# Solitary waves for a coupled nonlinear Schrödinger system with dispersion management

Panayotis Panayotaros * Mauricio Sepúlveda † Octavio Vera ‡

**Abstract**

We consider a system of coupled nonlinear Schrödinger equations with periodically varying dispersion coefficient that arises in the context of fiber-optics communication. We use P.L. Lions's Concentration Compactness principle to show the existence of standing waves with prescribed $L^2$ norm in an averaged equation that approximates the coupled system. We also use the Mountain Pass Lemma to prove the existence of standing waves with prescribed frequencies.

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## 1 Introduction

Over the last years, certain nonlinear dispersive equations with nonlocal nonlinearity have arisen in the context of optical communications and have become the subject of intense numerical and analytical study [1, 10, 15, 20, 32]. In 1981, I. P. Kaminow [14] showed that single-mode optical fibers are not really "single-mode" but actually bimodal due to the presence of birefringence. It can occur that the linear birefringence makes a pulse split in two pieces, while nonlinear birefringence can prevent splitting. C. R. Menyuk [19] showed that the evolution of two orthogonal pulse envelopes in birefringent optical fiber is governed by the Coupled Nonlinear Schrödinger System (CNLSS)

$$i\,u_t + u_{xx} + |u|^2\,u + \beta\,|v|^2\,u = 0 \tag{1.1}$$

$$i\,v_t + v_{xx} + |v|^2\,v + \beta\,|u|^2\,v = 0 \tag{1.2}$$

where $x \in \mathbb{R}$, $t \in \mathbb{R}$. $u = u(x,\,t)$ and $v = v(x,\,t)$ are complex unknown functions and $\beta$ is a real positive constant which depends on the anisotropy of the fiber. System (1.1), (1.2) is important for industrial applications in fiber communication systems [12], and all-optical switching devices [13]. Another motivation for studying the CNLSS arises from the Hartree-Fock theory for a binary mixture of Bose-Einstein condensates in two different hyperfine states, cf. [7].

In optical fiber devices a key goal is to transfer pulses over long distances. It is therefore important to stabilize pulses and counteract the effects of loss and dispersion along the fiber. Approaches to these problems rely mostly on techniques related to linear models. However, over the past two decades there have been suggested different approaches which intend to make use of the nonlinear effects [6]. As a model, we consider the Nonlinear Schrödinger (briefly NLS) equation

$$i\,u_t + d(t)\,u_{xx} + c(t)\,|u|^2\,u = 0 \tag{1.3}$$

for the envelope function $u = u(x,\,t)$ of the electromagnetic wave. $t \in \mathbb{R}$ is the distance along the fiber, whereas the coordinate $x \in \mathbb{R}$ is the physical time. The initial condition $u(x, t_0)$ describes a signal that

*Depto. Matemáticas y Mecánica, I.I.M.A.S.-U.N.A.M., Apdo. Postal 20-726, 01000 México D.F., México, panos@mym.iimas.unam.mx
†Depto. de Ingeniería Matemática, Universidad de Concepción, Chile, mauricio@ing-mat.udec.cl
‡Depto. de Matemática, Universidad del Bío-Bío, Chile, overa@ubiobio.cl

is given at all times $x$ at a point $t_0$ along the fiber. The dispersion and nonlinearity parameters $c$, $d$ respectively depend on properties of the fiber, and can be chosen to vary with $t$.

Varying the dispersion and nonlinearity parameters along the fiber is known as "dispersion management". The technique was introduced in the early eighties [16] and refined during the last decade [21], emerging as a dominant technology for high bandwidth data transmission through optical fibers, see [8, 9, 26] and references therein. In a dispersion managed fiber link, short segments of fiber with opposite linear dispersion are joined together in a periodically repeated structure, forming a fiber whose linear dispersion is effectively canceled out over each period of dispersion management. In such a system, the characteristic length of local dispersion is much shorter than that of nonlinearity or average dispersion, so that on the scale of a typical dispersion management segment, the effect of nonlinearity and average dispersion can be made small relative to those of the local dispersion.

A basic problem for NLS type equations such as (1.3) is to prove that they support solitary wave solutions. These are localized solutions that maintain their form and are expected play a important role in the dynamics, see e.g. T.-P. Tsai [31]. In V. Zharnitsky *et al.* [32] solutions of this type were found for an equation of NLS type whose solutions approximate those of (1.3). It is natural to ask whether similar solutions exist for the coupled NLS system.

The Cauchy problem for the system (1.1)-(1.2) was first studied by E. S. P. Siqueira [27, 28] who showed that, for initial data $u_0 \in H^1(\mathbb{R})$ and $v_0 \in H^1(\mathbb{R})$, the solution satisfies $u \in C(\mathbb{R} : H^1(\mathbb{R})) \cap C^1(\mathbb{R} : H^{-1}(\mathbb{R}))$ and $v \in C(\mathbb{R} : H^1(\mathbb{R})) \cap C^1(\mathbb{R} : H^{-1}(\mathbb{R})$. The proof uses techniques developed in [3, 4]. This CNLSS has been extensively studied for many authors, see [5, 14, 19] and references therein.

The starting point in this work is the nonautonomous CNLSS

$$i\,u_t + d(t)\,u_{xx} + \varepsilon\,|u|^2\,u + \varepsilon\,\beta\,|v|^2\,u + \varepsilon\,\alpha\,u_{xx} = 0 \tag{1.4}$$
$$i\,v_t + d(t)\,v_{xx} + \varepsilon\,|v|^2\,v + \varepsilon\,\beta\,|u|^2\,v + \varepsilon\,\alpha\,v_{xx} = 0 \tag{1.5}$$

where $d(t)$ is a periodically varying group velocity dispersion with zero average, $\varepsilon\,\alpha$ is the *average* (or residual) dispersion, and $x$ and $t$ correspond to the distance along the fiber and the retarded time respectively. System (1.4), (1.5) will be approximated by the autonomous *averaged* CNLSS

$$i\,w_t + \varepsilon\,\alpha\,w_{xx} + \varepsilon\,\langle Q_1\rangle(w,\,z) = 0 \tag{1.6}$$
$$i\,z_t + \varepsilon\,\alpha\,z_{xx} + \varepsilon\,\langle Q_2\rangle(w,\,z) = 0, \tag{1.7}$$

with $\langle Q_1\rangle$, $\langle Q_2\rangle$ nonlocal cubic nonlinearities given in section 2. The averaged system is derived from (1.4), (1.5) by a formal averaging argument we present in section 2. It is expected that, as $\epsilon \to 0$, solutions of (1.6), (1.7) should approximate solutions of (1.4), (1.5) over a time interval of size $O(\epsilon^{-1})$ (see [32] for the single NLS case). Extending results of [5] we can see that (1.6), (1.7) with initial data $(w(0), z(0)) = (w_0, z_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ has a unique solution $(w(t), z(t)) \in C(\mathbb{R}, H^1(\mathbb{R}) \times H^1(\mathbb{R}))$, assuming some mild regularity assumptions on $d$.

Following the general idea of seeking solitary waves, we are specifically interested in solutions of (1.6), (1.7) of the form

$$w(x,\,t) = e^{i\,\omega_1\,t}\varphi(x), \qquad z(x,\,t) = e^{i\,\omega_2\,t}\psi(x) \tag{1.8}$$

where $\varphi,\,\psi \in H^1(\mathbb{R})$, $\varphi,\,\psi \not\equiv 0$ and $\omega_1,\,\omega_2 \in \mathbb{R}$.

To state the main results, define the linear operators $T(t)$ by requiring that $T(t)u_0$ be the solution of $iu_t = d(t)u_{xx} = 0$, with $u(0) = u_0$, and consider the functional $\langle H\rangle : H^1(\mathbb{R}) \times H^1(\mathbb{R})) \to \mathbb{R}$ defined by

$$\langle H\rangle(u,\,v) = \int_0^1\int_{\mathbb{R}}\left[\alpha\,(|v_x|^2 + |v_x|^2 - \frac{1}{2}\,|T(t)u|^4 - \frac{1}{2}\,|T(t)v|^4 - \beta\,|T(t)u|^2\,|T(t)v|^2\right]dx\,dt, \tag{1.9}$$

We then have the following.

**Theorem 1.1** *Let $\alpha > 0$. Then for any $\lambda_1, \lambda_2 > 0$ (1.6), (1.7) has a solution of the form (1.8) that minimizes $\langle H \rangle$ over all $(u, v) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ satisfying $||u||_{L^2(\mathbb{R})} = \lambda_1$, $||v||_{L^2(\mathbb{R})} = \lambda_2$.*

The proof of Theorem 1.1 is based on minimization, and the Concentration-Compactness Principle and is given is section 4, where we also remark on the stability of the standing wave solutions. In Theorem 1.1 the frequencies $\omega_1$, $\omega_2$ are a-priori unspecified. It is also possible to obtain standing waves with prescribed frequencies.

**Theorem 1.2** *Let $\alpha > 0$. Consider any pair of $\omega_1$, $\omega_2 > 0$. Then (1.6), (1.7) has a solution of the form (1.8).*

The proof of Theorem 1.2 is based on the Mountain Pass Lemma applied to a functional obtained from the Hamiltonian of (1.6), (1.7) and is given is section 5.

The paper is organized as follows. In section 2 we formally derive the averaged system from a coupled NLS system with variable dispersion. Section 3 states some basic preliminary results used in the the subsequent proofs. In section 4 we formulate the constrained minimization problem for solutions of the form (1.8) and prove Theorem 1.1. by showing the existence of minimizers. We also comment on stability and the cases $\alpha = 0$, $\alpha < 0$. In section 5 we prove Theorem 1.2.

## 2 The averaged NLS system

From the point of view of modeling the starting point is the coupled nonlinear Schrödinger system system(CNLSS)

$$i\, u_\tau + \frac{1}{\varepsilon}\, d\left(\frac{\tau}{\varepsilon}\right) u_{zz} + c\left(\frac{\tau}{\varepsilon}\right) |u|^2\, u + \beta\, |v|^2\, u + \alpha\, u_{zz} = 0 \tag{2.1}$$

$$i\, v_\tau + \frac{1}{\varepsilon}\, d\left(\frac{\tau}{\varepsilon}\right) v_{zz} + c\left(\frac{\tau}{\varepsilon}\right) |v|^2\, v + \beta\, |u|^2\, v + \alpha\, v_{zz} = 0 \tag{2.2}$$

$$u(x,\, 0) = u_0(x) \tag{2.3}$$

$$v(x,\, 0) = v_0(x). \tag{2.4}$$

with $x \in \mathbb{R}$, $t \in \mathbb{R}$, and $u = u(x,\, t)$, $v = v(x,\, t)$ complex unknown functions. The real functions $d$, $c$ are $1-$periodic, piecewise-continuous and have vanishing average over their period. The real parameter $\alpha$ is the *average* dispersion coefficient. The parameter $\beta$ is real, positive and models the anisotropy of the fiber. $\varepsilon$ is a real positive parameter, and we are interested in the case where $\varepsilon$ is small; this implies that the functions $d$, $c$ have exhibit rapid, high amplitude oscillation.

Letting $\tau = \varepsilon\, \sigma$, (2.1)-(2.2) become

$$i\, u_\sigma + d(\sigma)\, u_{zz} + \varepsilon\, c(\sigma)\, |u|^2\, u + \varepsilon\, \beta\, |v|^2\, u + \varepsilon\, \alpha\, u_{zz} = 0$$
$$i\, v_\sigma + d(\sigma)\, v_{zz} + \varepsilon\, c(\sigma)\, |v|^2\, v + \varepsilon\, \beta\, |u|^2\, v + \varepsilon\, \alpha\, v_{zz} = 0.$$

Furthermore, letting $t = t(\sigma)$, $t'(\sigma) = c(\sigma)$ and $\tau = x$ we put (2.1)-(2.2) in the form

$$i\, u_t + d(t)\, u_{xx} + \varepsilon\, |u|^2\, u + \varepsilon\, \beta\, |v|^2\, u + \varepsilon\, \alpha\, u_{xx} = 0 \tag{2.5}$$

$$i\, v_t + d(t)\, v_{xx} + \varepsilon\, |v|^2\, v + \varepsilon\, \beta\, |u|^2\, v + \varepsilon\, \alpha\, v_{xx} = 0. \tag{2.6}$$

of (1.1)-(1.2). Equivalently the system is written as

$$i\, U_t + d(t)\, U_{xx} + \varepsilon\, F(u,\, v)\, U + \varepsilon\, \alpha\, U_{xx} = 0 \tag{2.7}$$

where

$$U = \begin{bmatrix} u \\ v \end{bmatrix} \qquad ; \qquad F(u,\,v) = \begin{bmatrix} |u|^2 & \beta\,|v|^2 \\ |v|^2 & \beta\,|u|^2 \end{bmatrix}$$

Consider equation (2.7) with $\varepsilon = 0$. Using Stone's theorem [23], we obtain $U(x,\,t) = T(t)U_0$, where $T(t)$ is the fundamental solution of $i\,U_t + d(t)\,U_{xx} = 0$. This operator is easily computed using the Fourier Transform $\mathcal{F}$

$$T(t)U_0(x) = \frac{1}{\sqrt{2\,\pi}} \int_{\mathbb{R}} e^{i\,x\,\xi} \varphi(\xi,\,t)\,\mathcal{F}U(\xi,\,0)\,d\xi$$

where $\varphi(\xi,\,t) = e^{-\,i\,\xi^2 \int_0^t d(\tau)\,d\tau}$. Moreover, due to the periodicity of $d(t)$, both $\varphi(\xi,\,t)$ and $T(t)$ are periodic in $t$. The family of unitary operators $T(t)$ is periodic $T(t+1) = T(t)$ since the average of $d$ over its period vanishes. We observe that $T(t)$ is an isometry on $H^s(\mathbb{R}) \times H^s(\mathbb{R})$ for all $s \in \mathbb{R}$.

Using the solution of the linear system, we define the functions $w,\ z$ by $u(x,\,t) = T(t)w(x,\,t)$ and $v(x,\,t) = T(t)z(x,\,t)$ respectively.

Then, (2.5)-(2.6) imply

$$i\,w_t + \varepsilon\,\alpha\,w_{xx} + \varepsilon\,Q_1(w,\,z,\,t) = 0 \tag{2.8}$$
$$i\,z_t + \varepsilon\,\alpha\,z_{xx} + \varepsilon\,Q_2(w,\,z,\,t) = 0 \tag{2.9}$$

where

$$\begin{aligned} Q_1(w,\,z,\,t) &= T^{-1}(t)\left(|T(t)w|^2 T(t)w + \beta\,|T(t)z|^2 T(t)w\right) \\ Q_2(w,\,z,\,t) &= T^{-1}(t)\left(|T(t)z|^2 T(t)z + \beta\,|T(t)w|^2 T(t)z\right). \end{aligned}$$

We now replace (2.5)-(2.6) by the *averaged* system

$$i\,w_t + \varepsilon\,\alpha\,w_{xx} + \varepsilon\,\langle Q_1\rangle(w,\,z) = 0 \tag{2.10}$$
$$i\,z_t + \varepsilon\,\alpha\,z_{xx} + \varepsilon\,\langle Q_2\rangle(w,\,z) = 0 \tag{2.11}$$

with

$$\langle Q_1\rangle(w,\,z) = \int_0^1 Q_1(w,\,z,\,t)\,dt \tag{2.12}$$
$$\langle Q_2\rangle(w,\,z) = \int_0^1 Q_2(w,\,z,\,t)\,dt. \tag{2.13}$$

System (2.10)-(2.11) is obtained by formally averaging the explicit time dependence in (2.8)-(2.9). This is motivated by the intuitive idea that in the limit $\varepsilon \to 0$, solutions of the averaged system should approximate solutions of (2.8)-(2.9), as in the classical averaging method for ODEs. In the context of the single NLS with time varying coefficients, the analogue of (2.10)-(2.11) was formally derived by [8], [1]. Zharnitsky et al. [32] give a precise statement justifying the averaging step.

