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Optical solitons in nematic liquid crystals: Arbitrary deviation angle model

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ABSTRACT

We study a coupled Schrödinger-elliptic evolution system that describes the propagation of a laser beam in nematic liquid crystals. The elliptic equation describes the effects of the beam electric field on the local orientation (director field) of the nematic liquid crystal and has an important regularizing effect, seen experimentally and understood theoretically in related models. In the present work we propose a new nonlinear elliptic equation for the director field that makes no assumption on the size of the director field angle. The analysis of this elliptic equation leads to an upper bound for the size of the director angle that we believe is optimal and physically relevant, and that implies that the elastic response of the medium prevents a complete alignment between the electric field and the orientation of the liquid crystal. The results on the elliptic problem are combined with arguments from dispersive wave theory to show the local and global well-posedness of the evolution problem and the decay of small initial conditions. We also show the existence of constrained minimizers of the Hamiltonian, assuming sufficiently large optical power (L^2−norm of the laser field). These minimizers are solitons with radial, monotonically decreasing profiles.

1. Introduction

We present results on well-posedness, decay and soliton solutions of the coupled nonlinear Schrödinger (NLS) equation

$$\partial_z u = \frac{1}{2}\mathrm{i}\nabla^2 u + \mathrm{i}\gamma(\sin^2(\psi+\theta_0) - \sin^2(\theta_0))u, \tag{1.1a}$$

$$\nu\nabla^2\psi = \frac{1}{2}E_0^2\sin(2\theta_0) - \frac{1}{2}(E_0^2+|u|^2)\sin(2(\psi+\theta_0)), \tag{1.1b}$$

where u and ψ depend on the "optical axis" coordinate $z \in \mathbb{R}$, and the "transverse coordinates" $(x, y) \in \mathbb{R}^2$. $\nabla^2 = \partial_x^2 + \partial_y^2$ is the Laplacian in the transverse directions. E_0, ν and γ are positive constants, and θ_0 is a constant satisfying $\theta_0 \in (\pi/4, \pi/2)$.

The model arises in the study of optical beam propagation in nematic liquid crystals and models a set of experiments by Assanto and collaborators [1–3]. The complex quantity u represents the electric field amplitude of a laser beam that propagates through a nematic liquid crystal along the optical axis z. The laser electric field has only one component, along the vertical axis x of the plane normal to the optical axis z. The quantity $\theta := \theta_0 + \psi$ (the "director field") describes the macroscopic orientation of the nematic liquid crystal molecules. The macroscopic molecular orientation vector field is assumed to lie on the plane defined by the optical axis z and the vertical direction x, and θ is the angle between the macroscopic molecular orientation vector and the z-axis.

System (1.1) is derived heuristically in the Appendix, where we also discuss further the physical meaning of the variables, the assumptions on the constants, and the experimental geometry.

The model (1.1) is a generalization of two related systems studied earlier, (1.2), (1.3), that were derived under the assumption that ψ is small [4]. In contrast, the derivation of (1.1) does not make any explicit assumptions on the size of ψ.

The first related model, see [3,4], is

$$\partial_z u = \frac{1}{2}\mathrm{i}\nabla^2 u + \frac{1}{2}\mathrm{i}\gamma u\sin(2\psi), \tag{1.2a}$$

$$\nu\nabla^2\psi = q\sin(2\psi) - 2|u|^2\cos(2\psi), \tag{1.2b}$$

with $q > 0$. A simpler model, obtained using $\sin\psi \approx \psi$, $\cos\psi \approx 1$, is

$$\mathrm{i}\partial_z u + \frac{1}{2}\nabla^2 u + \gamma\psi u = 0, \tag{1.3a}$$

$$\nu\nabla^2\psi - 2q\psi = -2|u|^2, \tag{1.3b}$$

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with $q > 0$. The constant q is related to θ_0 of (1.1) by $-2q = \cos 2\theta_0$ up to an error $O(\psi^2)$, see [4], i.e. $q > 0$ requires $\theta_0 > \pi/4$ up to this error, see the Appendix for further comments.

Model (1.3) captures the physical effect that a localized electric field u can produce a deformation of the director angle ψ at longer distances and is consistent with the experimental observation of stable optical solitons [3]. Partial theoretical explanations were given in [5–7].

Mathematical results on (1.3) include local and global well-posedness in H^1, existence of energy minimizing soliton solutions, and decay for small initial L^2 norm, see [8–10]. Similar results for related NLS-elliptic systems are shown in [11–14].

