamma_1,\gamma_2,\gamma_3} \left(\frac{u''_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)} - \frac{u'_{\gamma_1}(\rho)}{u_{\gamma_1}(\rho)} \frac{u'_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)} \right) \Phi_{\gamma_1} \Phi_{\gamma_2} \eta_{\gamma_3} \int Y_{\gamma_1} Y_{\gamma_2} Y_{\gamma_3} \\ &+ \frac{1}{2} \sum_{\gamma_1,\gamma_2,\gamma_3} \Phi_{\gamma_1} \Phi_{\gamma_2} \eta_{\gamma_3} \int Y_{\gamma_1} \nabla Y_{\gamma_2} \cdot \nabla Y_{\gamma_3}, \end{aligned} \tag{6.14}$$

$$\begin{aligned} H_2 = \frac{1}{2} \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} & \Phi_{\gamma_1} \Phi_{\gamma_2} \eta_{\gamma_3} \eta_{\gamma_4} \cdot \Bigg\{ \sum_\gamma \int Y_{\gamma_1} Y_{\gamma_4} Y_\gamma \int Y_{\gamma_2} Y_{\gamma_3} Y^*_\gamma \\ & \cdot \Bigg[-\rho \frac{u''_\gamma(\rho)}{u_\gamma(\rho)} + \frac{\rho}{2} \frac{u''_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)} + \frac{\rho}{2} \frac{u'_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)} \frac{u'_{\gamma_1}(\rho)}{u_{\gamma_1}(\rho)} + \frac{\rho^2}{2} \left(\frac{u'''_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)} - \frac{u''_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)} \frac{u'_{\gamma_1}(\rho)}{u_{\gamma_1}(\rho)} \right) \\ & - \frac{\rho^2}{2} \frac{u'_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)} \frac{u''_\gamma(\rho)}{u_\gamma(\rho)} + \frac{\rho^2}{2} \frac{u'_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)} \frac{u'_{\gamma_1}(\rho)}{u_{\gamma_1}(\rho)} \frac{u'_\gamma(\rho)}{u_\gamma(\rho)} \Bigg] \\ & + \left(\frac{1}{\rho} + \frac{u'_{\gamma_1}(\rho)}{u_{\gamma_1}(\rho)} \right) \sum_\gamma \int Y_{\gamma_1} \nabla Y_{\gamma_4} \cdot \nabla Y_\gamma \int Y_{\gamma_2} Y_{\gamma_3} Y^*_\gamma \\ & + \left(\frac{2}{\rho} - \frac{u'_{\gamma_2}(\rho)}{u_{\gamma_2}(\rho)} \right) \int Y_{\gamma_1} \nabla Y_{\gamma_2} \cdot \nabla Y_{\gamma_3} Y_{\gamma_4} \Bigg\}. \end{aligned} \tag{6.15}$$

In the rotating frame the quadratic Hamiltonian H_0 is modified to

$$H^\Omega_0 = \frac{1}{2R}\rho^2 \sum_\gamma \frac{u_\gamma{}'(\rho)}{u_\gamma(\rho)} \Phi_\gamma \Phi^*_\gamma + \frac{1}{2R}\rho^2 \sum_\gamma g \eta_\gamma \eta^*_\gamma - i\rho^2 \sum_\gamma m_\gamma \Phi_\gamma \eta^*_\gamma \tag{6.16}$$

while the cubic and quartic Hamiltonians are the same as above.

The dispersion relations implied by H_0 and H^Ω_0 are

$$\omega^2(\gamma) = \frac{u_\gamma{}'(\rho)}{u_\gamma(\rho)} \tag{6.17}$$

and

$$\omega_\Omega(\gamma) = m_\gamma + \frac{1}{R} \left(\frac{u_\gamma{}'(\rho)}{u_\gamma(\rho)} \right)^{\frac{1}{2}} \tag{6.18}$$

respectively.

From (5.17) we can establish that $\omega(\gamma) = \omega(l)$ viewed as a function of the positive real variable l is smooth, monotonically increasing with $\omega(0) = 0$. In addition, $\omega''(l) < 0$. The function $\omega(l)$ also depends on β, and note that in the limit $\beta \to 0$, $\omega''(l) \to 0$ (but not uniformly in l). For fixed β it is easy to see that for l large $\omega(l) \to \sqrt{l}$, which is the deep water dispersion relation.