Rescaling time in(2.10) and (2.11) by changing $t \to t/\varepsilon$ gives

$$i\,w_t + \alpha\,w_{xx} + \langle Q_1\rangle(w,\,z) = 0 \tag{2.14}$$
$$i\,z_t + \alpha\,z_{xx} + \langle Q_2\rangle(w,\,z) = 0. \tag{2.15}$$

The structure of (2.14)-(2.15) is very close to the structure of the coupled nonlinear Schrödinger system and we can extend the theory of existence for the coupled nonlinear Schrödinger system to (2.14)-(2.15) (See J. C. Ceballos *et al.* [5]) and references therein. In particular we similarly show that system (2.14), (2.15) with initial data $(w_0, z_0) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ has a unique solution $(w(t), z(t)) \in C(\mathbb{R}, H^1(\mathbb{R}) \times H^1(\mathbb{R})) \cap C^1(\mathbb{R}, H^{-1}(\mathbb{R}) \times H^{-1}(\mathbb{R}))$.

**Remark 2.1** Systems (2.5)-(2.6), (2.8)-(2.9), and the averaged system (2.14)-(2.15) are Hamiltonian. For instance (2.14)-(2.15) can be formally written as Hamilton's equations

$$w_t = -i \frac{\delta}{\delta z^*} \langle H \rangle, \quad z_t = -i \frac{\delta}{\delta w^*} \langle H \rangle, \tag{2.16}$$

with Hamiltonian

$$\langle H \rangle = \int_0^1 \int_{\mathbb{R}} \left[ \alpha \, |w_x|^2 + \alpha \, |z_x|^2 - \frac{1}{2} \, |T(t)w|^4 - \frac{1}{2} \, |T(t)z|^4 - \beta \, |T(t)w|^2 \, |T(t)z|^2 \right] dx \, dt, \tag{2.17}$$

see e.g. [30] for this notation. We furthermore check that $||w||_{L^2(\mathbb{R})}$, $||z||_{L^2(\mathbb{R})}$ are conserved quantities.

**Remark 2.2** For $\langle T \rangle$ defined by

$$\langle T \rangle u = \int_0^1 T(t) u \, dt$$

we have, using the Fourier Transform of the function $u$ as $\mathcal{F}u$ is

$$\mathcal{F}(\langle T \rangle u)(\xi) = \left( \int_0^1 e^{i \, \xi^2 \int_0^t d(\tau) \, d\tau} dt \right) \mathcal{F}u(\xi). \tag{2.18}$$

Indeed, using (2.7) with $\varepsilon = 0$, we have

$$\mathcal{F} \left( e^{-i \, \partial_x^2 \int_0^t d(\tau) \, d\tau} U \right) = e^{i \, \xi^2 \int_0^t d(\tau) \, d\tau} \mathcal{F}U.$$

**Remark 2.3** Let

$$\Theta(\eta) = e^{-i \, \eta^2 \int_0^t d(\tau) \, d\tau}.$$

Applying the Fourier transform we have

$$\begin{aligned} \mathcal{F}Q_1(w, \, z)(\xi) \quad &= \int_{\eta_1 - \eta_2 + \eta_3 = \xi} \Theta(\eta_1^2 - \eta_2^2 + \eta_3^2 - \xi^2) \, \mathcal{F}w_1(\eta_1) \, \mathcal{F}w_2^*(\eta_2) \, \mathcal{F}w_3(\eta_3) \, d\eta_1 \, d\eta_2 \, d\eta_3 \\ &+ \beta \int_{\mu_1 - \mu_2 + \eta_3 = \xi} \Theta(\mu_1^2 - \mu_2^2 + \eta_3^2 - \xi^2) \, \mathcal{F}z_1(\mu_1) \, \mathcal{F}z_2^*(\mu_2) \, \mathcal{F}w_3(\eta_3) \, d\mu_1 \, d\mu_2 \, d\eta_3 \end{aligned}$$

and

$$\begin{aligned} \mathcal{F}Q_2(w, \, z)(\xi) \quad &= \int_{\mu_1 - \mu_2 + \mu_3 = \xi} \Theta(\mu_1^2 - \mu_2^2 + \mu_3^2 - \xi^2) \, \mathcal{F}z_1(\mu_1) \, \mathcal{F}z_2^*(\mu_2) \, \mathcal{F}z_3(\mu_3) \, d\mu_1 \, d\mu_2 \, d\mu_3 \\ &+ \beta \int_{\eta_1 - \eta_2 + \mu_3 = \xi} \Theta(\eta_1^2 - \eta_2^2 + \mu_3^2 - \xi^2) \, \mathcal{F}w_1(\eta_1) \, \mathcal{F}w_2^*(\eta_2) \, \mathcal{F}z_3(\mu_3) \, d\eta_1 \, d\eta_2 \, d\mu_3. \end{aligned}$$

Using the fact that $T$ is an isometry in $H^1(\mathbb{R})$,

$$||Q_1(w, \, z)||_{H^1(\mathbb{R})} \leq c \, \left( ||w_1||_{H^1(\mathbb{R})} \, ||w_2||_{H^1(\mathbb{R})} \, ||w_3||_{H^1(\mathbb{R})} + \beta \, ||z_1||_{H^1(\mathbb{R})} \, ||z_2||_{H^1(\mathbb{R})} \, ||w_3||_{H^1(\mathbb{R})} \right),$$

and

$$||Q_2(w, \, z)||_{H^1(\mathbb{R})} \leq c \, \left( ||z_1||_{H^1(\mathbb{R})} \, ||z_2||_{H^1(\mathbb{R})} \, ||z_3||_{H^1(\mathbb{R})} + \beta \, ||w_1||_{H^1(\mathbb{R})} \, ||w_2||_{H^1(\mathbb{R})} \, ||z_3||_{H^1(\mathbb{R})} \right).$$

Hence,

$$||\langle Q_1 \rangle(w, \, z)||_{H^1(\mathbb{R})} \leq c \, \left( ||w_1||_{H^1(\mathbb{R})} \, ||w_2||_{H^1(\mathbb{R})} \, ||w_3||_{H^1(\mathbb{R})} + \beta \, ||z_1||_{H^1(\mathbb{R})} \, ||z_2||_{H^1(\mathbb{R})} \, ||w_3||_{H^1(\mathbb{R})} \right),$$

and

$$||\langle Q_2 \rangle(w, \, z)||_{H^1(\mathbb{R})} \leq c \, \left( ||z_1||_{H^1(\mathbb{R})} \, ||z_2||_{H^1(\mathbb{R})} \, ||z_3||_{H^1(\mathbb{R})} + \beta \, ||w_1||_{H^1(\mathbb{R})} \, ||w_2||_{H^1(\mathbb{R})} \, ||z_3||_{H^1(\mathbb{R})} \right).$$

Moreover

$$\langle Q_1 \rangle^*(w, \, z) =< Q_1 > (w^*, \, z^*), \quad \langle Q_2 \rangle^*(w, \, z) =< Q_1 > (w^*, \, z^*)$$

## 3 Some preliminary results

We state some basic results that will be used in sections 4, 5. We start with a technical lemma that is based on the Gagliardo-Nirenberg inequality.

**Lemma 3.1** *For all $u \in H^1(\mathbb{R})$ we have*

$$||u||^2_{L^\infty(\mathbb{R})} \leq 2\,||u||_{L^2(\mathbb{R})}\,||u_x||_{L^2(\mathbb{R})} \tag{3.1}$$

$$||u||^4_{L^4(\mathbb{R})} \leq 2\,||u||^3_{L^2(\mathbb{R})}\,||u_x||_{L^2(\mathbb{R})}. \tag{3.2}$$

**Lemma 3.2** (See [3], page 185). *Let $0 < \alpha < 4/n$. Let $u \in H^1(\mathbb{R}^n)$. Then there exists $c > 0$ such that*

$$\int_{\mathbb{R}^n} |u|^{\alpha+2}\,dx \leq c \left( \sup_{\phi \in \mathbb{R}^n} \int_{\{|x-\phi|\leq 1\}} |u(x)|^2\,dx \right)^{\alpha/2} ||u||^2_{H^1(\mathbb{R}^n)}. \tag{3.3}$$

To prove Theorem 1.1 we solve a minimization problem in unbounded domains. The main technical tool is Lemma 3.3 below. In general, the invariance of $\mathbb{R}^n$ by the non-compact groups of translations and dilations creates possible loss of compactness: as an illustration of these difficulties, recall that the Rellich-Kondrakov theorem [2] is no more valid in $\mathbb{R}^n$. The consequence of this fact is that, except for the special case of convex functionals, the standard convexity-compactness methods used in problems set in bounded domains fail to treat problems in unbounded domains.

**Lemma 3.3** (Lion's Concentration-Compactness Principle. See [17], Lemma III, page 135). *If $\lambda > 0$ and $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence of $H^1(\mathbb{R})$ with $P(u_k) \equiv ||u_k||^2_{L^2(\mathbb{R})} = \lambda$, then there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ for which one of the following properties holds:*

*1) (compactness) There exists a sequence $\{x_j\}_{j\in\mathbb{N}}$ such that for every $\varepsilon > 0$ there exists $0 < R < \infty$ so that*

$$\int_{x_j-R}^{x_j+R} |u_{k_j}|^2\,dx \geq \lambda - \varepsilon. \tag{3.4}$$

*2) (vanishing) For any $0 < R < +\infty$*

$$\lim_{j\to\infty} \sup_{\phi\in\mathbb{R}} \int_{\phi-R}^{\phi+R} |u_{k_j}|^2\,dx = 0. \tag{3.5}$$

*3) (splitting) There exists $0 < \gamma < \lambda$ such that for every $\varepsilon > 0$ there exists $j_0 \geq 0$ and two sequences $\{u'_j\}_{j\in\mathbb{N}} \subseteq H^1(\mathbb{R})$ and $\{u''_j\}_{j\in\mathbb{N}} \subseteq H^1(\mathbb{R})$ with compact disjoint supports, such that for $j \geq j_0$*

$$||u'_j||_{H^1(\mathbb{R})} + ||u''_j||_{H^1(\mathbb{R})} \leq 4 \sup_{j\in\mathbb{N}} ||u_{k_j}||_{H^1(\mathbb{R})}, \tag{3.6}$$

$$||u_{k_j} - u'_j - u''_j||_{L^2(\mathbb{R})} \leq \varepsilon, \tag{3.7}$$

$$\left| \int_{\mathbb{R}} |u'_j|^2\,dx - \gamma \right| \leq \varepsilon \tag{3.8}$$

$$\left| \int_{\mathbb{R}} |u''_j|^2\,dx + \gamma - \lambda \right| \leq \varepsilon, \tag{3.9}$$

$$\left|\left| \frac{\partial u'_j}{\partial x} \right|\right|_{L^2(\mathbb{R})} + \left|\left| \frac{\partial u''_j}{\partial x} \right|\right|_{L^2(\mathbb{R})} \leq \left|\left| \frac{\partial u_{k_j}}{\partial x} \right|\right|_{L^2(\mathbb{R})} + \varepsilon \tag{3.10}$$

*Moreover $dist(supp(u_j), supp(u''_j)) > 2\varepsilon^{-1}$.*

**Remark 3.3.1** In the case of splitting of Lemma 3.3 (i.e. case 3), V. Zharnitsky *et al* (see [32], Lemma 6.1) show that $u_j'$, $u_j''$ can be chosen to be of the form $u_j'(x) = \rho(x - x_j)u_m(x)$, $u_j''(x) = \theta(x - x_j)u_m(x)$, where $\{x_j\}_{j\in\mathbb{N}}$ is a sequence of points in $\mathbb{R}$, and the functions $\rho$, $\vartheta : \mathbb{R} \to [0, 1]$ are $C^\infty$, even and satisfy

$$
\begin{aligned}
&(i) \quad |\rho'(x)|, |\vartheta'(x)| < \epsilon, \quad \forall x \in \mathbb{R},\\
&(ii) \quad \rho(x) = 1, \quad \text{if} \quad |x| < t_1; \quad \rho(x) = 0, \quad \text{if} \quad |x| \geq t_1 + 2\epsilon^{-1},\\
&\qquad\ \ \vartheta(x) = 1, \quad \text{if} \quad |x| > t_2; \quad \vartheta(x) = 0, \quad \text{if} \quad |x| \leq t_2 - 2\epsilon^{-1},
\end{aligned}
$$

where $0 < t_1 < t_2$, $t_2 - t_1 > 6\epsilon^{-1}$. The above imply that $\text{supp}\rho \cap \text{supp}\vartheta = \emptyset$, $\text{dist}(\text{supp}\rho, \text{supp}\vartheta) > 2\epsilon^{-1}$. Moreover $1 - \rho(x - x_j) - \vartheta(x - x_j) \geq 0$, $\forall x$, $x_j \in \mathbb{R}$.

The proof of Theorem 1.2 is based on the Mountain Pass Lemma below. Let $E$ be a Banach space and $\mathbb{H} : E \longrightarrow \mathbb{R}$ a function which satisfies any of the following conditions:

$(PS)_a$ The Palais-Smale Compactness Condition at a value $a \in \mathbb{R}$ :

*Every sequence* $\{x_j\}_{j\in\mathbb{N}}$ *in* $E$, *such that* $\mathbb{H}(x_j) \longrightarrow a$ *and* $||\mathbb{H}'(x_j)|| \longrightarrow 0$, *has a convergent subsequence.*

$(PS)$ The Palais-Smale Compactness Condition:

$(PS)_a$ *holds for every* $a \in \mathbb{R}$.

$(MP)$ The Mountain Pass Condition:

*There is an open neighborhood* $U$ *of* $0$ *and some* $x_0 \neq \overline{U}$ *such that* $\max\{\mathbb{H}(0), \mathbb{H}(x_0)\} < m \equiv \inf\{\mathbb{H}(x) : x \in \partial U\}$. *Let* $A$ *denote the family of all continuous paths* $g : [0, 1] \longrightarrow \mathbb{H}$ *joining* $0$ *to* $x_0$, *and put* $c \equiv \inf_{g\in A} \mathbb{H}(g(t))$. *Clearly* $c \geq m$.

**Lemma 3.4**(Mountain Pass Lemma. See [24]). *Let* $\mathbb{H} : E \longrightarrow \mathbb{R}$ *be a* $C^1$ *function satisfying* $(MP)$. *Then there exists a sequence* $\{x_j\}_{j\in\mathbb{N}}$ *in* $E$ *such that*

$$
\mathbb{H}(x_j) \longrightarrow c \qquad \text{and} \qquad ||\mathbb{H}'(x_j)|| \longrightarrow 0. \tag{3.11}
$$

*If* $\mathbb{H}$ *also satisfies* $(PS)_c$ *with* $c$ *as in* $(MP)$, *then* $c$ *is a critical value: That is, for some* $x_c$ *in* $E$, $\mathbb{H}(x_c) = c$ *and* $\mathbb{H}'(x_c) = 0^T = (0, 0, \ldots, 0)$.

We also state three results of V. Zharnitsky *et al.* [32] that are used to apply the above results to nonlinearities that involve the operator $T$ of Section 2. Consider the linear part of the coupled free Schrödinger system

$$
i\, u_t + u_{xx} = 0 \tag{3.12}
$$

$$
i\, v_t + v_{xx} = 0, \tag{3.13}
$$

i.e. the two equations decouple and are the same. Consider (3.12). Let

$$
\varepsilon_n(t) = \sup_{\phi\in\mathbb{R}} \int_{\phi-1}^{\phi+1} |u_n(x, t)|^2\, dx. \tag{3.14}
$$

Recall that solutions exist in $C(\mathbb{R}, H^1(\mathbb{R}))$, and that the $L^2$ norm is conserved.