Recently, results of this type were also shown in [4] for (1.2). The motivation for the present study comes from a new effect seen in that work, namely that ψ takes values in $[0, \pi/4)$. This is a "saturation" effect due to the nonlinear equation (1.2b) for ψ. The bound on ψ, together with $\theta_0 > \pi/4$, allows for (but does not imply) a bound $\theta = \theta_0 + \psi < \pi/2$ on the total angle, implying that the molecular orientation cannot be along the laser electric field. This bound would be interesting because the interaction between ψ and u in (1.2b) describes the tendency of the electric field and the molecular orientation to align, see Appendix. On the other hand, the derivation of (1.2) uses the assumptions that $\theta_0 - \pi/4$ is positive and small, and that ψ is also small compared to $\theta_0 - \pi/4$ see [3,4]. Thus the physical relevance of this saturation effect motivates the study of systems derived without the assumption that ψ is small.

Our first result is that given $u \in L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, Eq. (1.1b) has a unique solution $\psi(|u|^2)$ that belongs to $H^2(\mathbb{R}^2)$ and satisfies $\psi(x) \in [0, \pi/2-\theta_0) \subset [0, \pi/4)$, for all $x \in \mathbb{R}^2$, see Proposition 2.1, Lemma 2.2, and Corollary 2.1. This is the saturation effect that the total angle $\theta = \theta_0 + \psi$ is less that $\pi/2$. This seems to be a sharp bound on the saturation of the nonlinearity. In particular it is more precise than the bound obtained in [4] and follows from a more general model that has no small size assumptions for $\theta_0 - \pi/4$ and ψ. The condition $\theta_0 > \pi/4$ is technical and was implicit in the other two models, see Appendix for the meaning of the assumptions on θ_0. The solution of (1.1b) uses a global continuation argument. The equation is rewritten around each solution as a linear regular elliptic operator plus a nonlinear term, see Proposition 2.1. The regularity of the linear part uses the assumptions $E_0 > 0$, $\theta_0 \in (\pi/4, \pi/2)$, and also the property that the range of ψ belongs to $[0, \pi/4)$, for all u. The fact that ψ remains in that interval as we vary u is shown in Lemmas 2.1, 2.2.

The second set of results concerns local and global existence of the initial value problem for (1.1), assuming initial conditions $u_0 \in H^1(\mathbb{R}^2)$, see Theorems 3.1, 3.2 respectively for precise statements in Section 3. We also prove that the solution decays for initial conditions $u_0 \in H^1$ with sufficiently small L^2-norm, see Proposition 3.3. The local existence theory uses the regularity of the map $\psi(u)$ and the right hand side of (1.1a), following from the results on (1.1b) in Section 2, and Strichartz estimates. The Strichartz estimates yield additional control $u(z) \in L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ so that (1.1b) can be solved for almost all z as the system evolves. Global existence uses the conservation of the Hamiltonian H and the optical power (L^2-norm of u) of the system, see Section 4. The bound of the H^1-norm follows from the conservation of energy and uses the assumption on θ_0 and the result that $\psi \in [0, \pi/4)$ everywhere from the analysis of Eq. (1.1b). The decay result for initial conditions with small L^2-norm uses Strichartz estimates to show that the solutions belong to $L^4([0, \infty), L^4(\mathbb{R}^2))$.

We also show existence and nonexistence results for soliton solutions (1.1). The main result is Proposition 4.3 of Section 3, showing the existence of minimizers of the Hamiltonian $H(u, \psi)$ over configurations $(u, \psi) \in H^1 \times H^1$ with the constraints that L^2-norm of u is fixed, and ψ is essentially bounded below by a suitable negative constant. The result holds under the assumption that L^2-norm of u is above a certain threshold. Standard arguments then imply that the minimizer is a smooth soliton solution, see Corollary 4.2. We note that H in $(u, \psi) \in H^1 \times H^1$, with the L^2-norm of u fixed, is not bounded below, see Lemma 4.1. This problem can be overcome by adding a pointwise restriction on ψ. The trigonometric form of the nonlinearity allows us to seek minimizers away from this constraint, see Lemma 4.2, and we also use radial symmetrization and rearrangements to reduce the problem to the radial case, Proposition 4.1. The existence of the minimizer also requires the existence of negative energy configurations. The energy estimates are similar to the ones for (1.2) [4], and lead to the assumption on L^2-norm of u, see Proposition 4.2. The existence of minimizers then follows from standard direct method arguments applied to radial configurations.