We now introduce some extra notation that will be useful in the next section. We define the variables a_γ, a_γ^* by

$$\eta_\gamma = \frac{\sqrt{2}}{2}\sqrt{\frac{\omega_\gamma}{g}}(a_\gamma + a^*_{\gamma^-}), \qquad \Phi_\gamma = -i\frac{\sqrt{2}}{2}\sqrt{\frac{g}{\omega_\gamma}}(a_\gamma - a^*_{\gamma^-}), \tag{6.19}$$

where if $\gamma = [l, m]$ then $\gamma^- = [l, -m]$. Hamilton's equations become

$$\dot{a}_\gamma = -i\frac{\partial H}{\partial a^*_\gamma}.$$

The Hamiltonian in these variables can be readily evaluated using the formulas above: the quadratic Hamiltonian is $H_0 = \sum_\gamma \omega_\gamma a_\gamma a^*_\gamma$ with ω_γ given by the dispersion relation. If we write

$$H_1 = \sum_{\gamma_1,\gamma_2,\gamma_3} \Phi_{\gamma_1}\Phi_{\gamma_2}\eta_{\gamma_3} I_{\gamma_1\gamma_2\gamma_3}, \qquad H_2 = \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} \Phi_{\gamma_1}\Phi_{\gamma_2}\eta_{\gamma_3}\eta_{\gamma_4} I_{\gamma_1\gamma_2\gamma_3\gamma_4}$$

with the $I_{\gamma_1\gamma_2\gamma_3}$, $I_{\gamma_1,\gamma_2\gamma_3\gamma_4}$ the coefficients appearing in (6.14) and (6.15), we also write

$$H_1 = \sum_{\gamma_1,\gamma_2,\gamma_3} (A_{\gamma_1\gamma_2\gamma_3} a_{\gamma_1} a_{\gamma_2} a_{\gamma_3} + A_{\gamma_1\gamma_2\gamma_3^-} a_{\gamma_1} a_{\gamma_2} a^*_{\gamma_3}) + c.c.,$$

$$H_2 = \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4} (B_{\gamma_1\gamma_2\gamma_3\gamma_4} a_{\gamma_1} a_{\gamma_2} a_{\gamma_3} a_{\gamma_4} + B_{\gamma_1\gamma_2\gamma_3\gamma_4^-} a_{\gamma_1} a_{\gamma_2} a_{\gamma_3} a^*_{\gamma_4}$$
$$+ B_{\gamma_1\gamma_2\gamma_3^-\gamma_4^-} a_{\gamma_1} a_{\gamma_2} a^*_{\gamma_3} a^*_{\gamma_4}) + c.c.$$

with the coefficients given by

$$A_{\gamma_1\gamma_2\gamma_3} = N_{\gamma_1\gamma_2\gamma_3} I_{\gamma_1\gamma_2\gamma_3},$$

$$A_{\gamma_1\gamma_2\gamma_3^-} = N_{\gamma_1\gamma_2\gamma_3}(-I_{\gamma_3^-\gamma_1\gamma_2} - I_{\gamma_1\gamma_3^-\gamma_2} + I_{\gamma_1\gamma_2\gamma_3^-}),$$

$$B_{\gamma_1\gamma_2\gamma_3\gamma_4} = N_{\gamma_1\gamma_2\gamma_3\gamma_4} I_{\gamma_1\gamma_2\gamma_3\gamma_4},$$

$$B_{\gamma_1\gamma_2\gamma_3\gamma_4^-} = N_{\gamma_1\gamma_2\gamma_3\gamma_4}(-I_{\gamma_4^-\gamma_1\gamma_2\gamma_3} - I_{\gamma_1\gamma_4^-\gamma_2\gamma_3} - I_{\gamma_1\gamma_2\gamma_4^-\gamma_3} + I_{\gamma_1\gamma_2\gamma_3\gamma_4^-}),$$

$$B_{\gamma_1\gamma_2\gamma_3^-\gamma_4^-} = N_{\gamma_1\gamma_2\gamma_3\gamma_4}(-I_{\gamma_3^-\gamma_1\gamma_2\gamma_4^-} - I_{\gamma_1\gamma_3^-\gamma_2\gamma_4^-} + I_{\gamma_1\gamma_2\gamma_3^-\gamma_4^-}),$$

$$N_{\gamma_1\gamma_2\gamma_3} = -\frac{\sqrt{2}}{2}\sqrt{\frac{\omega_{\gamma_3}}{\omega_{\gamma_1}\omega_{\gamma_2}}}, \qquad N_{\gamma_1\gamma_2\gamma_3\gamma_4} = -\frac{1}{4}\sqrt{\frac{\omega_{\gamma_3}\omega_{\gamma_4}}{\omega_{\gamma_1}\omega_{\gamma_2}}}.$$

Alternatively, define $z^k = a_1^{k_1} a_2^{k_2} \ldots$, $\overline{z}^{\overline{k}} = (a_1^*)^{\overline{k}_1} (a_2^*)^{\overline{k}_2} \ldots$ where the $k_i, \overline{k}_i$ are nonnegative integers and the subscripts i label (enumerate) the modes γ_i. Also let $|k| = k_1 + k_2 + \cdots$, $|\overline{k}| = \overline{k}_1 + \overline{k}_2 + \cdots$. We can write

$$H_1 = \sum_{\substack{k,\overline{k} \\ |k|+|\overline{k}|=3}} A_{k,\overline{k}} z^k \overline{z}^{\overline{k}}, \qquad H_2 = \sum_{\substack{k,\overline{k} \\ |k|+|\overline{k}|=4}} B_{k,\overline{k}} z^k \overline{z}^{\overline{k}}. \tag{6.20}$$