**Lemma 3.5** *Let* $u_n(x, 0)\}_{n\in\mathbb{N}}$ *be a sequence of vanishing initial data, i.e.,* $\varepsilon_n(0) \to 0$ *as* $n \to \infty$. *Consider corresponding solutions* $u_n = u_n(x, t)$ *and assume that* $||u_n||_{H^1(\mathbb{R})} \leq c$, *and* $||u_n||_{L^2(\mathbb{R})} = 1$, $\forall t \in \mathbb{R}$. *Then* $\{u_n(x, t)\}_{n\in\mathbb{N}}$ *is also vanishing and the following estimate holds:*

$$
\varepsilon_n(t) \leq 2\, \varepsilon_n(0) + 2\sqrt{c\, \varepsilon_n(0)\, t}, \quad \forall t \in \mathbb{R}. \tag{3.15}
$$

Similar bounds hold for the solutions of $iu_t + d(t)u_{xx} = 0$:

**Lemma 3.6** *Consider solutions $u_n = u_n(x,\, t)$ of*

$$i\, u_t + d(t)u_{xx} = 0 \tag{3.16}$$
$$\tag{3.17}$$

*with $d(t)$ piecewise smooth with a finite number of non-degenerate zeros. Assume vanishing initial data (as in Lemma 3.5). Define $\varepsilon_n(t)$ as in* (3.14) *and assume that $||u_n||_{H^1(\mathbb{R})} \leq c$, $||u_n||_{L^2(\mathbb{R})} = 1$. Then $\varepsilon_n(t)$ satisfies* (3.15)*.*

Considering the splitting case of Lemma 3.3, we see that $u_j$ splits, up a small error, to functions $u'_j$, $u''_j$ that have disjoint supports. The following lemma (see [32], Lemma 6.3) implies that products of $T(t)u'_j$, $T(t)u''_j$ are also small.

**Lemma 3.7** *Let $\lambda > 0$. Let $\{u_k\}_{k\in\mathbb{N}}$ be a bounded sequence in $H^1(\mathbb{R})$ with $||u_k||^2_{L^2(\mathbb{R})} = \lambda$, $\forall k \in \mathbb{N}$ that splits in the sense of Lemma 3.3. Then $\forall\epsilon > 0$ and corresponding subsequences $\{u'_j\}_{j\in\mathbb{N}}$ and $\{u''_j\}_{j\in\mathbb{N}}$ (as in Lemma 3.3) there exist disjoint sets $S_1$, $S_2$, $S_1 \cup S_2 = \mathbb{R}$ and a constant $C$ depending on $\lambda$ only such that*

$$\int_{S_1} |T(t)u'_j|^2 \leq C\epsilon, \quad \int_{S_2} |T(t)u''_j|^2 \leq C\epsilon, \quad \forall t \in [0, 1]. \tag{3.18}$$

# 4 Standing waves by constrained minimization

We seek solutions of (2.14), (2.15) of the standing wave form

$$w(x,\, t) = e^{i\,\omega_1\, t}\varphi(x), \qquad z(x,\, t) = e^{i\,\omega_2\, t}\psi(x) \tag{4.1}$$

where $\varphi,\, \psi \in H^1(\mathbb{R})$, $\varphi,\, \psi \not\equiv 0$ and $\omega_1,\, \omega_2 \in \mathbb{R}$. Inserting (4.1) into (2.14), (2.15) we obtain the nonlinear eigenvalue problem

$$-\,\omega_1\, \varphi + \alpha\, \varphi_{xx} + \langle Q_1\rangle(\varphi,\, \psi) = 0 \tag{4.2}$$
$$-\,\omega_2\, \psi + \alpha\, \psi_{xx} + \langle Q_2\rangle(\varphi,\, \psi) = 0. \tag{4.3}$$

Consider the $C^1$ functional $\langle H\rangle : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$ defined by

$$\langle H\rangle(w,\, z) = \int_0^1 \int_{\mathbb{R}} \left[\alpha\, |w_x|^2 + \alpha\, |z_x|^2 - \frac{1}{2}\, |T(t)w|^4 - \frac{1}{2}\, |T(t)z|^4 - \beta\, |T(t)w|^2\, |T(t)z|^2\right] dx\, dt. \tag{4.4}$$

Let $P(u) = ||u||^2_{L^2(\mathbb{R})}$ and define the $C^1$ functionals $\mathcal{P}^j : H^1(\mathbb{R}) \times H^1(\mathbb{R}) \to \mathbb{R}$, $j = 1, 2$, by $\mathcal{P}^1(w, z) = P(w)$, $\mathcal{P}^2(w, z) = P(z)$ respectively. Calculating the Fréchet derivatives of $\langle H\rangle$, $\mathcal{P}^1$, $\mathcal{P}^2$ we see that (4.2), (4.3) are the Euler-Lagrange equations for the extrema of $\langle H\rangle$ in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ with the constraints $\mathcal{P}^j(w, z) = \lambda_j > 0$, $j = 1, 2$. We shall seek solutions of (4.2), (4.3) by finding $(w, z) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, $P(w) = \lambda_1$, $P(z) = \lambda_2$ that attains

$$P_{\lambda_1,\lambda_2} = \inf\{\langle H\rangle(w,\, z) : \quad (w,\, z) \in H^1(\mathbb{R}) \times H^1(\mathbb{R}),\ P(w) = \lambda_1 \quad \text{and} \quad P(z) = \lambda_2\}. \tag{4.5}$$

The solution of the constrained minimization problem depends on the sign of the parameter $\alpha$. The case $\alpha > 0$ is examined in subsection 4.1. The cases $\alpha = 0$, $\alpha < 0$ are discussed in subsection 4.2.

In the proof of Theorem 1.1 we will use some facts about related minimization problems for single NLS equations. Define the $C^1$ functional $\langle H_1\rangle : H^1(\mathbb{R}) \to \mathbb{R}$ by

$$\langle H_1\rangle(w) = \int_0^1 \int_{\mathbb{R}} \left[\alpha |w_x|^2 - \frac{1}{2}|T(t)w|^4\right] dx\, dt, \quad \alpha > 0, \tag{4.6}$$

and let

$$P_\lambda = \inf\{\langle H_1\rangle(w) : \quad w \in H^1(\mathbb{R}),\ P(w) = \lambda\}. \tag{4.7}$$

Also, for $z \in H^1(\mathbb{R})$ define the $C^1$ functional $\langle H_{1,z}\rangle : H^1(\mathbb{R}) \to \mathbb{R}$ by

$$\langle H_{1,z}\rangle(w) = \int_0^1 \int_{\mathbb{R}} \left[\alpha |w_x|^2 - \frac{1}{2}|T(t)w|^4 - \beta |T(t)w|^2 |T(t)z|^2\right] dx\, dt, \quad \alpha, \beta > 0, \tag{4.8}$$

and let

$$P^1_\lambda(z) = \inf\{\langle H_{1,z}\rangle(w) : \quad w \in H^1(\mathbb{R}),\ P(w) = \lambda\}. \tag{4.9}$$

The general idea for proving Theorem 1.1 is to show that a minimizing sequence $\{(w_m, z_m)\}_{m\in\mathbb{N}}$ for $\langle H\rangle$ with the above $L^2-$norm constraints converges in $H^1(\mathbb{R})\times H^1(\mathbb{R})$. Our assumptions on $\{(w_m, z_m)\}_{m\in\mathbb{N}}$ are seen to imply that each of the sequences $\{w_m,\}_{m\in\mathbb{N}}$, $\{z_m,\}_{m\in\mathbb{N}}$ satisfies the assumptions of Lemma 3.3. In Lemma 4.1.4 we consider all combinations of three scenarios of Lemma 3.3 for each sequence and show that the only possibility is that both $\{w_m,\}_{m\in\mathbb{N}}$, and $\{z_m,\}_{m\in\mathbb{N}}$ follow the compactness scenario. Most of the effort in the proving this fact goes into ruling out the possibility that at least one of the sequences undergoes splitting in the sense of Lemma 3.3. The see that this does not happen we note from the definitions above that

$$\langle H\rangle(w_m, z_m) = \langle H_{1,z_m}\rangle(w_m) + \langle H_1\rangle(z_m) = \langle H_{1,w_m}\rangle(z_m) + \langle H_1\rangle(w_m). \tag{4.10}$$

In the case where, for instance, $\{w_m,\}_{m\in\mathbb{N}}$ splits, we consider the first equality and see that Lemma 4.1 below implies that there exists a $w \in H^1(\mathbb{R})$, $P(w) = \lambda_1$, such that $\langle H_{1,z_m}\rangle(w_m) > \langle H_{1,z_m}\rangle(w)$. Lemmas 4.2-4.4 below imply that even though $w$ will in general depend on $z_m$, $\langle H_{1,z_m}\rangle(w_m) - \langle H_{1,z_m}\rangle(w)$ is bounded away from zero by a positive constant that is independent of $z_m$. It then easily follows that $\{(w_m, z_m)\}_{m\in\mathbb{N}})$ is not minimizing. The proof of Theorem 1.1 is completed in Theorem 4.1.1, where we show that a minimizing sequence must in fact converge in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.

In Lemma 4.1 below we show the existence of the minimizer for $\langle H_{1,z}\rangle$. The proof is similar to the proof of the existence of a minimizer for $\langle H_1\rangle$ by Zharnitsky *et al*, see [32], and some details are omitted. In particular, all estimates that involve the operator $T$ are as in [32]. The proof is in Section 4.1. In Lemmas 4.2-4.3 we show that under $H^1$ boundedness conditions on the functions involved, the strict subadditivity inequalities for $\langle H_{1,z}\rangle$ can be made uniform in $z$. In Lemma 4.4 we show that a sequence $\{w_m,\}_{m\in\mathbb{N}}$ that splits misses the infimum of $\langle H_{1,z}\rangle$ (i.e. stays above it) by a quantity that is independent of $z$. The proof is in Section 4.1 and uses Lemmas 4.2, 4.3, and the observation that some estimates from part 7 of the proof of Lemma 4.1 are uniform in $z$.

**Lemma 4.1** *Let $z \in H^1(\mathbb{R})$, $\lambda > 0$. Then $P^1_\lambda(z) < 0$ and there exists $\tilde{w} \in H^1(\mathbb{R})$, $P(\tilde{w}) = \lambda$, satisfying $\langle H_{1,z}\rangle(\tilde{w}) = P^1_\lambda(z)$.*

**Lemma 4.2** *Let $\theta > 1$, $\lambda > 0$, $M > 0$, and $z \in H^1(\mathbb{R})$, with $||z||_{H^1(\mathbb{R})} \leq M$. Then there exists $K > 0$, independent of $z$, for which*

$$P^1_{\theta\lambda}(z) \leq \theta P^1_\lambda(z) + \theta(1-\theta)K.$$

*Proof.* Let $\tilde{w} \in H^1(\mathbb{R})$, $P(\tilde{w}) = \lambda$ satisfy $\langle H_{1,z}\rangle(\tilde{w}) = P^1_\lambda(z)$ (such $\tilde{w}$ exists by Lemma 4.1). Let $\theta > 1$. Then

$$\begin{aligned}
\langle H_{1,z}\rangle(\sqrt{\theta}\tilde{w}) &= \int_0^1 \int_{\mathbb{R}} \left[\theta\alpha\, |\tilde{w}_x|^2 - \frac{1}{2}\,\theta^2 |T(t)\tilde{w}|^4 - \beta\,\theta |T(t)\tilde{w}|^2\, |T(t)z|^2\right] dx\, dt \\
&= \theta\langle H_{1,z}\rangle(\tilde{w}) + \theta(1-\theta)\int_0^1 \int_{\mathbb{R}} |T(t)\tilde{w}|^4\, dx\, dt \\
&= \theta P^1_\lambda(z) + \theta(1-\theta)\int_0^1 \int_{\mathbb{R}} |T(t)\tilde{w}|^4\, dx\, dt. \qquad (4.11)
\end{aligned}$$

Therefore

$$P^1_{\theta\lambda}(z) \le \theta P^1_\lambda(z) + \theta(1-\theta)\int_0^1 \int_{\mathbb{R}} |T(t)\tilde{w}|^4 dx\, dt. \qquad (4.12)$$

We want to show that the integral in (4.12) is bounded below by some $K$ independent of $z$. Suppose on the contrary that there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \in H^1(\mathbb{R})$, with $||z_n||_{H^1(\mathbb{R})} \le M$, $\forall n \in \mathbb{N}$, for which the minimizers $\tilde{w}_n$ of $\langle H_{1,z_n}\rangle$ over $w \in H^1(\mathbb{R})$, $P(w) = \lambda$ satisfy

$$\lim_{n\to\infty} \int_0^1 \int_{\mathbb{R}} |T(t)\tilde{w}_n|^4\, dx\, dt = 0. \qquad (4.13)$$

By Lemma 3.1, the definition of $T$, and the boundedness of the sequence $\{z_n\}_{n\in\mathbb{N}}$ in $H^1(\mathbb{R})$ we have

$$||T(t)z_n||^2_{L^4(\mathbb{R})} \le C, \quad \forall n \in \mathbb{N}. \qquad (4.14)$$

Furthermore

$$\int_0^1 \int_{\mathbb{R}} |T(t)\tilde{w}_n|^2\, |T(t)z_n|^2\, dx\, dt \le \int_0^1 ||T(t)\tilde{w}_n||^2_{L^4(\mathbb{R})} ||T(t)z_n||^2_{L^4(\mathbb{R})} dt, \qquad (4.15)$$

hence

$$\lim_{n\to\infty} \int_0^1 \int_{\mathbb{R}} |T(t)\tilde{w}_n|^2\, |T(t)z_n|^2\, dx\, dt = 0. \qquad (4.16)$$

Thus the negative terms of $\langle H_{1,z_n}\rangle(\tilde{w}_n)$ vanish and for any $\epsilon > 0$ there exists $n_0 > 0$ such that $\langle H_{1,z_n}\rangle(\tilde{w}_n) > -\epsilon$, $\forall n > n_0$. On the other hand, for every $z$, $\tilde{w} \in H^1(\mathbb{R})$, $P(\tilde{w}) = \lambda$,

$$\begin{aligned}
P^1_\lambda(z) &\le \int_0^1 \int_{\mathbb{R}} \left[\theta\alpha\, |\tilde{w}_x|^2 - \frac{1}{2}\,\theta^2 |T(t)\tilde{w}|^4 - \beta\,\theta |T(t)\tilde{w}|^2\, |T(t)z|^2\right] dx\, dt \qquad (4.17) \\
&\le \int_0^1 \int_{\mathbb{R}} \left[\theta\alpha\, |\tilde{w}_x|^2 - \frac{1}{2}\,\theta^2 |T(t)\tilde{w}|^4\right] dx\, dt = \langle H_1\rangle(\tilde{w}) < P_\lambda, \qquad (4.18)
\end{aligned}$$

a contradiction, since by [32], $P_\lambda < 0$. □

**Lemma 4.3** *Let $\lambda_1$, $\lambda_2 > 0$, $M > 0$. Let $z \in H^1(\mathbb{R})$, with $||z||_{H^1(\mathbb{R})} \le M$. Define $\gamma$ by* $\min\{\lambda_1, \lambda_2\} = \gamma \max\{\lambda_1, \lambda_2\}$ *and let $\theta = 1 + \gamma$. Then there exists $K > 0$, independent of $z$, for which*

$$P^1_{\lambda_1+\lambda_2}(z) \le P^1_{\lambda_1}(z) + P^1_{\lambda_2}(z) + \theta(1-\theta)K.$$

*Proof.* The case $\lambda_1 = \lambda_2$ follows immediately by Lemma 4.2 with $\lambda = \lambda_1$, $\theta = 2$. Otherwise we may assume without loss of generality that $\lambda_1 = \gamma\lambda_2$ with $\gamma < 1$. Then, by Lemma 4.2

$$\begin{aligned}
P^1_{\lambda_1+\lambda_2}(z) &= P^1_{(1+\gamma)\,\lambda_2}(z) \le (1+\gamma)P^1_{\lambda_2}(z) + \theta(1-\theta)K \\
&= P^1_{\lambda_2}(z) + \gamma P^1_{\gamma^{-1}\lambda_1}(z) + \theta(1-\theta)K \le P^1_{\lambda_2}(z) + P^1_{\lambda_1}(z) + \theta(1-\theta)K. \quad \square
\end{aligned}$$

**Lemma 4.4** *Let $z \in H^1(\mathbb{R})$, $||z||_{H^1(\mathbb{R})} \le M_1$, and $\lambda > 0$. Consider sequence $\{w_j\}_{j\in\mathbb{N}}$ in $H^1(\mathbb{R})$ that satisfies $P(w_j) = \lambda$, $||w_j||_{H^1(\mathbb{R})} \le M_2$, $\forall j \in \mathbb{N}$ and splits in the sense of Lemma 3.3. Then there exists a subsequence $\{w_m\}_{m\in\mathbb{N}}$, and $\mu$, $m_0 > 0$, all independent of $z$, such that for $m > m_0$ we have $\langle H_{1,z}\rangle(w_m) \ge P_\lambda(z) + \mu$.*

**Remark 4.4.1** $\mu$ will in general depend on the sequence $\{w_j\}_{j\in\mathbb{N}}$ (through $\theta$, see the proof of Lemma 4.4 in Section 4.1).