The minimization proof in Section 4 implies that if the L^2-norm of u is sufficiently small then the infimum of the Hamiltonian H is not attained. Proposition 3.3 of Section 3 on the decay of solutions for u_0 of sufficiently small L^2-norm also implies the nonexistence of solitons. This result is similar to what we see for the systems (1.2) [4] and (1.3) [10] obtained under a small angle ψ assumption.

The paper is organized as follows. In Section 2 we show the existence of unique solutions for the director equation, Proposition 2.1, and the bound on θ, Corollary 2.1. In Section 3 we show local and global well-posedness for the initial value problem, Theorems 3.1, 3.2 and decay for small L^2-norm initial conditions, Proposition 3.3. In Section 4 we show the existence of constrained minimizers for the Hamiltonian, implying the existence of radially symmetric optical solitons, Proposition 4.3. In Section 5 we briefly discuss our results.

2. Solution of the director angle equation

The main result of this section is Proposition 2.1 on the existence of solutions to (1.1b), that can be written as

$$-\nu\nabla^2\psi = N(u, \psi) \tag{2.1}$$

where N is given by

$$N(u, \psi) = \frac{1}{2}(E_0^2 + |u|^2)\sin(2(\psi + \theta_0)) - \frac{1}{2}E_0^2\sin(2\theta_0) \tag{2.2}$$

with $\theta_0 \in (\pi/4, \pi/2)$ constant.

We see that $N(u, .)$ is decreasing on the interval $[\pi/4 - \theta_0, 3\pi/4 - \theta_0] \subset [-\pi/4, \pi/2]$. Also,

$$N(u, \psi) = \frac{1}{2}E_0^2\left(\sin(2(\psi + \theta_0)) - \sin(2\theta_0)\right) + \frac{1}{2}|u|^2\sin(2(\psi + \theta_0)) \tag{2.3}$$

implies

$$N(u, \psi) \le E_0^2\cos(2\theta_0)\psi + \frac{1}{2}|u|^2, \tag{2.4}$$

for all $\psi \in [0, \pi/2 - \theta_0]$, and $u \in \mathbb{C}$.

Note that from (2.4), we have $|N(u, \psi)| \le C\left(|\psi| + |u|^2\right)$, and therefore N is a Nemytskii operator for $u \in L^4(\mathbb{R}^2)$.

The proof of Proposition 2.1 uses a global continuation argument, and we will need to show that once a solution ψ exists for a given u it also satisfies some additional properties that allow us to find a unique solution for a nearby u. These properties are shown in the lemmas below. Lemma 2.1 and Corollary 2.1 establish the range of ψ, especially an upper bound that is independent of u. Lemma 2.2 shows how the range of ψ implies the monotonicity of the nonlinearity N of (2.2) as a function of ψ.

We will use the following facts. Given $v \in L^2(\mathbb{R}^2)$, we define $v^+ = \max(v,0)$ and $v^- = \max(-v,0)$. Then $\left\|v^\pm\right\|_{L^2}^2 = \pm\langle v, v^\pm\rangle$, $\langle v^+, v^-\rangle = 0$ and $\|v\|_{L^2}^2 = \left\|v^+\right\|_{L^2}^2 + \left\|v^-\right\|_{L^2}^2$. If $v \in H^1(\mathbb{R}^2)$, then $v^\pm \in H^1(\mathbb{R}^2)$. Moreover $\left\|\nabla v^\pm\right\|_{L^2}^2 = \pm\langle \nabla v, \nabla v^\pm\rangle$, $\langle \nabla v^+, \nabla v^-\rangle = 0$ and $\|\nabla v\|_{L^2}^2 = \left\|\nabla v^+\right\|_{L^2}^2 + \left\|\nabla v^-\right\|_{L^2}^2$.

Lemma 2.1. *Given $u \in L^4(\mathbb{R}^2)$, Eq. (1.1b) has at most one solution $\psi \in H^2(\mathbb{R}^2)$ satisfying $0 \le \psi(x) \le \pi/4$ for all $x \in \mathbb{R}^2$.*