The coefficients $A_{k,\overline{k}}$ are

$$A_{k_{\gamma_i} k_{\gamma_j} k_{\gamma_l}} = \sum_{\text{perm.} i,j,l} A_{\gamma_i \gamma_j \gamma_l}, \qquad A_{k_{\gamma_i} k_{\gamma_j} \overline{k}_{\gamma_l}} = \sum_{\text{perm.} i,j,l} A_{\gamma_i \gamma_j \overline{\gamma}_l}. \tag{6.21}$$

Similar formulas define the $B_{k,\overline{k}}$.

7. Normal Forms

We will now see that the cubic (3-wave) terms in the Hamiltonian can be eliminated by a formal canonical transformation. We will also try to eliminate the quartic (4-wave) terms, but it will turn out that some of them are resonant and can not be eliminated.

To construct canonical transformations, we will use the "Lie-series" method, which we now briefly review (see, for example [DF], [C]).

Consider a manifold M with a Poisson bracket J defined on $C^\infty(M)$ (we write $J(f,g) = [f,g]$). To every function $g \in C^\infty(M)$ we associate a map $\mathrm{Ad}_g\colon C^\infty(M) \to C^\infty(M)$ defined by $\mathrm{Ad}_g f = [g,f]$. We also formally define $\exp \mathrm{Ad}_g$ by

$$(\exp \mathrm{Ad}_g) f = f + \sum_k \frac{1}{k!} (\mathrm{Ad}_g)^k = f + [g,f] + \frac{1}{2}[g,[g,f]] + \cdots. \tag{7.1}$$

We can check that for every function g for which the series makes sense, the map $\exp \mathrm{Ad}_g$ preserves the Poisson bracket structure on $C^\infty(M)$, i.e., $(\exp \mathrm{Ad}_g)[f,h] = [(\exp \mathrm{Ad}_g) f, (\exp \mathrm{Ad}_g) h]$ and, therefore, defines a (local) canonical transformation by acting on the components of the coordinate charts of M.

The above computations can be given some meaning beyond formal manipulation, depending on the interpretation of the series of (7.1). For example, the series converges for analytic functions, and then sometimes it may be extended to the whole space of smooth functions. Note also that when this can be done, $\exp(\mathrm{Ad}_g) f$ is the time-1 map of the Hamiltonian vector field of g acting on the function f. Alternatively, if g contains a small parameter, we may consider the series (7.1) as an asymptotic expansion in the small parameter. These considerations pertain to the finite-dimensional case. Additional problems arise in infinite-dimensional systems. As we mentioned before, in this paper we will present only the formal calculations and postpone the analytic discussion to future work.

In the present application, M is the span of the η_γ, Φ_γ, and J is the Poisson bracket given in (6.11). We want a function g such that $\exp \mathrm{Ad}_g$ eliminates the cubic terms from the Hamiltonian. In particular, let $g = \epsilon S_0$, $H = H_0 + \epsilon H_1 + \epsilon^2 H_2$, then

$$(\exp\epsilon\,\mathrm{Ad}_{S_0})H = H_0 + \epsilon(H_1 + [S_0, H_0]) + \epsilon^2(H_2 + [S_0, H_1] + \tfrac{1}{2}[S_0, [S_0, H_0]]) + o(\epsilon^2), \tag{7.2}$$

i.e., we want S_0 such that

$$H_1 + [S_0, H_0] = 0. \tag{7.3}$$

If we can solve (7.3) the new Hamiltonian becomes

$$H_{\text{new}} = H_0 + \epsilon^2\left(H_2 + \frac{1}{2}[S_0, H_1]\right) + o(\epsilon^2). \tag{7.4}$$

To solve (7.3) and calculate $[S_0, H_1]$, we will use the spectral variables a_γ, a^*_γ and the notation developed at the end of the previous section.

Proposition 7.1. *Equation (7.3) can be formally solved or, equivalently, the cubic terms of the Hamiltonian can be eliminated by a formal canonical transformation.*

Proof. The cubic Hamiltonian ϵH_1 is a sum of terms $\epsilon A_{k,\bar{k}} z^k \bar{z}^{\bar{k}}$ with $|k| + |\bar{k}| = 3$. Let $s_0 = \sigma_{k,\bar{k}} z^k \bar{z}^{\bar{k}}$, $|k| + |\bar{k}| = 3$. Using the derivation property of the bracket and induction, we have the formula

$$[z^\mu \bar{z}^{\bar{\mu}}, z^k \bar{z}^{\bar{k}}] = i\sum_j (\bar{\mu}_j k_j - \bar{k}_j \mu_j)\frac{1}{z_j \bar{z}_j} z^{\mu+k} \bar{z}^{\bar{\mu}+\bar{k}} \tag{7.5}$$