## 4.1 Positive average dispersion

First, we prove the following.

*Claim.* $P_\lambda > -\infty$.

In fact, using Lemma 3.1, the definition of $T$, the Hölder inequality and straightforward calculations, we obtain

$$\begin{aligned}
\langle H\rangle(w,\, z) &\ge \int_0^1 \left( \alpha\, ||w_x||^2_{L^2(\mathbb{R})} + \alpha\, ||z_x||_{L^2(\mathbb{R})} - \frac{1}{2}\,(\beta+1)\, ||T(t)w||^4_{L^4(\mathbb{R})} - \frac{1}{2}\,(\beta+1)\, ||T(t)z||^4_{L^4(\mathbb{R})} \right) dt \\
&\ge \alpha\, ||w_x||^2_{L^2(\mathbb{R})} + \alpha\, ||z_x||^2_{L^2(\mathbb{R})} - (\beta+1)\, \lambda_1^{3/2}\, ||w_x||_{L^2(\mathbb{R})} - (\beta+1)\, \lambda_2^{3/2}\, ||z_x||_{L^2(\mathbb{R})} \\
&= \alpha \left[ ||w_x||^2_{L^2(\mathbb{R})} - \frac{(\beta+1)}{\alpha}\, \lambda_1^{3/2}\, ||w_x||_{L^2(\mathbb{R})} \right] + \alpha \left[ ||z_x||^2_{L^2(\mathbb{R})} - \frac{(\beta+1)}{\alpha}\, \lambda_2^{3/2}\, ||z_x||_{L^2(\mathbb{R})} \right] \\
&= \alpha \left( ||w_x||_{L^2(\mathbb{R})} - \frac{(\beta+1)}{2\,\alpha}\, \lambda_1^{3/2} \right)^2 - \frac{(\beta+1)^2}{4\,\alpha}\, \lambda_1^3 \\
&\quad + \alpha \left( ||z_x||_{L^2(\mathbb{R})} - \frac{(\beta+1)}{2\,\alpha}\, \lambda_2^{3/2} \right)^2 - \frac{(\beta+1)^2}{4\,\alpha}\, \lambda_2^3 \\
&\ge -\frac{(\beta+1)^2}{2\,\alpha}\, (\lambda_1^3 + \lambda_2^3) > -\infty, \qquad \forall\, w,\, z \in H^1(\mathbb{R}).
\end{aligned} \tag{4.1.1}$$

Taking the infimum, the claim follows.

**Lemma 4.1.1** *The minimization problem* (4.5) *with $\alpha > 0$ has negative infimum $P_{\lambda_1,\lambda_2} < 0$.*

*Proof.* Let $w = \sqrt{\lambda_1} v$, $z = \sqrt{\lambda_2} v$, with $P(v) = 1$. Let $\langle \tilde{H}_1\rangle(w) = \frac{1}{2}\langle H\rangle(w,z)$. The minimization problem for the functional $\langle \tilde{H}_1\rangle$, subject to $P(v) = \lambda$, arises in the averaged equation for the single NLS, considered in [32]. The existence of $v$, with $P(v) = 1$, and $\langle \tilde{H}_1\rangle(w) < 0$ is shown in [32], Theorem B.1 ($v$ is a Gaussian). □

The main statement of this section, leading immediately to Theorem 1.1, is the following.

**Theorem 4.1.2** (Existence). *There exists a solution to the problem* (4.5). *Moreover, every minimizing sequence has a subsequence which converges strongly in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$.*

*Proof of Theorem 1.1.* By the $C^1$ regularity of $\langle H\rangle$ the minimizers of Theorem 4.1.2 satisfy (2.14), (2.15). □

**Remark 4.1.3** In the special case $\lambda_1 = \lambda_2 = \lambda$ we have

$$P_{\lambda,\lambda} = 2\tilde{P}_1 = 2\langle \tilde{H}_1\rangle(\phi), \quad \text{where} \tag{4.1.2}$$

$$\langle \tilde{H}_1 \rangle(w) = \int_0^1 \int_{\mathbb{R}} \left[ \alpha \, |w_x|^2 - \frac{\beta+1}{2} \, |T(t)w|^4 \right] dx \, dt, \quad P_\lambda = \inf\{ \langle \tilde{H}_1 \rangle(w) : \quad w \in H^1(\mathbb{R}), \; P(w) = \lambda \},$$

and $\phi =\in H^1(\mathbb{R})$ satisfies $P(\phi) = \lambda$. An analogous result for (2.8), (2.2) with $T = \text{id}$ (i.e. $d \equiv 0$) was shown in [22]. The existence of $\phi$ follows from [32], since $\langle \tilde{H}_1 \rangle$ is $\langle H_1 \rangle$ with a different parameter in front of the nonlinearity. To see (4.1.2), we observe that by the first line of (4.1.1) $\langle H \rangle(w, z) \geq \langle \tilde{H}_1 \rangle(w) + \langle \tilde{H}_1 \rangle(z)$, $\forall w, \; z \in H^1(\mathbb{R})$. Taking a minimizing sequence for $\langle H \rangle$ we then have $P_{\lambda,\lambda} \geq 2\tilde{P}_1$. On the other hand, $P_{\lambda,\lambda} \leq \langle H \rangle(\phi, \phi) = 2\tilde{P}_1$.

To prove Theorem 4.1.2 we first show strong convergence in $L^2(\mathbb{R})$ using Lemma 3.3. We shall use the following lemma.

**Lemma 4.1.4** *In the constrained minimization problem* (4.5) *with positive average dispersion* $\alpha > 0$, *there exists a minimizing sequence* $\{(w_j, z_j)\}_{j \in \mathbb{N}}$ *where the components* $\{w_j\}_{j \in \mathbb{N}}$, $\{z_j\}_{j \in \mathbb{N}}$ *are neither vanishing nor splitting in the sense of Lemma 3.3.*

The lemma uses structural properties of the Hamiltonian and is proved below. We examine combinations of the scenarios of Lemma 3.3 for each sequence and we conclude that both $\{w_j\}_{j \in \mathbb{N}}$, and $\{z_j\}_{j \in \mathbb{N}}$ follow the compactness scenario. In Theorem 4.1.2 we show that each sequence converges strongly in $L^2(\mathbb{R})$, and that the limits are concentrated in a common interval. This implies strong convergence of $\{(w_j, z_j\}_{j \in \mathbb{N}}$ in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$, which is used to show convergence of the quartic term in the Hamiltonian. These results, in combination with lower semicontinuity of the $H^1(\mathbb{R}) \times H^1(\mathbb{R})$-norm, give the existence of a minimizer.

**Remark 4.1.5** By the second line of (4.1.1), we have that for any minimizing sequences of $w_k$, $z_k$, the norms $||w_k||_{H^1(\mathbb{R})}$ and $||z_k||_{H^1(\mathbb{R})}$ are bounded by constants that depend on $\lambda_1$, $\lambda_2$.

*Proof of the Theorem 4.1.2.* Let $\{(w_j, \, z_j)\}_{j \in \mathbb{N}}$ be a minimizing sequence for $\langle H \rangle(w, \, z)$. By inequality (4.1.1), $||w_j||_{H^1(\mathbb{R})}$ and $||z_j||_{H^1(\mathbb{R})}$ must be bounded. From the Alaoglu's Theorem (See [25], page 66), there exists a weakly converging subsequences $w_{j_m}$ and $z_{j_m}$ so that

$$(w_{j_m}, z_{j_m}) \rightharpoonup (w^*, z^*) \quad \text{weakly on} \quad H^1(\mathbb{R}) \times H^1(\mathbb{R}) \tag{4.1.3}$$

for some $(w^*, \, z^*)$ in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Applying Lemma 4.1.4, and Lemma 3.3, we have that the minimizing sequences remains localized as $m \to \infty$. That is, for any $\varepsilon > 0$ there exist $R_1, R_2 > 0$ and sequences $\{x_m\}$, $\{y_m\}$ so that

$$\int_{x_m - R_1}^{x_m + R_1} |w_m(x)|^2 \, dx > \lambda_1 - \varepsilon, \qquad \int_{y_m - R_2}^{y_m + R_2} |z_m(x)|^2 \, dx > \lambda_2 - \varepsilon. \tag{4.1.4}$$

The distance $|x_m - y_m|$ will either remain bounded, $\forall m \in \mathbb{N}$, or diverge. In the case where $|x_n - y_m|$ diverges, $w_m$, $z_m$ are concentrated in finite intervals whose distance diverges. Then the normalized sums $u_m = N_m(w_m + z_m)$, $N_m = (\lambda_1 + \lambda_2)/||w_m + z_m||_{L^2(\mathbb{R})}$, define a sequence $\{u_m\}_{m \in \mathbb{N}} \in H^1(\mathbb{R})$, $P(u_m) = \lambda_1 + \lambda_2$, $\forall m \in \mathbb{N}$, splits in the sense of Lemma 3.3, for we easily check that $w_m$, $z_m$ correspond to the pieces $u'_m$, $u''_m$ of Lemma 3.3. Applying Lemma 3.7, and Lemmas 3.1, 3.3, we have that, for any $\epsilon > 0$

$$\begin{aligned}
& \int_0^1 \int_{\mathbb{R}} |T(t)w'_m|^2 \, |T(t)w''_m|^2 \, dx \, dt \\
= \; & \int_0^1 \left( \int_{S_1} |T(t)w'_m|^2 \, |T(t)w''_m|^2 \, dx + \int_0^1 \int_{S_2} |T(t)w'_m|^2 \, |T(t)w''_m|^2 \, dx \right) dt \\
\leq \; & C\epsilon^{1/2} \int_0^1 (||T(t)w''_m||^2_{L^\infty(\mathbb{R})} + ||T(t)w'_m||^2_{L^\infty(\mathbb{R})}) \, dt \leq c_1 \, \epsilon^{1/2},
\end{aligned} \tag{4.1.5}$$

with $C$, $c_1$ that depend on $\lambda_1$, $\lambda 2$. Thus the coupling term vanishes and the infimum $P_{\lambda_1,\lambda_2}$ of $\langle H \rangle$ is attained by the nontrivial $w$, $z$ that minimize $\langle H_1 \rangle$ over $H^1(\mathbb{R})$ functions with $P(w) = \lambda_1$, $P(z) = \lambda_2$

respectively. Moreover $P_{\lambda_1,\lambda_2} = P^1_{\lambda_1} + P^1_{\lambda_2}$. This value is also attained by arbitrary independent translates $w_X(x) = w(x - X)$, $z_Y(x) = w(x - Y)$ of $w$, $z$ respectively. Since $T(t)$ is an isomorphism in $L^2(\mathbb{R})$, for $w, z \neq 0$, there exist $X$, $Y$ for which

$$-\beta \int_0^1 \int_{\mathbb{R}} |T(t)w_X|^2 |T(t)z_Y|^2 \, dx \, dt < 0. \tag{4.1.6}$$

But then $\langle H\rangle(w_X, z_Y) < P_{\lambda_1,\lambda_2}$, a contradiction.

Thus $|x_m - y_m|$ remain bounded, $\forall m \in \mathbb{N}$. Then we can translate both $w_m$, and $z_m$ by $\tilde{x}_m$ so that $\tilde{w}_m(x) = w_m(x - \tilde{x}_m)$, $\tilde{z}_m(x) = z_m(x - \tilde{x}_m)$ satisfy

$$\int_{-R}^{+R} |\tilde{w}_m(x)|^2 \, dx > \lambda_1 - \varepsilon, \qquad \int_{-R}^{+R} |\tilde{z}_m(x)|^2 \, dx > \lambda_2 - \varepsilon. \tag{4.1.7}$$

The fact that for any $R_j > 0$, the embedding $H^1(\mathbb{R}) \hookrightarrow L^2([-R_j, R_j])$ is compact (see [3], page 21) implies that $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \hookrightarrow L^2([-R_j, R_j]) \times L^2([-R_j, R_j])$ is also compact. It then follows that

$$\int_{-R_1}^{R_1} |w^*(x)|^2 \, dx = \lim_{m\to\infty} \int_{-R_1}^{R_1} |\widetilde{w}_m(x)|^2 \, dx \tag{4.1.8}$$

$$\int_{-R_2}^{R_2} |z^*(x)|^2 \, dx = \lim_{m\to\infty} \int_{-R_2}^{R_2} |\widetilde{z}_m(x)|^2 \, dx. \tag{4.1.9}$$

Using (4.1.8), (4.1.9) in (4.1.4) we then have that for any $\varepsilon > 0$

$$\int_{\mathbb{R}} |w^*(x)|^2 \, dx > \lambda_1 - \varepsilon, \qquad \int_{\mathbb{R}} |z^*(x)|^2 \, dx > \lambda_2 - \varepsilon$$

and therefore $||w^*||^2_{L^2(\mathbb{R})} = \lambda_1$, $||z^*||^2_{L^2(\mathbb{R})} = \lambda_2$. This implies convergence of the $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ norm, which together with weak convergence in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ yields strong convergence of $\{(\tilde{w}_m, \tilde{z}_m)\}_{m\in\mathbb{N}}$ in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$.

*Claim.* $||(\widetilde{w}_m, \widetilde{z}_m) - (w^*, z^*)||_{H^1(\mathbb{R})\times H^1(\mathbb{R})} \to 0$.

In fact, using Lemma 3.1 and the fact that $\widetilde{w}_m$ and $w^*$ are bounded in $H^1(\mathbb{R})$ we obtain

$$||\widetilde{w}_m - w^*||^4_{L^4(\mathbb{R})} \le 2\, ||\widetilde{w}_m - w^*||^3_{L^2(\mathbb{R})} \, ||\partial_x \widetilde{w}_m - \partial_x w^*||_{L^2(\mathbb{R})} \le c\, ||\widetilde{w}_m - w^*||^3_{L^2(\mathbb{R})}$$

and taking the limit we have that $||\widetilde{w}_m - w^*||^4_{L^4(\mathbb{R})} \longrightarrow 0$. In a similar way $||\widetilde{z}_m - z^*||^4_{L^4(\mathbb{R})} \longrightarrow 0$.