Proof. Let $\psi_1, \psi_2 \in H^2(\mathbb{R}^2)$ be solutions of (1.1b) taking values in the interval $[0, \pi/4] \subset [\pi/4-\theta_0, 3\pi/4-\theta_0]$. These solutions are also in $C^0(\mathbb{R}^2)$ by the Sobolev inequalities. By (2.1) their difference satisfies

$$-\nu\nabla^2(\psi_1-\psi_2) = N(x,\psi_1) - N(x,\psi_2), \tag{2.5}$$

and since $N(x, .)$ is decreasing in $[0, \pi/4]$, we also have

$$(N(x,\psi_1) - N(x,\psi_2))(\psi_1-\psi_2)^+ \le 0,$$

a.e. in $\mathbb{R}^2$. Multiplying (2.5) by $(\psi_1-\psi_2)^+$ and integrating we therefore have

$$\left\|\nabla(\psi_1-\psi_2)^+\right\|_{L^2}^2 \le 0.$$

Interchanging ψ_1 and ψ_2, we similarly have $\left\|\nabla(\psi_2-\psi_1)^+\right\|_{L^2}^2 \le 0$. From the decomposition $\nabla(\psi_1-\psi_2) = \nabla(\psi_1-\psi_2)^+ - \nabla(\psi_2-\psi_1)^+$, it follows that $\nabla(\psi_1-\psi_2) = 0$ a.e. in $\mathbb{R}^2$. Since ψ_1, ψ_2 are continuous and decay at infinity, $C_0(\mathbb{R}^2)$, we obtain $\psi_1 \equiv \psi_2$. □

Lemma 2.2. *Consider $u \in L^4(\mathbb{R}^2)$ and let $\psi \in H^2(\mathbb{R}^2)$ be a corresponding solution of (1.1b) that also satisfies $\pi/2 - 2\theta_0 \le \psi(x) \le \pi - \theta_0$, for all $x \in \mathbb{R}^2$. Then $0 \le \psi(x) \le \pi/4$, for all $x \in \mathbb{R}^2$.*

Proof. By (2.3), $\pi/2-2\theta_0 \le \psi \le 0$ implies $N(x,\psi) \ge 0$, therefore $N(x,\psi)\psi^- \ge 0$ a.e. in $\mathbb{R}^2$. Multiplying (2.1) by ψ^-, integrating and using $\nabla\psi \cdot \nabla\psi^- = -\left|\nabla\psi^-\right|^2$ we have

$$-\nu\int_{\mathbb{R}^2}\left|\nabla\psi^-\right|^2 dx = \int_{\mathbb{R}^2} N(x,\psi)\psi^- \,dx \ge 0.$$

It follows that $\psi^- \equiv 0$.

For $\psi \in [\pi/4, \pi-\theta_0]$, from (2.2) we have $N(u,\psi) \le 0$ and therefore $N(x,\psi)(\psi-\pi/4)^+ \le 0$ a.e. in $\mathbb{R}^2$. Multiplying (2.1) by $(\psi-\pi/4)^+$, integrating and using $\nabla(\psi-\pi/4)\cdot\nabla(\psi-\pi/4)^+ = \left|\nabla(\psi-\pi/4)^+\right|^2$, we similarly obtain that $(\psi-\pi/4)^+$ is a constant and therefore $(\psi-\pi/4)^+ \equiv 0$. □

Corollary 2.1. *Let ψ be as in Lemma 2.2, with $u \in L^4(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$. Then $0 \le \psi(x) \le \psi_{\max} < \pi/2 - \theta_0 < \pi/4$, for all $x \in \mathbb{R}^2$, where*

$$\psi_{\max} = \frac{\pi}{2} - \theta_0 - \frac{1}{2}\arcsin\left(\frac{E_0^2\sin(2\theta_0)}{E_0^2 + \|u\|_\infty^2}\right). \tag{2.6}$$

Proof. Consider $u \in L^4(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$ and define $\psi_{\max}$ as in (2.6). We observe that $0 < E_0^2\sin(2\theta_0) < E_0^2 + |u(x)|^2$, therefore $\psi_{\max} \in (\pi/4-\theta_0, \pi/2-\theta_0) \subset (-\pi/4, \pi/4)$. Let ψ be a solution of (2.1) as in the hypothesis. We need to show that $\psi(x) \le \psi_{\max}$, for all $x \in \mathbb{R}^2$. By (2.2), $N(u(x), \psi(x)) \le 0$ is equivalent to

$$\frac{\pi}{2} - \theta_0 - \frac{1}{2}\arcsin\left(\frac{E_0^2\sin(2\theta_0)}{E_0^2 + |u(x)|^2}\right) \le \psi(x).$$

Since arcsin is increasing,

$$\frac{\pi}{2} - \theta_0 - \frac{1}{2}\arcsin\left(\frac{E_0^2\sin(2\theta_0)}{E_0^2 + |u(x)|^2}\right) < \frac{\pi}{2} - \theta_0 - \frac{1}{2}\arcsin\left(\frac{E_0^2\sin(2\theta_0)}{E_0^2 + \|u\|_\infty^2}\right).$$