for arbitrary $\mu, \bar{\mu}, k, \bar{k}$. In particular, $[s_0, H_0] = -i\sigma_{k,\bar{k}}(\sum_i \omega_i(k_i - \bar{k}_i)) z^k \bar{z}^{\bar{k}}$. If we set

$$\sigma_{k,\bar{k}} = \frac{1}{i}\frac{A_{k,\bar{k}} z^k \bar{z}^{\bar{k}}}{\sum_i \omega_i(k_i - \bar{k}_i)} \tag{7.6}$$

the term $\epsilon A_{k,\bar{k}} z^k \bar{z}^{\bar{k}}$ is eliminated, provided that the "3-wave resonance" condition

$$A_{k,\bar{k}} \neq 0, \qquad \sum_i \omega_i(k_i - \bar{k}_i) = 0, \tag{7.7}$$

is not satisfied.

Letting $S_0 = -i\sum_{k,\bar{k}} \frac{A_{k,\bar{k}}}{\sum_i \omega_i(k_i - \bar{k}_i)} z^k \bar{z}^{\bar{k}}$, all the nonresonant terms are thus eliminated. Therefore, it is enough to show that (7.7) is never satisfied.

From (6.20) and (6.21) the resonances (7.7) occur if and only if

$$I_{\gamma_1\gamma_2\gamma_3} \neq 0, \qquad \omega_{\gamma_1} + \omega_{\gamma_2} + \omega_{\gamma_3} = 0, \tag{7.8}$$

$$I_{\gamma_1\gamma_2\gamma_3^-} \neq 0, \qquad \omega_{\gamma_1} + \omega_{\gamma_2} - \omega_{\gamma_3} = 0. \tag{7.9}$$

To show that none of the above equations are satisfied, we examine the dispersion ω_γ and the terms $I_{\gamma_1\gamma_2\gamma_3}$. First, since $\omega_\gamma > 0$, (7.8) cannot be satisfied. Also, $\frac{d\omega}{dl} > 0$ and $\frac{d^2\omega}{dl^2} < 0$, so that

$$\omega(l_3) = \omega(l_1) + \omega(l_2) \Rightarrow l_3 > l_1 + l_2. \tag{7.10}$$

On the other hand, from (6.14)

$$I_{\gamma_1\gamma_2\gamma_3} = b_{\gamma_1\gamma_2\gamma_3} \int Y_{\gamma_1} Y_{\gamma_2} Y_{\gamma_3} + c_{\gamma_1\gamma_2\gamma_3} \int Y_{\gamma_1} \nabla Y_{\gamma_2} \cdot \nabla Y_{\gamma_3} \tag{7.11}$$

with the $b_{\gamma_1\gamma_2\gamma_3}$, $c_{\gamma_1\gamma_2\gamma_3}$ determined by (6.14). Now, $Y_\gamma = Y_l^m(\vartheta, \varphi) = e^{im\varphi} P_l^{|m|}(\mu)$ with $\mu = \cos\vartheta$ and $P_l^m(\mu)$ the associated Legendre functions (for material related to spherical harmonics, we refer to [BL], [McR]). We have

$$\int Y_{\gamma_1} Y_{\gamma_2} Y_{\gamma_3} = \int_0^{2\pi} e^{i\left(\sum_{i=1}^{3} m_i\right)\varphi} \, d\varphi \int_{-1}^{1} P_{\gamma_1} P_{\gamma_2} P_{\gamma_3} \, d\mu$$

and

$$\int_{-1}^{1} P_{\gamma_1} P_{\gamma_2} P_{\gamma_3} d\mu \neq 0 \quad \text{only if} \quad |l_1 - l_2| \leq l_3 \leq l_1 + l_2 \quad \text{and} \quad l_1 + l_2 + l_3 = \text{even}.$$

Hence, when the frequency addition rule holds, $\int Y_{\gamma_1} Y_{\gamma_2} Y_{\gamma_3}$ vanishes. For the second integral of (7.11), we have

$$\int Y_{\gamma_1} \nabla Y_{\gamma_2} \cdot \nabla Y_{\gamma_3} = \delta_{123} \int_{-1}^{1} \left[(1-\mu^2) P_{\gamma_1} \frac{dP_{\gamma_2}}{d\mu} \frac{dP_{\gamma_3}}{d\mu} - \frac{m_2 m_3}{1-\mu^2} P_{\gamma_1} P_{\gamma_2} P_{\gamma_3} \right] d\mu$$

with $\delta_{123} = \int_0^{2\pi} e^{i\left(\sum_{i=1}^{3} m_i\right)\varphi} d\varphi$. Using

$$\frac{dP_l^{|m|}}{d\mu} = (1-\mu^2)^{1/2} P_l^{|m|+1} - |m|\mu(1-\mu^2)^{1/2} P_l^{|m|}$$