Applying the same argument to $T(t)\widetilde{w}_m - T(t)w^*$ we obtain $T(t)\widetilde{w}_m \longrightarrow T(t)w^*$ and hence

$$||T(t)\widetilde{w}_m||^4_{L^4([0,\,1]\times\mathbb{R})} \longrightarrow ||T(t)w^*||^4_{L^4([0,\,1]\times\mathbb{R})}.$$

In a similar way we obtain that $T(t)\widetilde{z}_m \longrightarrow T(t)z^*$ and hence $||T(t)\widetilde{z}_m||^4_{L^4([0,\,1]\times\mathbb{R})} \to ||T(t)z^*||^4_{L^4([0,\,1]\times\mathbb{R})}$. Using (4.1.7), and the fact that the Sobolev norm $||\cdot||_{H^1(\mathbb{R})\times H^1(\mathbb{R})}$ is weakly lower semi-continuous(See [3], page 19), it follows that $||(w^*, z^*||_{H^1(\mathbb{R})\times H^1(\mathbb{R})} \le \lim_{m\to\infty} \inf ||(\widetilde{w}_m, \widetilde{z}_m)||_{H^1(\mathbb{R})\times H^1(\mathbb{R})}$ Moreover, using the convergence of the $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ norm of $(\widetilde{w}_m, \widetilde{z}_m)$ to the $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ norm of $(w^*, z^*)$ we conclude that $P(w^*_x) \le \inf \lim_{m\to\infty} P(\partial_x \widetilde{w}_m)$ and $P(z^*_x) \le \inf \lim_{m\to\infty} P(\partial_x \widetilde{z}_m)$. Therefore $\langle H\rangle(w^*, z^*) \le \lim_{m\to\infty} \langle H\rangle(\widetilde{w}_m, \widetilde{z}_m)$, which can only happen if $\langle H\rangle(w^*, z^*) = \lim_{m\to\infty} \langle H\rangle(\widetilde{w}_m, \widetilde{z}_m)$. Since the negative terms of $\langle H\rangle$ converge to their values at $(w^*, z^*)$ we have that the $L^2(\mathbb{R} \times L^2(\mathbb{R})$ norm of $(\partial_x \widetilde{w}_m, \partial_x \widetilde{z}_m)$ converges to the $L^2(\mathbb{R} \times L^2(\mathbb{R})$ of $(w^*, z^*)$. Combining with the weak convergence, we have that $(\widetilde{w}_m, \widetilde{z}_m)$ converges to $(w^*, z^*)$ strongly in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$. $\square$

*Proof of Lemma 4.1.4. Vanishing does not occur.* We first consider the case where both sequences $\{(w_j\}_{j\in\mathbb{N}}$, $\{z_j\}_{j\in\mathbb{N}}$ vanish in the sense of Lemma 3.3. Then the nonpositive terms of $\langle H\rangle$ must vanish: by Lemma 3.2

$$\int_0^1\int_{\mathbb{R}}|T(t)w_j|^4\,dx\,dt \le \int_0^1\left(C||T(t)||^2_{H^1(\mathbb{R})}\sup_{\phi\in\mathbb{R}}\int_{\phi-1}^{\phi+1}|T(t)w_j|^2dx\right)dt \longrightarrow 0$$

by the assumption that $\{w_j\}_{j\in\mathbb{N}}$ is vanishing, and Lemma 3.6. Similarly,

$$\int_0^1\int_{\mathbb{R}}|T(t)w_j|^2|T(t)w_j|^2\,dx\,dt \le \int_0^1\left(\int_{\mathbb{R}}|T(t)w_j|^4\,dx\int_{\mathbb{R}}|T(t)w_j|^2\,dx\right)^{\frac{1}{2}}dt \longrightarrow 0.$$

Thus $P_{(\lambda_1,\lambda_2)}\ge 0$, contradicting Lemma 4.1.1.

Consider the case where only $\{z_j\}_{j\in\mathbb{N}}$ is vanishing. Then nonpositive terms involving $z_j$ vanish by the above and $P_{\lambda_1,\lambda_2}\ge P^1_{\lambda_1}$, since $|\partial_x z_m|\ge 0$. Using appropriate test functions for $z_m$ that vanish in the sense of Lemma 3.3 we see that $P_{\lambda_1,\lambda_2}=P^1_{\lambda_1}$. Let $w$ satisfy $\langle H_1\rangle(w)=P^1_{\lambda_1}$. But then setting $z=\frac{\sqrt{\lambda_2}}{\sqrt{\lambda_1}}w$, i.e. $P(z)=\lambda_2$, we have $\langle H\rangle(w,z)<P^1_{\lambda_1}=P_{\lambda_1,\lambda_2}$, a contradiction. The argument for the case where only $\{w_j\}_{j\in\mathbb{N}}$ is vanishing is identical.

*Splitting does not occur.* Consider the scenario where at least one component of the minimizing sequence $\{(w_n,z_n)\}_{n\in\mathbb{N}}$ splits. By (4.1.1), $||w_n||_{H^1(\mathbb{R})}$, $||z_n||_{H^1(\mathbb{R})}$ are bounded, $\forall n\in\mathbb{N}$. We may assume that the one that splits is $\{w_n)\}_{n\in\mathbb{N}}$. Using the definitions of $\langle H\rangle$, $\langle H_{1,z}\rangle$, $\langle H_1\rangle$, and Lemma 4.4, we can choose a subsequence $\{w_m\}_{m\in\mathbb{N}}$, $\mu$, $m_0$, all independent of $\{z_n)\}_{n\in\mathbb{N}}$, so that for $m>m_0$ we have

$$\langle H\rangle(w_m,z_m)=\langle H_{1,z_m}\rangle(w_m)+\langle H_1\rangle(z_m)\ge P_{\lambda_1}(z_m)+\mu+\langle H_1\rangle(z_m) \tag{4.1.10}$$

with $\mu>0$, independent of $z_n$ ($\mu$ will in general depend on $\{w_n)\}_{n\in\mathbb{N}}$). Letting $\tilde{w}_m\in H^1(\mathbb{R})$, $P(\tilde{w}_m)=\lambda_1$, satisfy $\langle H_{1,z_m}\rangle(\tilde{w}_m)=P_{\lambda_1}(z_m)$ we therefore have

$$\langle H\rangle(w_m,z_m)\ge\langle H_{1,z_m}\rangle(\tilde{w}_m)+\mu+\langle H_1\rangle(z_n)\ge P_{\lambda_1,\lambda_2}+\mu,\quad\forall m>m_0, \tag{4.1.11}$$

a contradiction with the assumption that $\{(w_n,z_n)\}_{n\in\mathbb{N}}$ is a minimizing sequence. The argument for the case where $\{z_n)\}_{n\in\mathbb{N}}$ is assumed to split is similar, use instead $\langle H\rangle(w_n,z_n)=\langle H_{1,w_n}\rangle(z_n)+\langle H_1\rangle(w_n)$. □

*Proof of Lemma 4.1.* We outline the steps.

*1. Claim.* $P_\lambda(z)>-\infty$.

Let $w$, $z\in H^1(\mathbb{R})$, $P(w)=\lambda$. By Lemma 3.1

$$\int_{\mathbb{R}}|T(t)w|^4dx\,dt\le 2||T(t)w||_{L^2(\mathbb{R})}||\partial_x(T(t)w)||_{L^2(\mathbb{R})}\le 2\lambda^{3/2}||\partial_x w||_{L^2(\mathbb{R})}, \tag{4.1.12}$$

and

$$\int_{\mathbb{R}}|T(t)w|^2|T(t)z|^2dx\,dt\le||T(t)w||^2_{L^4(\mathbb{R})}|T(t)z|^2_{L^4(\mathbb{R})}\le 2C(z)\lambda^{3/4}||\partial_x w||^{1/2}_{L^2(\mathbb{R})}, \tag{4.1.13}$$

where $C(z)$ is a function of $||z||_{H^1(\mathbb{R})}$. Therefore

$$P^1_\lambda(z)\ge||\partial_x w||^2_{L^2(\mathbb{R})}-\lambda^{3/2}||\partial_x w||_{L^2(\mathbb{R})}-2\beta\lambda^{3/4}C(z)||\partial_x w||^{1/2}_{L^2(\mathbb{R})} \tag{4.1.14}$$

which is bounded below by a constant that depends on $\lambda$, and $||z||_{H^1(\mathbb{R})}$.

*2. Claim.* $P_\lambda(z)<0$.

Let $w,\, z \in H^1(\mathbb{R})$. Then $\langle H_{1,z}\rangle(w) \le \langle H_1\rangle(w) \le P_\lambda$. But $P_\lambda < 0$ by [32].

*3. Claim.* Let $z \in H^1(\mathbb{R})$. Let $\lambda,\, \theta > 0$. Then $P_{\theta\lambda}(z) < \theta P_\lambda(z)$.

Let $\theta > 1$, $w \in H^1(\mathbb{R})$. Then

$$\begin{aligned}\langle H_{1,z}\rangle(\sqrt{\theta}w) &= \int_0^1\int_{\mathbb{R}}\left[\theta\alpha\,|w_x|^2 - \frac{1}{2}\,\theta^2|T(t)w|^4 - \beta\,\theta|T(t)w|^2\,|T(t)z|^2\right]dx\,dt \qquad (4.1.15)\\ &= \theta\langle H_{1,z}\rangle(w) + \theta(1-\theta)\int_0^1\int_{\mathbb{R}}|T(t)w|^4 dx\,dt. \qquad (4.1.16)\end{aligned}$$

Let $\{w_n\}_{n\in\mathbb{N}} \in H^1(\mathbb{R})$, $P(w_n) = \lambda$, be a minimizing sequence for $\langle H_{1,z}\rangle$. We check that $\int_{\mathbb{R}}|T(t)w_n|^4dx$ is bounded away form zero: otherwise, by (4.1.13) both negative terms of $\langle H_{1,z}\rangle$ vanish and we have a contradiction with $P^1_\lambda(z) < 0$. Then (4.1.15) implies that there exists a $k > 0$ such that $P^1_{\theta\lambda}(z) \le \theta P^1_\lambda(z) + k < \theta P^1_\lambda(z)$.

*4. Claim.* Let $z \in H^1(\mathbb{R})$. Then $P_{\lambda_1+\lambda_2}(z) < P_{\lambda_1}(z) + P_{\lambda_2}(z)$

This follows immediately from 3, see [32] (the argument also appears in the proof of Lemma 4.3).

*5. Claim:* Let $\{w_n\}_{n\in\mathbb{N}} \in H^1(\mathbb{R})$, $P(w_n) = \lambda$, be a minimizing sequence for $\langle H_{1,z}\rangle$. Then $||w_n||_{H^1(\mathbb{R})} \le M$, $\forall n \in \mathbb{N}$. The constant $M$ is a function of $\lambda$, and $||z||_{H^1(\mathbb{R})}$.

This follows immediately from (4.1.14).

*6.* By Claim 5 the minimizing sequence satisfies the hypothesis of Lemma 3.3. We eliminate the vanishing scenario by combining Lemmas 3.2, 3.6 to show that if $\{w_n\}_{n\in\mathbb{N}}$ vanishes in the sense of Lemma 3.3 then the negative terms of $\langle H_{1,z}\rangle$ vanish and we contradict the fact that $P^1_\lambda(z) < 0$.

*7.* We consider the splitting scenario: $\forall\epsilon > 0$ there exist an $m_0 > 0$ , and a subsequence $\{w_m\}_{m\in\mathbb{N}}$ such that $m > m_0$ implies that $w_m = w'_m + w''_m + h_m$, with $w'_m$, $w''_m$ as in Lemma 3.3. We then have

$$\langle H_{1,z}\rangle(w_m) = \langle H_{1,z}\rangle(w'_m) + \langle H_{1,z}\rangle(w''_m) + R_m, \qquad (4.1.17)$$

where $R_m = R^1_m + R^2_m + R^2_m$, and

$$\begin{aligned}R^1_m &= \int_0^1\int_{\mathbb{R}}(2\alpha\mathrm{Re}[(\partial_x w'_m)^*\partial_x w''_m] - 2(\mathrm{Re}[(T(t)w'_m)^*(T(t)w''_m)])^2 - |T(t)w'_m|^2|T(t)w''_m|^2\\ &\quad -(|T(t)w'_m|^2 + |T(t)w''_m|^2 + 2\beta|T(t)z|^2)\mathrm{Re}[(T(t)w'_m)^*(T(t)w''_m)])\,dx\,dt, \qquad (4.1.18)\end{aligned}$$

$$R^2_m = \int_0^1\int_{\mathbb{R}}(2\alpha\mathrm{Re}[(\partial_x w_m)^*\partial_x h_m])\,dx\,dt, \qquad (4.1.19)$$

$$\begin{aligned}R^3_m &= \int_0^1\int_{\mathbb{R}}(-2(\mathrm{Re}[(T(t)w'_m)^*(T(t)h_m)])^2 - 2(\mathrm{Re}[(T(t)w''_m)^*(T(t)h_m)])^2)\,dx\,dt\\ &\quad \int_0^1\int_{\mathbb{R}}(-(|T(t)w'_m|^2 + |T(t)w''_m|^2)|T(t)h_m|^2 - \frac{1}{2}|T(t)h_m|^4)\,dx\,dt\\ &\quad \int_0^1\int_{\mathbb{R}}(-(|T(t)w'_m|^2 + |T(t)w''_m|^2)\mathrm{Re}[(T(t)(w'_m + w''_m))^*(T(t)h_m)])\,dx\,dt\\ &\quad \int_0^1\int_{\mathbb{R}}(-(\mathrm{Re}[(T(t)w'_m)^*(T(t)w''_m)] + \mathrm{Re}[(T(t)(w'_m + w''_m))^*T(t)h_m])|T(t)h_m|^2)\,dx\,dt\\ &\quad \int_0^1\int_{\mathbb{R}}(-\beta|T(t)z|^2||T(t)h_m|^2 - 2\beta(\mathrm{Re}[(T(t)(w'_m + w''_m))^*T(t)h_m])|T(t)z|^2)\,dx\,dt. \qquad (4.1.20)\end{aligned}$$

To estimate $R^1_m$ we first observe that $(\partial_x w'_m)^* \partial_x w''_m$ vanishes by Lemma 3.3. The remaining terms involve products of $T(t)w'_m$, and $T(t)w''_m$ and are bounded using Lemma 3.7 (as in (4.1.5)). Estimating the other terms in a similar way we find $R^1_m \le C_1 \epsilon^{1/2}$ (assuming $\epsilon \le 1$), where $C_1$ depends on $||z||_{H^1(\mathbb{R})}$, and $||w_m||_{H^1(\mathbb{R})}$.

$R^3_m$ contains terms proportional to $T(t)h_m$ or its modulus. These can be estimated using Lemma 3.1 and the fact that, by Lemma 3.3, $||h_m||_{L^2(\mathbb{R})} \le \epsilon$, and $||\partial_x h_m||_{L^2(\mathbb{R})} \le 5||w_m||_{H^1(\mathbb{R})}$. For instance, on line 3 of (4.1.20), by Lemmas 3.1, 3.3,

$$\begin{aligned} & \int_{\mathbb{R}} ||T(t)w'_m|^2 + |T(t)w''_m|^2||\mathrm{Re}[(T(t)(w'_m + w''_m))^*(T(t)h_m)]|\,dx \\ \le\ & 4(||T(t)w'_m|^2|_{L^\infty(\mathbb{R})} + ||T(t)w''_m|^2|_{L^\infty(\mathbb{R})}) \int_{\mathbb{R}} |T(t)(w'_m + w''_m)||T(t)h_m|\,dx \\ \le\ & c_3||T(t)h_m||_{L^2(\mathbb{R})} \le c_3\,\epsilon, \quad \forall t \in [0,1], \end{aligned} \tag{4.1.21}$$

where $c_3$ depends on $||w_m||_{H^1(\mathbb{R})}$. Other terms are estimated similarly, and we see that $R^1_m \le C_3\epsilon$ (assuming $\epsilon \le 1$), where $C_3$ depends on $||z||_{H^1(\mathbb{R})}$, and $||w_m||_{H^1(\mathbb{R})}$.