Thus if $\psi_{\max} \le \psi(x) \le \pi/4$, then $N(u(x), \psi(x)) \le 0$ and therefore we have that $N(x,\psi)(\psi-\psi_{\max})^+ \le 0$ a.e. in $\mathbb{R}^2$. Multiplying (2.1) by $(\psi-\psi_{\max})^+$ and arguing as in Lemma 2.2, we see that $(\psi-\psi_{\max})^+ \equiv 0$. □

Lemma 2.3. *There exists a constant $C_{\theta_0,E_0} > 0$ such that if $\psi \in H^2(\mathbb{R}^2)$ is a solution of (1.1b) and satisfies $0 \le \psi(x) < \pi/2 - \theta_0$, for all $x \in \mathbb{R}^2$, then $\|\psi\|_{H^2} \le C_{\theta_0,E_0}\|u\|_{L^4}^2$.*

Proof. Multiplying (1.1b) by ψ, integrating, using the assumption $\psi \in [0, \pi/2-\theta_0)$ and inequality (2.4) we obtain

$$\nu\|\nabla\psi\|_{L^2}^2 \le E_0^2\cos(2\theta_0)\|\psi\|_{L^2}^2 + \frac{1}{2}\int_{\mathbb{R}^2}|u|^2\psi \,dx.$$

If we define $\alpha = -E_0^2\cos(2\theta_0) > 0$, using the Cauchy–Schwarz and Young inequalities with $\varepsilon^2 = \alpha/2$, this implies

$$\nu\|\nabla\psi\|_{L^2}^2 + \frac{\alpha}{2}\|\psi\|_{L^2}^2 \le \frac{1}{2\alpha}\|u\|_{L^4}^4. \tag{2.7}$$

By (2.1), (2.3), (2.4) we also have

$$\left\|\nabla^2\psi\right\|_{L^2} \le E_0^2\|\psi\|_{L^2} + \frac{1}{2}\|u\|_{L^4}^2 \le \left(\frac{E_0^2}{\alpha} + \frac{1}{2}\right)\|u\|_{L^4}^2. \tag{2.8}$$

The lemma follows from (2.7) and (2.8). □

In order to solve (1.1b) we use the following definition. Let X be a Banach space, and consider a map $F : X \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ that satisfies $F(u,0) = 0$ and is continuous in a neighborhood of $(u,0)$. Then we will consider the property that for any $\varepsilon > 0$, there exists a neighborhood $\mathscr{V} \subset X \times H^2(\mathbb{R}^2)$ of $(u,0)$ for which

$$\|F(w,\psi_1) - F(w,\psi_2)\|_{L^2} \le \varepsilon\|\psi_1-\psi_2\|_{H^2}, \tag{2.9}$$

for all $(w,\psi_1), (w,\psi_2) \in \mathscr{V}$.

Property (2.9) combines Lipschitz continuity and superlinearity for the second component of F near $(u,0)$. In Lemma 2.5 we will see that (2.9) implies the existence of a unique solution $\psi(w)$ of $-\nabla^2\psi + V\psi = F(w,\psi)$ near $(u,0)$, V as in Lemma 2.4. Continuity of F in the first component makes ψ continuous in w. This setup will be then used to solve (1.1b). Lemmas 2.4, 2.5 are shown in [4].

Lemma 2.4. *Let $V \in L^\infty(\mathbb{R}^2)$, with $V \ge 0$ and $\liminf_{|x|\to\infty} V(x) \ge a > 0$. Then $-\nabla^2\psi + V\psi = f$ with $f \in L^2(\mathbb{R}^2)$ has a unique solution $\psi \in H^2(\mathbb{R}^2)$. Moreover, there exists a constant $K = K(V) > 0$ such that $\|\psi\|_{H^2} \le K\|f\|_{L^2}$.*

Lemma 2.5. *Let X be a Banach space and consider a map $F : X \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. Assume that V satisfies the conditions of Lemma 2.4 and that F is continuous in a neighborhood of $(u,0)$ and satisfies (2.9) at $(u,0)$. Then there exists a neighborhood $\mathscr{U} \subset X$ of u and $\delta > 0$ such that for any $w \in \mathscr{U}$ the equation $-\nabla^2\psi + V\psi = F(w,\psi)$ has a unique solution $\psi \in H^2(\mathbb{R}^2)$ with $\|\psi\|_{H^2} < \delta$. Furthermore, the map $w \mapsto \psi$ from X to $H^2(\mathbb{R}^2)$ is continuous in $\mathscr{U}$.*

The existence of solutions (1.1b) is shown in Proposition 2.1, using a continuation idea and the setup of Lemma 2.5. We will use technical Lemmas 2.6, 2.7 on property (2.9) for the nonlinear terms, see [4] for proofs.