(note that $P_l^{|m|} = 0$ for $|m| > l$), we have

$$\begin{aligned}
&\int Y_{\gamma_1} \nabla Y_{\gamma_2} \cdot \nabla Y_{\gamma_3} \\
&= -\delta_{123} \int_{-1}^{1} \frac{\mu}{(1-\mu^2)^{1/2}} (|m_3| P_{l_1}^{|m_1|} P_{l_2}^{|m_2|+1} P_{l_3}^{|m_3|} - |m_2| P_{l_1}^{|m_1|} P_{l_2}^{|m_2|} P_{l_3}^{|m_3|+1}) d\mu \\
&\quad + \delta_{123} \int_{-1}^{1} (P_{l_1}^{|m_1|} P_{l_2}^{|m_2|+1} P_{l_3}^{|m_3|+1} - m_2 m_3 P_{l_1}^{|m_1|} P_{l_2}^{|m_2|} P_{l_3}^{|m_3|}) d\mu \\
&\quad - \delta_{123} \int_{-1}^{1} \frac{\mu}{(1-\mu^2)^{1/2}} (m_2 m_3 P_{l_1}^{|m_1|} P_{l_2}^{|m_2|} P_{l_3}^{|m_3|} - |m_2 m_3| P_{l_1}^{|m_1|} P_{l_2}^{|m_2|} P_{l_3}^{|m_3|}) d\mu.
\end{aligned}$$

The terms $\frac{\mu}{(1-\mu^2)^{1/2}} P_{l_i}^{|m_{\alpha_i}|+1}$ in the integral can be written as a linear combination of $P_{l_i}^{|m_{\alpha_i}|}$ and $P_{l_i}^{|m_{\alpha_i}|+2}$ (see [McR, p. 115]). This is also the case for the terms of the last integral, unless one of the m_2, m_3 is zero, in which case the integral is zero. Therefore, the triple integral $\int Y_{\gamma_1} \nabla Y_{\gamma_2} \cdot \nabla Y_{\gamma_3}$ can be expressed as a sum of

$$\int_{-1}^{1} P_{l_1}^{m_{\alpha_1}} P_{l_2}^{m_{\alpha_2}} P_{l_3}^{m_{\alpha_3}} \, d\mu$$

and vanishes when the frequency addition rule holds. □

Note that the size of the denominators of S_0 depends on β. In the $\beta \to 0$ limit the linearised system is dispersionless and the denominators of S_0 will vanish. Thus, the proposition concerns more the intermediate (β of order 1) and deep water regimes. We remark that, using the fact that $\omega' > 0$ and $\omega'' < 0$, it is easy to show that all terms $|\omega_{l_1} - \omega_{l_2} - \omega_{l_3}|^{-1}$ appearing in S_0 can be bounded by $|\omega_1 - 2\omega_2|^{-1}$, which can be computed easily for given β.

In the rotating frame we can obtain a similar result for the Hamiltonian $H^{\Omega} = H_0^{\Omega} + \frac{1}{R}(\epsilon H_1 + \epsilon^2 H_2 + \cdots)$. Here the quadratic Hamiltonian H_0^{Ω} involves the dispersion ω_{Ω} of (6.18).

Corollary 7.2. *In the rotating frame, the cubic terms in the Hamiltonian can be eliminated by a canonical transformation.*

Proof. The normal form calculation is as above, and we have to check that there are no resonances. From (6.18), the resonance condition is now

$$I_{\gamma_1\gamma_2\gamma_3} \neq 0, \qquad Rm_{\gamma_1} + \omega_{\gamma_1} + Rm_{\gamma_2} + \omega_{\gamma_2} + Rm_{\gamma_3} + \omega_{\gamma_3} = 0, \tag{7.12}$$

$$I_{\gamma_1\gamma_2\gamma_3^-} \neq 0, \qquad Rm_{\gamma_1} + \omega_{\gamma_1} + Rm_{\gamma_3} + \omega_{\gamma_2} - Rm_{\gamma_3} - \omega_{\gamma_3} = 0. \tag{7.13}$$

The coefficients are as in (7.11). From the above discussion of the triple integrals of harmonics, $I_{\gamma_1\gamma_2\gamma_3} \neq 0 \Rightarrow m_{\gamma_1} + m_{\gamma_1} + m_{\gamma_1} = 0$ and $I_{\gamma_1\gamma_2\gamma_3^-} \neq 0 \Rightarrow m_{\gamma_1} + m_{\gamma_1} - m_{\gamma_1} = 0$, and, therefore, the resonance conditions (7.12), (7.13) are equivalent to the rest frame resonance conditions. □

We now consider the problem of eliminating the quartic terms. However, this time there are resonances.