The integrand in $R^2_m$ is proportional to $\partial_x h_m$. This is not necessarily small, however it can be written as small plus nonegative: using Remark 3.3.1 we may write $h_m = (1 - \rho_m + \vartheta_m)u_m$, where $\rho_m(x) = \rho(x - x_m)$, $\vartheta_m(x) = \vartheta(x - x_m)$. Then

$$R^2_m = \int_0^1 \int_{\mathbb{R}} (-\mathrm{Re}[(\partial_x w_m)^*(\partial_x \rho_m + \partial_x \vartheta_m)u_m])\,dx\,dt + \int_0^1 \int_{\mathbb{R}} (1 - \rho_m + \vartheta_m)|\partial_x u_m|^2)\,dx\,dt. \tag{4.1.22}$$

Using the bounds on $\partial_x \rho_m$, $\partial_x \vartheta_m$ from Remark 3.3.1, the first integral, denoted by $\tilde{R}^2_m$, is estimated as

$$\begin{aligned} |\tilde{R}^2_m| &\le \int_{\mathbb{R}} |\mathrm{Re}[(\partial_x w_m)^*(\partial_x \rho_m + \partial_x \vartheta_m)u_m]|\,dx \\ &\le ||\partial_x u_m||_{L^2(\mathbb{R})}||(\partial_x \rho_m + \partial_x \vartheta_m)u_m||_{L^2(\mathbb{R})} \\ &\le ||\partial_x u_m||_{L^2(\mathbb{R})}||\partial_x \rho_m + \partial_x \vartheta_m)||_{L^\infty(\mathbb{R})}||u_m||_{L^2(\mathbb{R})} \le \tilde{C}_2\epsilon,. \end{aligned} \tag{4.1.23}$$

with $\tilde{C}_2$ depending on $||w_m||_{H^1(\mathbb{R})}$. The second integral in (4.1.22) is nonegative.

Also, by Lemma 3.3, $P(w'_m) = \lambda_1 + \beta'_m$, $P(w''_m) = \lambda_2 + \beta''_m$, with $\lambda_1 + \lambda_2 = \lambda$, $|\beta'_m|$, $|\beta''_m| < \epsilon$. Letting

$$\tilde{w}'_m = \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1 + \beta'_m}} w'_m, \quad \tilde{w}''_m = \frac{\sqrt{\lambda_2}}{\sqrt{\lambda_2 + \beta''_m}} w''_m, \tag{4.1.24}$$

$$r'_m = \langle H_{1,z}\rangle(w'_m) - \langle H_{1,z}\rangle(\tilde{w}'_m), \quad r''_m = \langle H_{1,z}\rangle(w''_m) - \langle H_{1,z}\rangle(\tilde{w}''_m) \tag{4.1.25}$$

we easily check that $|r'_m|$, $|r''_m| \le C\epsilon$, with $C$ depending on $||z||_{H^1(\mathbb{R})}$, $||w_m||_{H^1(\mathbb{R})}$.

Collecting the above we have

$$\langle H_{1,z}\rangle(w_m) \ge \langle H_{1,z}\rangle(\tilde{w}'_m) + \langle H_{1,z}\rangle(\tilde{w}''_m) + \tilde{R}_m, \quad \text{with} \tag{4.1.26}$$

$$\tilde{R}_m = R^1_m + \tilde{R}^2_m + R^3_m + r'_m + r''_m, \quad |\tilde{R}_m| \le \tilde{C}\epsilon^{1/2}, \tag{4.1.27}$$

where using also the boundedness of $||w_m||_{H^1(\mathbb{R})}$ (Claim 6), $\tilde{C}$ depends on $\lambda$, and $||z||_{H^1(\mathbb{R})}$. Taking $\epsilon$ sufficiently small and using strict subadditivity (Claim 5), (4.1.26), (4.1.27) imply $\langle H_{1,z}\rangle(w_m) > P_\lambda(z)$, a contradiction.

*8.* Once the vanishing and splitting scenarios are eliminated strong convergence in $H^1(\mathbb{R})$ up to translations follows as in [32]. $\square$

*Proof of Lemma 4.4.* By the splitting assumption we have that $\forall\epsilon > 0$ there exist an $\tilde{m} > 0$ , and a subsequence $\{w_m\}_{m\in\mathbb{N}}$ such that $m > \tilde{m}$ implies that $w_m = w'_m + w''_m + h_m$, with $w'_m$, $w''_m$ as in Lemma 3.3. Then $\langle H_{1,z}\rangle(w_m) \geq \langle H_{1,z}\rangle(w'_m) + \langle H_{1,z}\rangle(w''_m) + \tilde{R}_m$, with $\tilde{R}_m$ as in (4.1.27). Bounding $\tilde{R}_m$ as in Lemma 4.1 we additionally check that $\tilde{R}_m \leq \tilde{C}\epsilon^{1/2}$, where $\tilde{C}$ only depends only on $M_1$, $M_2$. By Lemma 4.3 we then have

$$\langle H_{1,z}\rangle(w_m) \geq P^1_\lambda(z) + \theta(1-\theta)K + \tilde{R}_m,$$

with $K$ independent of $z$. By Lemmas 3.3, 4.3 $\theta$ is determined by the sequence $\{w_j\}_{j\in\mathbb{N}}$. The statement follows by setting $\mu = \frac{1}{2}\theta(1-\theta)K$, and choosing $\epsilon$ (sufficiently small) and a corresponding subsequence of $\{w_j\}_{j\in\mathbb{N}}$. $\square$

We add some remarks on the stability of standing wave solutions. Let $M_{(\lambda_1,\lambda_2)}$ by the set of $(u,v)$ that minimize $\langle H\rangle$ over $(w,z) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, with $P(w) = \lambda_1$, $P(z) = \lambda_2$. Let $(\tau_{a,y}\phi)(x) = e^{ia}\phi(x-y)$, $a, y \in \mathbb{R}$, and for $U \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Also let $\tau(U)$ be the set of all $(\tau_{a_1,y}u_1, \tau_{a_2,y}u_2)$, with $a_1$, $a_2$, $y \in \mathbb{R}$, $u_1$, $u_2 \in H^1(\mathbb{R})$. Note that $U \in M_{(\lambda_1,\lambda_2)}$ implies that $\tau(U) \in M_{(\lambda_1,\lambda_2)}$. For $(x_1,x_2) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, $U \subset H^1(\mathbb{R}) \times H^1(\mathbb{R})$ we say that $x$ is $\epsilon-$close to $U$ is there exists $(y_1,y_2) \in U$ such that $||x_1 - y_1||_{H^1(\mathbb{R}} + ||x_2 - y_2||_{H^1(\mathbb{R}} < \epsilon$.

A solution of the form (1.8) of (1.6), (1.7) is *orbitally stable* if $\forall\epsilon > 0$ there exists a neighborhood $U_\epsilon$ of $(\phi,\psi) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ so that any $(w(t), z(t))$ satisfying (1.6), (1.7) with initial condition $(w(0), z(0)) \in U_\epsilon$, remains $\epsilon-$close to $\tau((\phi,\psi))$, $\forall t \in \mathbb{R}$.

**Proposition 4.1.6** Let $x = (\varphi,\psi) \in M_{\lambda_1,\lambda_2}$, and assume that $\tau(x) = M_{(\lambda_1,\lambda_2)}$. Then the corresponding standing wave solution of (1.6), (1.7) is orbitally stable.

*Proof.* The statement follows from the continuity of the solutions $(w(t), z(t))$ of (2.14), (2.15), the conservation of $\langle H\rangle$, $\mathcal{P}^1$, $\mathcal{P}^2$, see Remark 2.1, and the argument of Ohta, see [22], p. 937. $\square$

Since the validity of assumption $\tau(x) = M_{(\lambda_1,\lambda_2)}$ is not known, Theorem 1.1 only implies a weaker stability statement below, using essentially the argument of [22], p. 937 (we omit the proof). In particular, given $x \in M_{(\lambda_1,\lambda_2)}$, let $x \in M_{(\lambda_1,\lambda_2),c}(x)$ be the set of $y \in M_{(\lambda_1,\lambda_2)}$ that can be connected to $x$ by a continuous path $\gamma : [0,1] \to H^1(\mathbb{R}) \times H^1(\mathbb{R})$ satisfying $P_1(\gamma(t)) = \lambda_1$, $P_2(\gamma(t)) = \lambda_2$, $\forall t \in [0,1]$. Note that $\tau(x) \subset M_{(\lambda_1,\lambda_2),c}(x)$. Then we have the following.

**Proposition 4.1.7** Let $x = (\varphi,\psi) \in M_{(\lambda_1,\lambda_2)}$. Then $\forall\epsilon > 0$ there exists a neighborhood $U_\delta$ of $x \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$ such that any $(u(t), v(t))$ satisfying (1.6), (1.7) with initial condition $(u(0), v(0)) \in U_\epsilon$ remains $\epsilon-$close to $M_{(\lambda_1,\lambda_2),c}(x)$, $\forall t \in \mathbb{R}$.

## 4.2 Nonpositive average dispersion

In the case $\alpha = 0$ we can use Strichartz-type estimates to bound the Hamiltonian from below in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$. However, we do not have an $H^1$ bound of minimizing sequences (i.e. as in Remark 4.1.5) and we may have loss of compactness due to loss of control of derivatives. For the single NLS equation this problem was analyzed successfully in [15], where it was shown that vanishing and splitting of the minimizing sequence is not possible in neither Fourier nor physical space and that we are back to the classical situation where Sobolev's embedding theorem can be applied. Similar ideas seem to apply for the system as well, however the arguments are more lengthy and technical and we will not pursue this case here.

For the case $\alpha < 0$, the minimization problem can not have a globally minimizing ground state solution since $P_\lambda = -\infty$. We show that any critical points that may exist can not be local minima.

**Theorem 4.3.1.** *Let $(w,\, z)$ be a critical point of the constrained averaged variational principle* (4.5) *with negative average dispersion. Then, for any $\varepsilon > 0$, there exists $(\phi,\, \psi) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, such that $||\phi||^2_{L^2(\mathbb{R})} = \lambda_1$, $||\psi||^2_{L^2(\mathbb{R})} = \lambda_2$, $||w - \phi||_{H^1(\mathbb{R})} < \varepsilon$, $||z - \psi||_{H^1(\mathbb{R})} < \varepsilon$ and $\langle H\rangle(w,\, z) > \langle H\rangle(\phi,\, \psi)$.*

*Proof.* If $(w,\, z)$ is a critical point of the constrained averaged principle (4.5) with $\alpha < 0$ then by Lemma 3.1, $(w,\, z) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, otherwise $\langle H\rangle$ would be unbounded.

On the other hand, we perturb $(w,\, z)$ with an arbitrary small high frequency radiation at the tails, which will produce a smaller change in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ but yet an even small change in $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and $L^4([0,\, 1] :\, L^4(\mathbb{R})) \times L^4([0,\, 1] :\, L^4(\mathbb{R}))$. Let $\rho \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp} \rho \subseteq [-\frac{1}{2},\, \frac{1}{2}]$, $0 \le \rho \le 1$ and $a_n > 0$ is specifically large and will be chosen later. We define $\phi_n = \frac{1}{n^2} e^{in(x-a_n)} \rho_{a_n}(x)$ with $\rho_{a_n}(x) = \rho(x - a_n)$. Then $\phi_n \in \mathcal{D}(\mathbb{R})$, $\operatorname{supp} \phi_n \subseteq \operatorname{supp} \rho_{a_n}(x) \subseteq \left[a_n - \frac{1}{2},\, a_n + \frac{1}{2}\right]$. Let $w_n = w + \phi_n$ and $z_n = z + \phi_n$. Using that $|w_n|^2 = w_n \cdot \overline{w}_n = (w + \phi_n)(\overline{w} + \overline{\phi}_n) = |w|^2 + |\phi_n|^2 + 2\, Re\, (\overline{w}\, \phi_n)$ we have the following estimates

$$
\begin{aligned}
||w_n||^2_{L^2(\mathbb{R})} &= ||w||^2_{L^2(\mathbb{R})} + ||\phi_n||^2_{L^2(\mathbb{R})} + 2\, Re \int_{\mathbb{R}} \overline{w}\, \phi_n\, dx \\
&\approx ||w||^2_{L^2(\mathbb{R})} + \mathcal{O}\left(\frac{1}{n^4}\right) + \mathcal{O}\left(\frac{1}{n^{2+q_n}}\right) \\
&\approx \lambda_1 + \varphi(n)
\end{aligned} \tag{4.2.1}
$$

where $\varphi(n) \approx \mathcal{O}\left(\frac{1}{n^4}\right)$, $|w| < 1/n^{q_n}$ for $x \in \operatorname{supp} \rho_{a_n}$.

$$
\begin{aligned}
||\partial_x w_n||^2_{L^2(\mathbb{R})} &= ||w_x||^2_{L^2(\mathbb{R})} + ||\partial_x \phi_n||^2_{L^2(\mathbb{R})} + 2\, Re \int_{\mathbb{R}} \partial_x \overline{w}\, \partial_x \phi_n\, dx \\
&\le ||w_x||^2_{L^2(\mathbb{R})} + \frac{c_\rho}{n^2} - 2\, Re \int_{\mathbb{R}} \overline{w}\, \partial_{xx} \phi_n\, dx \\
&\approx ||w_x||^2_{L^2(\mathbb{R})} + \frac{c_\rho}{n^2} + \mathcal{O}\left(\frac{1}{n^{q_n}}\right)
\end{aligned} \tag{4.2.2}
$$

since $\partial_x \phi_n = \frac{e^{i\, n\, (x-a_n)}}{n^2} \left(i\, n\, \rho_{a_n}(x) + \partial_x \rho_{a_n}(x)\right)$. Moreover,

$$
||T w_n||^4_{L^4([0,\,1]:\; L^4(\mathbb{R}))} = ||T w||^4_{L^4([0,\,1]:\; L^4(\mathbb{R}))} + R
$$

where

$$
\begin{aligned}
|R| \;\le\;& ||T\phi_n||^4_{L^4([0,\,1]:\; L^4(\mathbb{R}))} + 4 \int_0^1 \int_{\mathbb{R}} |Tw|^3\, |T\phi_n|\, dx\, dt \\
&+ 6 \int_0^1 \int_{\mathbb{R}} |Tw|^2\, |T\phi_n|^2\, dx\, dt + 4 \int_0^1 \int_{\mathbb{R}} |Tw|\, |T\phi_n|^3\, dx\, dt.
\end{aligned}
$$

We estimate these integrals using Lemma 3.6 and they turn out to be small. In a similar way we obtain the same estimates for $z_n$. From (4.2.1), by scaling the sequences with $\sqrt{\lambda_j + \varphi(n)}$, we obtain new sequences $w_n'$ and $z_n'$ respectively satisfying the constraint $||w_n'||^2_{L^2(\mathbb{R})} = \lambda_1$, $||z_n'||^2_{L^2(\mathbb{R})} = \lambda_2$, and satisfying the following $\langle H\rangle(w_n',\, z_n') < \langle H\rangle(w,\, z)$ with $||w_n' - w||_{H^1(\mathbb{R})} \to 0$ and $||z_n' - z||_{H^1(\mathbb{R})} \to 0$ as $n \to \infty$. □

# 5 Standing waves with prescribed frequencies

In this section we find solutions of the nonlinear eigenvalue problem (4.2), (4.3) with $\omega_1,\, \omega_2 > 0$. It will be assumed that $\alpha > 0$.