Remark 2.1. Suppose $F_1, F_2 : X \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ satisfy (2.9) at u. Then for A_1, A_2 bounded operators in $L^2(\mathbb{R}^2)$, the map $F = A_1F_1 + A_2F_2 : X \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is continuous in a neighborhood of $(u, 0)$ and satisfies property (2.9) at $(u, 0)$.

Lemma 2.6. *Let $\alpha \in C^1(\mathbb{R})$, $u \in L^4(\mathbb{R}^2)$ and define $F : L^4(\mathbb{R}^2) \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by $F(w, \psi) = \left(|w|^2 - |u|^2\right)\alpha(\psi)$. Then F is continuous in a neighborhood of $(u, 0)$ and satisfies property (2.9) at $(u, 0)$.*

Lemma 2.7. *Let $u \in L^4(\mathbb{R}^2)$ and define $F : L^4(\mathbb{R}^2) \times H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ by $F(w, \psi) = G(\psi)$, and $G(\psi) = (h_1 + h_2|u|^2)\beta(\psi)$, where $h_1, h_2 \in L^\infty(\mathbb{R}^2)$, and $\beta \in C^1(\mathbb{R})$ with $\beta(0) = 0$, $\beta'(0) = 0$. Then $F(w, \psi) = G(\psi)$ is continuous in a neighborhood of $(u, 0)$ and satisfies (2.9) at $(u, 0)$.*

Proposition 2.1. *Let $u \in L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, then there exists a unique solution $\psi \in H^2(\mathbb{R}^2)$ of (1.1b) satisfying $0 \le \psi(x) < \pi/2 - \theta_0$, for all $x \in \mathbb{R}^2$. Furthermore $\|\psi\|_{H^2} \le C\,\|u\|_{L^4}^2$.*

Proof. Let $\mathscr{U} \subset L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ be the set of functions u for which there exists a solution $\psi \in H^2(\mathbb{R}^2)$ of (1.1b), with the property that $0 \le \psi < \pi/2 - \theta_0$ everywhere in $\mathbb{R}^2$. We will prove that $\mathscr{U}$ is a nonempty open and closed subset of $L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Uniqueness and the bound on $\|\psi\|_{H^2}$ would then follow from Lemmas 2.1, 2.3 respectively.

The set $\mathscr{U}$ is nonempty since $u = 0 \in \mathscr{U}$, with $\psi = 0$. We will prove that $\mathscr{U}$ is closed. Let $\{u_n\}_{\mathbb{N}} \subset \mathscr{U}$ be a sequence that converges to u in $L^4(\mathbb{R}^2)$. By Lemma 2.3 we see that the corresponding sequence of solutions $\{\psi_n\}_{\mathbb{N}}$ of (1.1b) is bounded in $H^2(\mathbb{R}^2)$. Then there exist $\psi \in H^2(\mathbb{R}^2)$ and a subsequence that converges weakly to ψ in $H^2(\mathbb{R}^2)$. Since for any given compact set $\Omega \subset \mathbb{R}^2$, $H^2(\Omega)$ is compactly embedded in $C(\Omega)$, using a diagonal argument for a nested sequence of compact sets, we can conclude that there exists a subsequence of $\{\psi_n\}_{\mathbb{N}}$ that converges uniformly to ψ in any compact set in $\mathbb{R}^2$. Thus for any $\varphi \in C_0^\infty(\mathbb{R}^2)$ we have $\lim_{n\to\infty}\langle \nabla^2\psi_n, \varphi\rangle = \langle \nabla^2\psi, \varphi\rangle$ and that $\sin(2\psi_n)\varphi$ converge uniformly to $\sin(2\psi)\varphi$. It follows that ψ is a solution of (1.1b) corresponding to u. Since the ψ_n converge pointwise to ψ, we also have $0 \le \psi \le \pi/2 - \theta_0$ in $\mathbb{R}^2$. From Corollary 2.1 we deduce that $0 \le \psi < \pi/2 - \theta_0$ in $\mathbb{R}^2$, which implies $\mathscr{U}$ is closed.