As previously, we try to find a function S_1 such that $(\exp \epsilon^2 \operatorname{Ad}_{S_1})(\exp \epsilon \operatorname{Ad}_{S_0})H$ has no ϵ^2 terms, and we are led to the equation

$$\left(H_2 + \frac{1}{2}[S_0, H_1]\right) + [S_1, H_0] = 0. \tag{7.14}$$

The contribution from $[S_0, H_1]$ can be computed with the aid of the formula (7.5). The calculation is long but straightforward and can be simplified with the use of a diagrammatic method that will appear elsewhere. We write the result as

$$[S_0, H_1] = K + L$$

with

$$K = -i \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_q} \Big\{ a_{\gamma_1} a_{\gamma_2} a_{\gamma_3} a_{\gamma_4} \big[A'_{\gamma_q\gamma_1\gamma_2} A_{\gamma_3\gamma_4\gamma_q} + A'_{\gamma_1\gamma_q\gamma_2} A_{\gamma_3\gamma_4\gamma_q}$$
$$+ A'_{\gamma_1\gamma_2\gamma_q} A_{\gamma_3\gamma_4\gamma_q^-} + A_{\gamma_q\gamma_1\gamma_2} A'_{\gamma_3\gamma_4\gamma_q^-} + A_{\gamma_1\gamma_q\gamma_2} A'_{\gamma_3\gamma_4\gamma_q^-} + A_{\gamma_1\gamma_2\gamma_q} A'_{\gamma_3\gamma_4\gamma_q^-} \big]$$
$$+ a_{\gamma_1} a_{\gamma_2} a_{\gamma_3} a^*_{\gamma_4} \big[A_{\gamma_3\gamma_4^-\gamma_q^-} (A'_{\gamma_q\gamma_1\gamma_2} + A'_{\gamma_1\gamma_q\gamma_2} + A'_{\gamma_1\gamma_2\gamma_q})$$
$$+ A'_{\gamma_3\gamma_q^-\gamma_4^-} (A_{\gamma_q\gamma_1\gamma_2} + A_{\gamma_1\gamma_q\gamma_2} + A_{\gamma_1\gamma_2\gamma_q})$$
$$+ A'_{\gamma_3\gamma_4^-\gamma_q^-} (A_{\gamma_q\gamma_1\gamma_2} + A_{\gamma_1\gamma_q\gamma_2} + A_{\gamma_1\gamma_2\gamma_q}) + A_{\gamma_3\gamma_q^-\gamma_4^-} (A'_{\gamma_q\gamma_1\gamma_2} + A'_{\gamma_1\gamma_q\gamma_2} + A'_{\gamma_1\gamma_2\gamma_q})$$
$$+ A_{\gamma_3\gamma_4^-\gamma_q^-} (A_{\gamma_q\gamma_1\gamma_2^-} - A_{\gamma_1\gamma_q\gamma_2^-}) + A'_{\gamma_1\gamma_2\gamma_q^-} (A_{\gamma_q\gamma_3\gamma_4^-} - A_{\gamma_3\gamma_q\gamma_4^-}) \big] \Big\},$$

where $A'_{\gamma_1\gamma_2\gamma_5} = A_{\gamma_1\gamma_2\gamma_5}(\omega_{\gamma_1} + \omega_{\gamma_2} + \omega_{\gamma_5})^{-1}$, $A'_{\gamma_1\gamma_2\gamma_5^-} = A_{\gamma_1\gamma_2\gamma_5^-}(\omega_{\gamma_1} + \omega_{\gamma_2} - \omega_{\gamma_5})^{-1}$ and so forth, and

$$L = i \sum_{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_q} a_{\gamma_1} a_{\gamma_2} a^*_{\gamma_3} a^*_{\gamma_4} C_{\gamma_1\gamma_2\gamma_3\gamma_4},$$

with

$$C_{\gamma_1\gamma_2\gamma_3\gamma_4} = \left(\frac{1}{\omega_{\gamma_1} + \omega_{\gamma_2} + \omega_{\gamma_q}} + \frac{1}{\omega_{\gamma_1} + \omega_{\gamma_2} + \omega_{\gamma_5}} \right)$$
$$\times (A_{\gamma_q\gamma_1\gamma_2} + A_{\gamma_1\gamma_q\gamma_2} + A_{\gamma_1\gamma_2\gamma_q})(A_{\gamma_q^-\gamma_3^-\gamma_4^-} + A_{\gamma_1^-\gamma_q^-\gamma_4^-} + A_{\gamma_3^-\gamma_4^-\gamma_q^-})$$
$$+ \left(\frac{1}{\omega_{\gamma_1} + \omega_{\gamma_2} - \omega_{\gamma_q}} - \frac{1}{\omega_{\gamma_q} - \omega_{\gamma_3} - \omega_{\gamma_4}} \right) A_{\gamma_1\gamma_2\gamma_q^-} A_{\gamma_q\gamma_3^-\gamma_4^-}$$
$$- \left(\frac{1}{\omega_{\gamma_1} + \omega_{\gamma_q} - \omega_{\gamma_3}} - \frac{1}{\omega_{\gamma_2} - \omega_{\gamma_q} - \omega_{\gamma_4}} \right)$$
$$\times (A_{\gamma_1\gamma_q\gamma_3^-} + A_{\gamma_q\gamma_1\gamma_3^-})(A_{\gamma_2\gamma_q^-\gamma_4^-} + A_{\gamma_2\gamma_4^-\gamma_q^-}).$$