Consider the $C^1$ functional $\mathbb{H}: H^1(\mathbb{R}) \times H^1(\mathbb{R}) \longrightarrow \mathbb{R}$, $\mathbb{H} \in C^1$ defined by

$$\begin{aligned}\mathbb{H}(\varphi,\, \psi) \quad &= \quad \int_{\mathbb{R}} (\, \omega_1\, |\varphi|^2 + \omega_2\, |\psi|^2 + \alpha\, |\varphi_x|^2 + \alpha\, |\psi_x|^2\, )\, dx \\ & \quad - \int_0^1 \int_{\mathbb{R}} \left[ \frac{1}{2}\, |T(t)\varphi|^4 + \frac{1}{2}\, |T(t)\psi|^4 + \beta\, |T(t)\varphi|^2\, |T(t)\psi|^2 \right] dx\, dt, \end{aligned} \tag{5.1}$$

for $(\varphi,\, \psi) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$. Calculating the Fréchet derivative of $\mathbb{H}$ we see that critical points of $\mathbb{H}$ must satisfy (4.2), (4.3).

We will find critical points of $\mathbb{H}$ by applying the Mountain Pass Lemma.

Consider the norm $\mathbb{H}$

$$||(\varphi,\, \psi)||_{\mathbb{H}}^2 = \int_{\mathbb{R}} (\, \omega_1\, |\varphi|^2 + \omega_2\, |\psi|^2 + \alpha\, |\varphi_x|^2 + \alpha\, |\psi_x|^2\, )\, dx \tag{5.2}$$

for $(\varphi,\, \psi) \in H^1(\mathbb{R},\, \mathbb{R}^2)$.
Let $E = H^1(\mathbb{R},\, \mathbb{R}^2)$, and $U = \mathbb{B}_\rho(0)$. Let

$$||(\varphi,\, \psi)||_E = ||(\varphi,\, \psi)||_{H^1(\mathbb{R})} = \int_{\mathbb{R}} (\, |\varphi|^2 + |\psi|^2 + |\varphi_x|^2 + |\psi_x|^2\, )\, dx. \tag{5.3}$$

Note that the norms $||\cdot||_{\mathbb{H}}$ and $||\cdot||_E$ are equivalent. Also note that

$$\mathbb{H}(\varphi,\, \psi) = ||(\varphi,\, \psi)||_{\mathbb{H}}^2 - \int_0^1 \int_{\mathbb{R}} \left[ \frac{1}{2}\, |T(t)\varphi|^4 + \frac{1}{2}\, |T(t)\psi|^4 + \beta\, |T(t)\varphi|^2\, |T(t)\psi|^2 \right] dx\, dt. \tag{5.4}$$

*Proof Of Theorem 1.2.* We have

$$\beta \int_0^1 \int_{\mathbb{R}} |T(t)\varphi|^2 |T(t)\psi|^2 dx\, dt \le \frac{\beta}{2} \int_0^1 \left( ||T(t)\varphi||_{L^4(\mathbb{R})}^4 + ||T(t)\psi||_{L^4(\mathbb{R})}^4 \right) dt.$$

Then

$$\begin{aligned} & - \int_0^1 \int_{\mathbb{R}} \left[ \frac{1}{2}\, |T(t)\varphi|^4 + \frac{1}{2}\, |T(t)\psi|^4 + \beta\, |T(t)\varphi|^2\, |T(t)\psi|^2 \right] dx\, dt \\ \ge & \quad - \frac{(\beta+1)}{2} \int_0^1 \left( ||T(t)\varphi||_{L^4(\mathbb{R})}^4 + ||T(t)\psi||_{L^4(\mathbb{R})}^4 \right) dt. \end{aligned}$$

Using Lemma 3.1

$$- \int_0^1 \int_{\mathbb{R}} \left[ \frac{1}{2}\, |T(t)\varphi|^4 + \frac{1}{2}\, |T(t)\psi|^4 + \beta\, |T(t)\varphi|^2\, |T(t)\psi|^2 \right] dx\, dt \ge -\, (\beta+1)\, ||(\varphi,\, \psi)||_E^3.$$

Then in (5.4) we have

$$\mathbb{H}(\varphi,\, \psi) \ge ||(\varphi,\, \psi)||_{\mathbb{H}}^2 - (\beta+1)\, ||(\varphi,\, \psi)||_E^3. \tag{5.5}$$

By the equivalence of the norms $||\cdot||_E$, $||\cdot||_{\mathbb{H}}$

$$\mathbb{H}(\varphi,\, \psi) \ge \frac{1}{c_2}\, ||(\varphi,\, \psi)||_E^2 - (\beta+1)\, ||(\varphi,\, \psi)||_E^3 = \frac{1}{c_2}\, ||(\varphi,\, \psi)||_E^2\, (1 - c_2\, (\beta+1)\, ||(\varphi,\, \psi)||_E)\, .$$

Let $||(\varphi,\, \psi)||_E = \rho$ and $c = c_2\, (\beta+1)$, then the graph $\frac{\rho^2}{c}\, (1 - c\, \rho)$ is strictly positive for $\rho \in \left(0,\, \frac{1}{c}\right)$ $\left(1 - c\, \rho > 0 \Longleftrightarrow \rho < \frac{1}{c}\right)$. Take $\rho = \frac{2}{3\, c}$. Hence, for $(\varphi,\, \psi) \in \partial\mathbb{B}_\rho$, i. e., $||(\varphi,\, \psi)||_E = \rho$, we have

$$\mathbb{H}(\varphi,\, \psi) \ge \frac{1}{c} \left( \frac{2}{3\, c} \right)^2 \left( 1 - c\, \frac{2}{3\, c} \right) = \frac{4}{27\, c^3} = a > 0.$$

Moreover, $\mathbb{H}(0, 0) = 0$. Hence, the functional $\mathbb{H}$ has a strict local minimum at 0 in the function space $E = H^1(\mathbb{R}, \mathbb{R}^2)$.

*Claim.* We have

$$\mathbb{H}(\theta\,\varphi_0,\,\theta\,\psi_0) \longrightarrow -\,\infty \quad \text{as} \quad \theta \longrightarrow +\,\infty. \tag{5.6}$$

In fact,

$$\begin{aligned}\mathbb{H}(\theta\,\varphi_0,\,\theta\,\psi_0) &= ||(\theta\,\varphi_0,\,\theta\,\psi_0)||_{\mathbb{H}}^2 - \int_0^1\int_{\mathbb{R}}\left[\frac{1}{2}\,\theta^4\,|T(t)\varphi_0|^4 + \frac{1}{2}\,\theta^4\,|T(t)\psi_0|^4 + \beta\,\theta^4\,|T(t)\varphi_0|^2\,|T(t)\psi_0|^2\right]dx\,dt \\ &= \theta^2\,||(\varphi_0,\,\psi_0)||_{\mathbb{H}}^2 - \theta^4\int_0^1\int_{\mathbb{R}}\left[\frac{1}{2}\,|T(t)\varphi_0|^4 + \frac{1}{2}\,|T(t)\psi_0|^4 + \beta\,|T(t)\varphi_0|^2\,|T(t)\psi_0|^2\right]dx\,dt.\end{aligned}$$

Then

$$\frac{\mathbb{H}(\theta\,\varphi_0,\,\theta\,\psi_0)}{\theta^2} = ||(\varphi_0,\,\psi_0)||_{\mathbb{H}}^2 - \theta^2\int_0^1\int_{\mathbb{R}}\left[\frac{1}{2}\,|T(t)\varphi_0|^4 + \frac{1}{2}\,|T(t)\psi_0|^4 + \beta\,|T(t)\varphi_0|^2\,|T(t)\psi_0|^2\right]dx\,dt. \tag{5.7}$$

Choose $(\varphi_0,\,\psi_0) \in E$ fixed, we obtain in (5.7)

$$\lim_{\theta\to+\,\infty}\frac{\mathbb{H}(\theta\,\varphi_0,\,\theta\,\psi_0)}{\theta^2} = -\,\infty.$$

The claim follows.

Therefore, $\mathbb{H}(\varphi,\,\psi)$ satisfies the conditions of the Mountain Pass Lemma. Hence, applying the Mountain Pass Lemma we obtain a subsequence $\{(\varphi_j,\,\psi_j)\}_{j\in\mathbb{N}}$ in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ with the following properties:

$$\mathbb{H}(\varphi_j,\,\psi_j) \longrightarrow c \qquad \text{and} \qquad ||\mathbb{H}'(\varphi_j,\,\psi_j)|| \longrightarrow 0 \qquad \text{as} \quad j \longrightarrow +\,\infty \tag{5.8}$$

where $c$ is a positive constant.

*Claim.* Any sequence $\{(\varphi_j,\,\psi_j)\}_{j\in\mathbb{N}}$ in $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ satisfying (5.8) must be bounded.

In fact, suppose that $\{(\varphi_j,\,\psi_j)\}_{j\in\mathbb{N}}$ satisfies (5.8), but $||(\varphi_j,\,\psi_j)|| \longrightarrow \infty$ as $j \to \infty$ where $||\cdot||$ can be either $||\cdot||_{\mathbb{H}}$ or $||\cdot||_{H^1(\mathbb{R})\times H^1(\mathbb{R})}$. It follows that

$$\frac{\mathbb{H}(\varphi_j,\,\psi_j)}{||(\varphi_j,\,\psi_j)||^2} \longrightarrow 0 \qquad \text{as} \quad n \to \infty \tag{5.9}$$

and also

$$\frac{\mathbb{H}'(\varphi_j,\,\psi_j)\cdot(\varphi_j,\,\psi_j)}{||(\varphi_j,\,\psi_j)||^2} \longrightarrow 0 \qquad \text{as} \quad n \to \infty. \tag{5.10}$$

On the other hand

$$\begin{aligned}\mathbb{H}(\varphi_j,\,\psi_j) &= \int_{\mathbb{R}}(\,\omega_1\,|\varphi_j|^2 + \omega_2\,|\psi_j|^2 + \alpha\,|\varphi_{j_x}|^2 + \alpha\,|\psi_{j_x}|^2\,)\,dx \\ &\quad - \frac{1}{2}\int_0^1\int_{\mathbb{R}}\left[\,|T(t)\varphi_j|^4 + |T(t)\psi_j|^4 + 2\,\beta\,|T(t)\varphi_j|^2\,|T(t)\psi_j|^2\,\right]dx\,dt,\end{aligned}$$

for $(\varphi_j,\,\psi_j) \in H^1(\mathbb{R}) \times H^1(\mathbb{R})$, and

$$\begin{aligned}\mathbb{H}'(\varphi_j,\,\psi_j)\cdot(\varphi_j,\,\psi_j) &= 2\int_{\mathbb{R}}(\,\omega_1\,|\varphi_j|^2 + \omega_2\,|\psi_j|^2 + \alpha\,|\varphi_{j_x}|^2 + \alpha\,|\psi_{j_x}|^2\,)\,dx \\ &\quad - 2\int_0^1\int_{\mathbb{R}}\left[\,|T(t)\varphi_j|^4 + |T(t)\psi_j|^4 + 2\,\beta\,|T(t)\varphi_j|^2\,|T(t)\psi_j|^2\,\right]dx\,dt\end{aligned}$$

for $(\varphi_j,\,\psi_j)\in H^1(\mathbb{R})\times H^1(\mathbb{R})$.

Moreover,

$$2\,\mathbb{H}(\varphi_j,\,\psi_j)-\mathbb{H}'(\varphi_j,\,\psi_j)\cdot(\varphi_j,\,\psi_j)=\int_0^1\int_{\mathbb{R}}\big[\,|T(t)\varphi_j|^4+|T(t)\psi_j|^4+2\,\beta\,|T(t)\varphi_j|^2\,|T(t)\psi_j|^2\,\big]\,dx\,dt.$$

Using (5.9) and (5.10) we have

$$0=\lim_{j\to\infty}\frac{\int_0^1\int_{\mathbb{R}}\big[\,|T(t)\varphi_j|^4+|T(t)\psi_j|^4+2\,\beta\,|T(t)\varphi_j|^2\,|T(t)\psi_j|^2\,\big]\,dx\,dt}{||(\varphi_j,\,\psi_j)||^2}.$$

Moreover

$$\mathbb{H}'(\varphi_j,\,\psi_j)\cdot(\varphi_j,\,\psi_j)=2\,||(\varphi_j,\,\psi_j)||^2_{\mathbb{H}}-2\int_0^1\int_{\mathbb{R}}\big[\,|T(t)\varphi_j|^4+|T(t)\psi_j|^4+2\,\beta\,|T(t)\varphi_j|^2\,|T(t)\psi_j|^2\,\big]\,dx\,dt.$$

Dividing by $||(\varphi_j,\,\psi_j)||^2$ and letting $j\to\infty$ gives $0=2-0=2$, which is a contradiction. Thus $\{(\varphi_j,\,\psi_j)\}_{j\in\mathbb{N}}$ must be bounded. The claim follows.

Thus, there exists a subsequence, still denoted by $\{(\varphi_j,\,\psi_j)\}_{j\in\mathbb{N}}$ such that

$$\begin{aligned}\varphi_j &\rightharpoonup \varphi \quad \text{weakly on} \quad H^1(\mathbb{R})\\ \psi_j &\rightharpoonup \psi \quad \text{weakly on} \quad H^1(\mathbb{R}).\end{aligned}$$

*Claim.* $(\varphi,\,\psi)$ is nontrivial.

In fact, since $||\mathbb{H}'(\varphi_j,\,\psi_j)||\longrightarrow 0$ and $\{(\varphi_j,\,\psi_j)\}$ is bounded, then $\mathbb{H}'(\varphi_j,\,\psi_j)\cdot(\varphi_j,\,\psi_j)\longrightarrow 0$. Hence,

$$\begin{aligned}&2\,\mathbb{H}(\varphi_j,\,\psi_j)-\mathbb{H}'(\varphi_j,\,\psi_j)\cdot(\varphi_j,\,\psi_j)\\ &=\int_0^1\int_{\mathbb{R}}[\,|T(t)\varphi_j|^4+|T(t)\psi_j|^4+2\,\beta\,|T(t)\varphi_j|^2|T(t)\psi_j|^2\,]\,dx\,dt\longrightarrow 0.\end{aligned}$$

Using the Palais-Smale condition we have that the sequence cannot be vanishing.

Indeed,

$$\begin{aligned}\frac{c}{2} &< \int_{\mathbb{R}}[\,|T(t)\varphi_j|^4+|T(t)\psi_j|^4+2\,\beta\,|T(t)\varphi_j|^2|T(t)\psi_j|^2\,]\,dx\,dt\\ &\le (\beta+1)\int_{\mathbb{R}}[\,|T(t)\varphi_j|^4+|T(t)\psi_j|^4\,]\,dx\,dt\\ &< (\beta+1)\,\Big(||T(t_j)\varphi_j||^2_{L^\infty(\mathbb{R})}\,||\varphi_j||^2_{L^2(\mathbb{R})}+||T(t_j)\varphi_j||^2_{L^\infty(\mathbb{R})}\,||\varphi_j||^2_{L^2(\mathbb{R})}\Big)\end{aligned}$$

for some $t_j\in[0,\,1]$ given that $j$ is sufficiently large and therefore

$$||T(t_j)\varphi_j||_{L^\infty(\mathbb{R})}>\frac{c_1}{2}>0 \qquad \text{and} \qquad ||T(t_j)\psi_j||_{L^\infty(\mathbb{R})}>\frac{c_1}{2}>0.$$

Therefore, by rearranging the sequence $(\varphi_j,\,\psi_j)$ so that the maxima are assumed at $x=0$, we obtain that the weak limit $(\varphi,\,\psi)$ is nontrivial.

Finally, we show that $\mathbb{H}'(\varphi,\,\psi)\cdot(u,\,v)=0$ for any $(u,\,v)\in H^1(\mathbb{R})\times H^1(\mathbb{R})$.