To see that $\mathscr{U}$ is open, it is enough to consider $u_0 \in \mathscr{U}$ and the corresponding solution ψ_0 of (1.1b), and prove that there exists $\delta > 0$ such that if $v \in L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ with $\|v\|_{L^4 \cap L^\infty} < \delta$ then $u_0 + v \in \mathscr{U}$. Assume that the solution for $u_0 + v$ of (1.1b) is written as $\psi_0 + \sigma$. Then σ must satisfy

$$
\begin{aligned}
-\nu\nabla^2\sigma &= -\nu\nabla^2(\sigma + \psi_0) + \nu\nabla^2\psi_0 \\
&= \frac{1}{2}\left(E_0^2 + |u_0 + v|^2\right)\sin(2(\sigma + \psi_0 + \theta_0)) \\
&\quad - \frac{1}{2}\left(E_0^2 + |u_0|^2\right)\sin(2(\psi_0 + \theta_0)) \\
&= \frac{1}{2}\left(|u_0 + v|^2 - |u_0|^2\right)\sin(2(\sigma + \psi_0 + \theta_0)) \\
&\quad + \frac{1}{2}\left(E_0^2 + |u_0|^2\right)(\sin(2(\sigma + \psi_0 + \theta_0)) - \sin(2(\psi_0 + \theta_0))) \\
&= \frac{1}{2}\left(|u_0 + v|^2 - |u_0|^2\right)\sin(2(\sigma + \psi_0 + \theta_0)) \\
&\quad + \frac{1}{2}\left(E_0^2 + |u_0|^2\right)(\cos(2(\psi_0 + \theta_0))\sin(2\sigma) \\
&\quad + \sin(2(\psi_0 + \theta_0))(\cos(2\sigma) - 1)).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&-\nu\nabla^2\sigma + \left(E_0^2 + |u_0|^2\right)(-\cos(2(\psi_0 + \theta_0)))\sigma \\
&= \frac{1}{2}\left(|u_0 + v|^2 - |u_0|^2\right)\sin(2(\sigma + \psi_0 + \theta_0)) \\
&+ \frac{1}{2}\left(E_0^2 + |u_0|^2\right)(\cos(2(\psi_0 + \theta_0))(\sin(2\sigma) - 2\sigma) \\
&+ \sin(2(\psi_0 + \theta_0))(\cos(2\sigma) - 1)),
\end{aligned}
$$

and then

$$-\nu\nabla^2\sigma + V\sigma = F(u_0 + v, \sigma),$$

where V is the potential given by

$$V = \left(E_0^2 + |u_0|^2\right)(-\cos(2(\psi_0 + \theta_0))),$$

and

$$
\begin{aligned}
F(w, \sigma) =& \frac{1}{2}\left(|w|^2 - |u_0|^2\right)\sin(2(\sigma + \psi_0 + \theta_0)) \\
&+ \frac{1}{2}\left(E_0^2 + |u_0|^2\right)\cos(2(\psi_0 + \theta_0))(\sin(2\sigma) - 2\sigma) \\
&+ \frac{1}{2}\left(E_0^2 + |u_0|^2\right)\sin(2(\psi_0 + \theta_0))(\cos(2\sigma) - 1)
\end{aligned}
$$

We can see that $V \in L^\infty(\mathbb{R}^2)$ and, since $0 \le \psi_0 \le \pi/2 - \theta_0$, $V \ge 0$. As $\psi_0 \in H^2(\mathbb{R}^2)$, we also have $\lim_{|x|\to\infty}\psi_0(x) = 0$, hence

$$\liminf_{|x|\to\infty} V(x) \ge \frac{1}{2}\,|E_0|^2\,(-\cos(2\theta_0)) > 0.$$

Therefore V verifies the conditions of Lemma 2.4. By Lemmas 2.6, 2.7 and Remark 2.1, we see that F is continuous from $L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \times H^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^2)$ and verifies (2.9) at $(u, 0)$. Using Lemma 2.5, there exists $r > 0$ such that if $\|v\|_{L^4} < r$ then $-\nabla^2\sigma + V\sigma = F(u_0 + v, \sigma)$ has a unique solution $\sigma \in H^2(\mathbb{R}^2)$ with $\|\sigma\|_{H^2} \le \delta$. Taking $r > 0$ small enough, we can assume $|\sigma| < \pi/4$ and then $-\pi/4 < \psi_0 + \sigma < 3\pi/4 - \theta_0$ for all $x \in \mathbb{R}^2$. Then Corollary 2.1 implies that $0 \le \psi_0 + \sigma < \pi/2 - \theta_0$ everywhere $\mathbb{R}^2$. Thus $\mathscr{U}$ is open. Since $\mathscr{U}$ is closed, open and nonempty, we conclude $\mathscr{U} = L^4(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. $\square$