Writing $H_2 + \frac{1}{2}[S_0, H_1]$ as $\sum_{k,\bar{k}} \tilde{B}_{k,\bar{k}} z^k \bar{z}^{\bar{k}}$ with $|k| + |\bar{k}| = 4$, the resonance condition is now

$$\tilde{B}_{\gamma_1\gamma_2\gamma_3\gamma_4} \neq 0, \qquad \omega_{\gamma_1} + \omega_{\gamma_2} + \omega_{\gamma_3} + \omega_{\gamma_4} = 0, \tag{7.15}$$

$$\tilde{B}_{\gamma_1\gamma_2\gamma_3\gamma_4^-} \neq 0, \qquad \omega_{\gamma_1} + \omega_{\gamma_2} + \omega_{\gamma_3} - \omega_{\gamma_4} = 0, \tag{7.16}$$

$$\tilde{B}_{\gamma_1\gamma_2\gamma_3^-\gamma_4^-} \neq 0, \qquad \omega_{\gamma_1} + \omega_{\gamma_2} - \omega_{\gamma_3} - \omega_{\gamma_4} = 0. \tag{7.17}$$

Proposition 7.3. *The resonance conditions (7.15) and (7.16) are never satisfied while condition (7.17) is satisfied for suitable* $\gamma_1, \gamma_2, \gamma_3, \gamma_4$.

Proof. Since $\omega_\gamma > 0$, (7.15) is never satisfied. For (7.16), we have that $\tilde{B}_{\gamma_1\gamma_2\gamma_3\gamma_4^-}$ is of the form

$$\tilde{B}_{\gamma_1\gamma_2\gamma_3\gamma_4^-} = \sum_{Q,R} b^{Q,R}_{\gamma_1\gamma_2\gamma_3\gamma_4^-} \sum_{\gamma} Q_{\gamma_i\gamma_j\gamma} R_{\gamma_n\gamma_m^-\gamma^-}$$

where $Q_{Y_iY_jY}$ and $R_{Y_iY_jY}$ are $\int Y_{Y_i} Y_{Y_j} Y_{Y}$ or $\int (\nabla Y_{Y_i} \cdot \nabla Y_{Y_j}) Y^*_{Y}$. From our previous discussion of the triple integrals of spherical harmonics, a necessary condition for $\tilde{B}_{\gamma_1\gamma_2\gamma_3\gamma_4}$ not to vanish is that there exist indices $\gamma = [l, m]$ such that

$$|l_i - l_j| \le l \le |l_i + l_j| \qquad \text{and} \qquad |l_n - l_m| \le l \le |l_n + l_m| \tag{7.18}$$

(and permutations on the i, j, n, m). However, since $\frac{d\omega}{dl} > 0$ and $\frac{d^2\omega}{dl^2} < 0$, we have that $\omega_{\gamma_i} = \omega_{\gamma_j} + \omega_{\gamma_n} + \omega_{\gamma_m} \Rightarrow l_i > l_j + l_n + l_m$ or $|l_i - l_j| > |l_n + l_m|$ (and permutations) so that (7.18) is not satisfied, and (7.16) cannot be satisfied either. For (7.17), pick $\gamma_1 = \gamma_3$ and $\gamma_2 = \gamma_4$, then the frequency sum rule is satisfied; moreover $\tilde{B}_{\gamma_1\gamma_2\gamma_3\gamma_4}$ is (generically) nonzero (a few of them have been computed). □

Therefore the resonant part of the Hamiltonian is of the form

$$H_{2.res} = \sum_{\gamma_1.\gamma_2.\gamma_3.\gamma_4} R_{\gamma_1\gamma_2\gamma_3\gamma_4} a_{\gamma_1} a_{\gamma_2} a^*_{\gamma_3} a^*_{\gamma_4}$$

with

$$R_{\gamma_1\gamma_2\gamma_3\gamma_4} = B_{\gamma_1\gamma_2\gamma_3^-\gamma_4^-} + \frac{1}{2} C_{\gamma_1\gamma_2\gamma_3\gamma_4}. \tag{7.19}$$

We note that $H_{2.res}$ should include all the "generic" resonances corresponding to $l_3 = l_1, l_4 = l_2$ or $l_3 = l_2, l_4 = l_1$. In addition the resonance condition (7.17) may be satisfied for other integers. A preliminary numerical search has not found any so far, but note that for large l we have $\omega(l) \to \sqrt{l}$, and since $\sqrt{a} + \sqrt{b} = \sqrt{c} + \sqrt{d}$ has other types of integer solutions, we expect that there are quartets of integers that are arbitrarily close to resonance. A more detailed study of this issue and its dynamical implications will be considered in the future.