In fact, we show that the expression

$$
\begin{aligned}
&\mathbb{H}'(\varphi,\,\psi)\cdot(u,\,v)-\mathbb{H}'(\varphi_n,\,\psi_n)\cdot(u,\,v)\\
=\;& \int_{\mathbb{R}}[\omega_1\cdot 2Re((\varphi-\varphi_n)\overline{u})+\omega_2\cdot 2Re((\psi-\psi_n)\overline{v})+\alpha\cdot 2Re((\varphi-\varphi_n)_x\,\overline{u}_x)+\alpha\cdot 2Re((\psi-\psi_n)_x\,\overline{v}_x)]\,dx\\
&-2\int_0^1\int_{\mathbb{R}}\Big[|T(t)\varphi|^2\cdot Re(T(t)\varphi\cdot\overline{T(t)u})-|T(t)\varphi_n|^2\cdot Re(T(t)\varphi_n\cdot\overline{T(t)u})\Big]\,dx\,dt\\
&-2\int_0^1\int_{\mathbb{R}}\Big[|T(t)\psi|^2\cdot Re(T(t)\psi\cdot\overline{T(t)v})-|T(t)\psi_n|^2\cdot Re(T(t)\psi_n\cdot\overline{T(t)v})\Big]\,dx\,dt\\
&-2\,\beta\int_0^1\int_{\mathbb{R}}\Big[|T(t)\varphi|^2\cdot Re(T(t)\psi\cdot\overline{T(t)v})-|T(t)\varphi_n|^2\cdot Re(T(t)\psi_n\cdot\overline{T(t)v})\Big]\,dx\,dt\\
&-2\,\beta\int_0^1\int_{\mathbb{R}}\Big[|T(t)\psi|^2\cdot Re(T(t)\varphi\cdot\overline{T(t)u})-|T(t)\psi_n|^2\cdot Re(T(t)\varphi_n\cdot\overline{T(t)u})\Big]\,dx\,dt
\end{aligned}
\tag{5.11}
$$

converges to zero as $n\to\infty$. The first integral on the right-hand side in (5.11) tends to zero, because $\varphi-\varphi_n\rightharpoonup 0$ and $\psi-\psi_n\rightharpoonup 0$ in $H^1(\mathbb{R})$ respectively. The other integrals are estimated as follows.

*Claim.* We have

$$
\begin{aligned}
&|T(t)\varphi|^2\cdot Re(T(t)\varphi\,\overline{T(t)u})-|T(t)\varphi_n|^2\cdot Re(T(t)\varphi_n\,\overline{T(t)u})\\
&\quad\le\;|T(t)u|\,|T(t)(\varphi-\varphi_n)|\,\big[\,|T(t)\varphi|^2+(\,|T(t)\varphi|+|T(t)\varphi_n|\,)\;|T(t)\varphi_n|\,\big]\,.
\end{aligned}
\tag{5.12}
$$

In fact,

$$
\begin{aligned}
&|T(t)\varphi|^2\cdot Re(T(t)\varphi\,\overline{T(t)u})-|T(t)\varphi_n|^2\cdot Re(T(t)\varphi_n\,\overline{T(t)u})\\
=\;&|T(t)\varphi|^2\cdot Re(T(t)(\varphi-\varphi_n)\,\overline{T(t)u}+T(t)\varphi_n\,\overline{T(t)u})-|T(t)\varphi_n|^2\cdot Re(T(t)\varphi_n\,\overline{T(t)u})\\
=\;&|T(t)\varphi|^2\,(Re(T(t)(\varphi-\varphi_n)\,\overline{T(t)u}+Re(T(t)\varphi_n\,\overline{T(t)u}))-|T(t)\varphi_n|^2\cdot Re(T(t)\varphi_n\,\overline{T(t)u})\\
=\;&|T(t)\varphi|^2\,Re(T(t)(\varphi-\varphi_n)\,\overline{T(t)u})+Re((|T(t)\varphi|^2-|T(t)\varphi_n|^2)\,T(t)\varphi_n\,\overline{T(t)u})\\
=\;&|T(t)\varphi|^2\,Re(T(t)(\varphi-\varphi_n)\,\overline{T(t)u})+Re((|T(t)\varphi|-|T(t)\varphi_n|)\,(|T(t)\varphi|+|T(t)\varphi_n|)\,T(t)\varphi_n\,\overline{T(t)u})\\
\le\;&|T(t)\varphi|^2\,|T(t)(\varphi-\varphi_n)|\,|T(t)u|+|\,|T(t)\varphi|-|T(t)\varphi_n|\,|\,(|T(t)\varphi|+|T(t)\varphi_n|)\,|T(t)\varphi_n|\,|T(t)u|\\
\le\;&|T(t)\varphi|^2\,|T(t)(\varphi-\varphi_n)|\,|T(t)u|+|\,T(t)\varphi-T(t)\varphi_n\,|\,(|T(t)\varphi|+|T(t)\varphi_n|)\,|T(t)\varphi_n|\,|T(t)u|\\
=\;&|T(t)\varphi|^2\,|T(t)(\varphi-\varphi_n)|\,|T(t)u|+|T(t)(\varphi-\varphi_n)|\,(|T(t)\varphi|+|T(t)\varphi_n|)\,|T(t)\varphi_n|\,|T(t)u|\\
=\;&|T(t)u|\,|T(t)(\varphi-\varphi_n)|\,\big(\,|T(t)\varphi|^2+(|T(t)\varphi|+|T(t)\varphi_n|)\,|T(t)\varphi_n|\,\big)\,.
\end{aligned}
$$

In a similar way we obtain

$$
\begin{aligned}
&|T(t)\varphi|^2\cdot Re(T(t)\psi\cdot\overline{T(t)v})-|T(t)\varphi_n|^2\cdot Re(T(t)\psi_n\cdot\overline{T(t)v})\\
&\quad\le\;|T(t)v|\,|T(t)\varphi|^2\,|T(t)(\psi-\psi_n)|+|T(t)v|\,|T(t)\psi|\,|T(t)(\varphi-\varphi_n)|\,(\,|T(t)\varphi|+|T(t)\varphi_n|\,).
\end{aligned}
\tag{5.13}
$$

Hence in (5.11) we have

$$
\begin{aligned}
&\mathbb{H}'(\varphi,\,\psi)\cdot(u,\,v)-\mathbb{H}'(\varphi_n,\,\psi_n)\cdot(u,\,v)\\
\le\;\;&2\int_0^1\int_{\mathbb{R}}\big[|T(t)u|\,|T(t)(\varphi-\varphi_n)|\,\big(\,|T(t)\varphi|^2+(\,|T(t)\varphi|+|T(t)\varphi_n|\,)\;|T(t)\varphi_n|\,\big)\,\big]\,dx\,dt \qquad (5.14)\\
&-2\int_0^1\int_{\mathbb{R}}\big[|T(t)v|\,|T(t)(\psi-\psi_n)|\,\big(\,|T(t)\psi|^2+(\,|T(t)\psi|+|T(t)\psi_n|\,)\;|T(t)\psi_n|\,\big)\,\big]\,dx\,dt\\
&-2\,\beta\int_0^1\int_{\mathbb{R}}\big[|T(t)v|\,|T(t)\varphi|^2\,|T(t)(\psi-\psi_n)|+|T(t)v|\,|T(t)\psi|\,|T(t)(\varphi-\varphi_n)|\,(\,|T(t)\varphi|+|T(t)\varphi_n|\,)\big]\,dx\,dt\\
&-2\,\beta\int_0^1\int_{\mathbb{R}}\big[|T(t)u|\,|T(t)\psi|^2\,|T(t)(\varphi-\varphi_n)|+|T(t)v|\,|T(t)\varphi|\,|T(t)(\psi-\psi_n)|\,(\,|T(t)\psi|+|T(t)\psi_n|\,)\big]\,dx\,dt.
\end{aligned}
$$

We estimate the first integral on the right-hand side in (5.14). We take a sufficiently large interval $K = [-R,\, R]$, so that $|u(x)| < \varepsilon$ (respectively $|v(x)| < \varepsilon$), for all $x \in K$. Thus, using Lemma 3.6 and the boundedness of $\varphi$, $\varphi_n$ (respectively $\psi$, $\psi_n$) in $H^1(\mathbb{R})$ we obtain the bound

$$\left| \int_0^1 \int_{\mathbb{R}\setminus K} \big[ |T(t)u|\, |T(t)(\varphi - \varphi_n)|\, \big( |T(t)\varphi|^2 + (\, |T(t)\varphi| + |T(t)\varphi_n|\,)\, |T(t)\varphi_n|\, \big) \big]\, dx\, dt \right| \leq c\, \varepsilon$$

that is uniform in time. To estimate on the remaining interval $K = [-R,\, R]$, using $H^1([-R,\, R]) \overset{c}{\hookrightarrow} C^0([-R,\, R])$ we have that the sequences converges strongly $\varphi_n \to \varphi$ for $x \in K$ (respectively $\psi_n \to \psi$ for $x \in K$). Therefore, we can show that

$$\sup_{x\in K,\, t\in[0,\,1]} |T(t)(\varphi - \varphi_n)| < \varepsilon \qquad \left( \text{respectively,} \quad \sup_{x\in K,\, t\in[0,\,1]} |T(t)(\psi - \psi_n)| < \varepsilon \right)$$

provided $n$ is sufficiently large.

In fact, we take a large set $K_\varepsilon = [R - 1/\varepsilon,\, R + 1/\varepsilon]$, then choosing $n$ so large that

$$\sup_{x\in K_\varepsilon} |\varphi - \varphi_n| < \varepsilon, \qquad \left( \text{respectively,} \quad \sup_{x\in K_\varepsilon} |\psi - \psi_n| < \varepsilon \right)$$

we can apply Theorem 3.1 to show the localization does not occur.

Now, we can estimate the integral on the remaining interval

$$\begin{aligned} \int_K \big[ |T(t)u|\, |T(t)(\varphi - \varphi_n)|\, \big( |T(t)\varphi|^2 + (\, |T(t)\varphi| + |T(t)\varphi_n|\,)\, |T(t)\varphi_n|\, \big) \big]\, dx \;&\leq\; C \sup_{x\in K} |T(\varphi - \varphi_n)| \\ &\leq\; C\, \varepsilon, \end{aligned}$$

where $C$ does not depend on $n$. The other terms in (5.14) are estimated in a similar way. □

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# References

[1] M. J. Ablowitz and G. Biondini, *Multiscale pulse dynamics in communication systems with strong dispersion management,* Opt. Lett. 23(1998) 1668-1670.

[2] R. A. Adams, *Sobolev spaces,* Academic Press, New York (1975).

[3] T. Cazenave, *An introduction to Nonlinear Schrödinger Equations,* UFRJ, Rio de Janeiro, Brasil, 1993.

[4] T. Cazenave and F. B. Weissler, *The Cauchy problem for the critical nonlinear Schrödinger equation in $H^s$*, Nonlinear Anal. Theory, Meth. Appl. 100(1990) 807-836.

[5] J. C. Ceballos, M. Sepúlveda and O. Vera. *Gain in regularity for a Coupled Schrodinger System.* Bol. Soc. Paran. Mat. Vol. 24, 1-2(2006) 41-68.

[6] A. Chraplyvy, *8X10 GB/s transmission through 280 Km. of dispersion managed fiber,* IEEE Photonics Tech. Lett. 5(1993) 1233-1235.

[7] B.D. Esry, C.H. Greene, J.Ñ. Burke and J.L.Bhon, *Hartree-Fock Theory for Double Condensates,* Phys. Rev. Lett. 78(1997) 3594-3597.

[8] I. Gabitov and S. K. Turitsyn, *Averaged pulse dynamics in a cascaded transmission system with passive dispersion compensation,* Opt. Lett. 21(1996) 327-329.

[9] I. Gabitov and S. K. Turitsyn, *Breathing solitons in optical fiber link,* Pis'ma v ZhETF, Vol. 63, 10(1996) 814-819.

[10] I. Gabitov and S. K. Turitsyn, *Asymptotic breathing pulse in optical communication systems with dispersion compensation,* Opt. Lett. 21,(1995) 327.

[11] R. Glassey, *On the assymptotic behavior of nonlinear wave equations,* Trans. AMS, Vol. 182, 187(1973) 187-200.

[12] A. Hasegawa, Y. Kodama. *Solitons in optical communications,* Academic Press, San Diego, 1995.

[13] M. N. Islam, *Ultrafast fiber switching devices and system,* Cambridge University Press, New York, 1992.

[14] I. P. Kaminow. *Polarization in optical fibers,* IEEE J. Quantum Electron, 17(1981) 15-22.

[15] M. Kunze, *On a variational problem with lack of compactness related to the Strichartz inequality,* Calc. Var. Partial differential Equations 19(2004) 307-336.

[16] C. Lin, H. Kogelnik and L. G. Cohen, *Optical pulse equalization and low dispersion transmission in the single-mode fiber in the 1.3-1.7 μm. spectral region,* Opt. Lett. 5(1980) 476-478.

[17] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part. 1,* Ann. Inst. Henri Poincaré, Analyse non linéaire, 1(1984) 109-145.

[18] P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part. 1,* Ann. Inst. Henri Poincaré, Analyse non linéaire, 1(1984) 223-283.

[19] C. R. Menyuk. *Nonlinear pulse propagation in birefringence optical fiber,* IEEE J. Quantum Electron, 23(1987) 174-176.

[20] J. Moeser, I. Gabitov and C.K.R.T. Jones, *Pulse stabilization by high order dispersion management,* Opt. Lett. 27(24)(2002) 2206-2208.

[21] M. Nakazawa, H. Kubota, K. Suzuki and E. Yamida, *Recent progress in soliton transmission technology,* Chaos Vol. 10, 3(2000) 486-514.

[22] M. Ohta, *Stability of solitary waves for coupled nonlinear Schrödinger equations*, Nonlinear Analysis, Theory Meth. Appl. 26 (1995) 933-939.

[23] A. Pazy, *Semigroups of linear operators and applications to partial differential equations,* Springer-Verlag, New York, 1983.

[24] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations,* CBMS Regional Conference Series in Mathematics, Vol. 65, American Mathematical Society, Providence, RI, 1986.

[25] W. Rudin, *Functional Analysis,* Tata McGraw-Hill, 1974.

[26] T. Schafer, V. Mezentsev, K. H. Spatschek and S. K. Turitsyn, *The dispersion-managed soliton as a ground states of a macroscopic nonlinear quantum oscillator,* R. Soc. Lond. Proc. Ser. A, Math. Phys. Eng. Sci. 457(2001) 273-282.

[27] E. S. P. Siqueira, *Comportamento assintótico das soluções de um sistema não linear de equações de Schrödinger.* Tese de doutorado. U.F.R.J. 1990.

[28] E. S. P. Siqueira, *Asymptotic behavior of the solutions of a nonlinear system of Schrödinger equations,* Nonlinear Analysis, Theory Meth. Appl. 10, 20(1993) 1231-1243.

[29] R. S. Strichartz, *Restriction of Fourier transforms to quadratic surfaces and decay of solutions of wave equations,* Duke Math. J., 44(1977) 705-714.

[30] C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation,* Springer, Berlin-New York, 1999.

[31] T.-P. Tsai and H.-T. Yau, *Asymptotic dynamics of Nonlinear Schrödinger equations: resonance-dominated and dispersion-dominated solutions,* Comm. Pure Appl. Math. 55(2002) 153-216.

[32] V. Zharnitsky, E. Grenier, C.K.R.T. Jones and S.K. Turitsyn, *Stabilizing effects of dispersion management,* Physica D 152-153(2001) 794-817.

[33] V. Zharnitsky, E. Grenier, C.K.R.T. Jones and J.S. Hesthaven, *Ground states of the dispersion managed nonlinear Schrödinger equation,* Phys. Rev. E 62(2000) 7358-7364.