The equations arising from the above second-order normal form Hamiltonian $H = H_0 + H_{2.res}$ are easily seen to have families of periodic orbits. To simplify the argument we will assume that $H_{2.res}$ contains only the "generic" 4-wave resonances, in which case the subspaces $M_L = a(l, m) = 0$ for $l \neq L$ are invariant under the flow. On these subspaces we can find families of periodic orbits.

Proposition 7.4. *Let ϕ_H be the flow generated by the Hamiltonian vector field of $H = H_0 + H_{2.res}$. Then, (a) the subspaces $T_{(L.M)} = \{a_{l.m} = 0\, for\, l \neq L, m \neq M\}$ with $L = 1, 2, 3, \ldots,$ $M = -L, -L+1, \ldots, L$ are invariant under ϕ_H. Moreover, the restriction of the flow ϕ_H to $T_{(L.M)}$ foliates $T_{(L.M)}$ by periodic orbits. (b) The subspaces $S_{(L.\bar{M})} = \{a_{l.m} = 0 \quad for \quad l \neq L, m \neq \pm M\}$ with $L = 1, 2, 3, \ldots,$ $M = -L, -L+1, \ldots, L$ and $3|M| > L$ are invariant under ϕ_H and the restriction of ϕ_H to $S_{(L.\bar{M})}$ foliates $S_{(L.\bar{M})}$ by periodic orbits.*

Proof. (a) From the form of $H_{2.res}$ in (7.19) observe that in order for the coefficients of the terms $a_{(l_1.m_1)} a_{(l_2.m_2)} a^*_{(l_1.m_3)} a^*_{(l_2.m_4)}$ to be nonzero, the m_i must satisfy the relation

$$m_1 + m_2 - m_3 - m_4 = 0. \tag{7.20}$$

To show that the $T_{(L.M)}$ are invariant, first note that $\dot{a}_{(l.m)} = 0$ if $l \neq L$. Also, for $m \neq M$ we have $\dot{a}_{(L.m)} = iC(M, M, m, M) a_{(L.M)} a_{(L.M)} a^*_{(L.M)}$ with the coefficient

$C(M, M, m, M)$ given by (7.19). But the condition $C(M, M, m, M) \neq 0$ and (7.20) require that $m = M$. Thus the $T_{(L,M)}$ are invariant and

$$\dot{a}_{(L,M)} = i\omega_L a_{(L,M)} + iC(M, M, M, M)a_{(L,M)}a_{(L,M)}a^*_{(L,M)}.$$

This equation is a Hamiltonian system generated by

$$H_{L,M} = \omega_L a_{(L,M)}a^*_{(L,M)} + C_M a_{(L,M)}a_{(L,M)}a^*_{(L,M)}a^*_{(L,M)}$$

with $C_M = C(M, M, M, M)$. The Hamiltonian $H_{L,M}$ depends only on the "action" $J_{L,M} = a_{(L,M)}a^*_{(L,M)}$. By a canonical transformation to the variables $J_{L,M}$ and $\theta_{L,M}$ ("action-angle" variables), the equation becomes

$$\dot{J}_{L,M} = 0, \qquad \dot{\theta}_{L,M} = \omega_L + C_M J_{L,M}.$$

The solutions of this equation are all periodic and correspond to traveling waves with amplitude dependent phase velocity.

(b) Similarly, for $a_{(l,m)} \in S_{(L,\tilde{M})}$, $\dot{a}_{(L,\pm M)}$ has terms of the form

$$C(\tilde{M}, \tilde{M}, m, \tilde{M})a_{(L,M)}a_{(L,M)}a^*_{(L,M)}$$

with $\tilde{M}$ either $+M$ or $-M$. From (7.20) the coefficients $C(\tilde{M}, \tilde{M}, m, \tilde{M})$ vanish unless $m = \pm M$ or $m = \pm 3M$. Thus for $3|M| > L$ the subspaces $S_{(L,\tilde{M})}$ are invariant. On $S_{(L,\tilde{M})}$ the equation of motion for the restricted flow is generated by the Hamiltonian

$$H_{L,\tilde{M}} = i\omega_L(a^*_{(L,M)}a^*_{(L,M)} + a^*_{(L,-M)}a^*_{(L,-M)}) + C_{M,M}J_{L,M}J_{L,M}$$
$$+2C_{M,-M}J_{L,M}J_{L,-M} + C_{-M,-M}J_{L,-M}J_{L,-M}$$

with $J_{L,\pm M} = a_{(L,\pm M)}a^*_{(L,\pm M)}$, $C_{A,B} = C(A, B, A, B)$. Using "action-angle" variables we again have

$$\dot{J}_{L,\pm M} = 0, \qquad \dot{\theta}_{L,\pm M} = \omega_L + 2C_{\pm M,\pm M}J_{L,M} + 2C_{M,-M}J_{L,\mp M}.$$

These periodic orbits are superpositions of the previous traveling waves, and, if the amplitudes $J_{L,\pm M}$ are equal, the result is standing waves. □

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