3. Well-posedness of the evolution problem

We now consider the initial value problem for system (1.1), written as

$$
\begin{aligned}
u(z) =& W(z)u_0 + i\gamma\int_0^z W(z - z')u(z')(\sin^2(\psi(z') + \theta_0) \\
&- \sin^2(\theta_0))dz', && (3.1a) \\
-\nu\nabla^2\psi =& N(u, \psi), && (3.1b)
\end{aligned}
$$

where $\{W(z) : z \in \mathbb{R}\}$ is the unitary group in $L^2(\mathbb{R}^2)$ generated by $\frac{i}{2}\nabla^2$, and $N(u, \psi)$ is given by (2.2) with $\theta_0 \in (\pi/4, \pi/2)$ constant.

The main results of this section are local existence of solutions of (3.1) given in Theorem 3.1, global existence of solutions, shown in Theorem 3.2, and decay for small initial conditions proved in Proposition 3.3.

To show local existence we use the Banach space Y_ζ, $\zeta > 0$, defined by

$$Y_\zeta = \left\{u \in C([0, \zeta], H^1(\mathbb{R}^2)) : \nabla u \in L^4([0, \zeta], L^4(\mathbb{R}^2))\right\}, \quad (3.2)$$

with the norm

$$\|u\|_{Y_\zeta} = \|u\|_{C([0,\zeta],H^1(\mathbb{R}^2))} + \|\nabla u\|_{L^4([0,\zeta],L^4(\mathbb{R}^2))}. \quad (3.3)$$

Also we will make repeated use of the Gagliardo–Nirenberg inequalities

$$\|v\|_{L^\infty} \le C\,\|v\|_{L^2}^{1/3}\,\|\nabla v\|_{L^4}^{2/3}, \quad (3.4)$$

$$\|v\|_{L^4} \le C\,\|v\|_{L^2}^{1/2}\,\|\nabla v\|_{L^2}^{1/2}. \quad (3.5)$$

We note that for $u \in Y_z$, for some $z > 0$, (3.4) and (3.5) imply that u satisfies the conditions for solving the director equation a.e. in $[0,z]$, following Proposition 2.1. The analysis of the previous section is also used to prove suitable boundedness and Lipschitz properties of the right hand side of (3.1a), shown in Proposition 3.1, Lemmas 3.3 and 3.4.

We also recall the Strichartz estimates (see [9]). Let v be the integral solution of the inhomogeneous problem $\mathrm{i}\partial_z v + \frac{1}{2}\nabla^2 v + f = 0$, $v(0) = v_0$. Then $v = h + g$ with

$$h(z) = W(z)v_0, \quad g(z) = \mathrm{i}\int_0^z W(z-z')f(z')dz'. \tag{3.6}$$

Let $1 < r \leq 2 \leq p < \infty$, $q = 2p/(p-2)$ and $\gamma = 2r/(3r-2)$. Then there exist $C_p, C_{p,r} > 0$ such that

$$\|h\|_{L^q(I,L^p)} \leq C_p \|u_0\|_{L^2}, \tag{3.7}$$

$$\|g\|_{L^q(I,L^p)} \leq C_{p,r} \|f\|_{L^\gamma(I,L^r)}, \tag{3.8}$$

for any interval $I \subset \mathbb{R}$.

The local existence theorem uses the Strichartz estimates in Lemmas 3.1, 3.2, where we see that $\|u\|_{Y_\zeta}$ can be controlled by $\|u(0)\|_{H^1}$.

Global existence uses the fact that the conservation of the energy (Hamiltonian H) and the L^2—norm of u imply a bound on $\|u\|_{H^1}$. The local solution can be therefore always extended to a larger interval, see Theorem 3.2. Decay follows from Strichartz estimates for z on the half-line, see Proposition 3.2.

Proposition 3.1. *The map $\Psi : H^1(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2) \to H^2(\mathbb{R}^2)$ defined by $\Psi(u) = \psi$, where ψ is the solution of (3.1b) satisfies*

$$\begin{aligned}\|\Psi(u_1) - \Psi(u_2)\|_{H^2} \leq C_{\nu,\theta_0,E_0}&\left(\|u_1\|_{H^1}, \|u_2\|_{H^1}\right)\\ &\times (1 + \|u_1\|_{L^\infty}^2 + \|u_2\|_{L^\infty}^2)\\ &\times \|u_1 - u_2\|_{H^1},\end{aligned} \tag{3.9}$$

and is therefore locally Lipschitz continuous.

Proof. Let $u_1, u_2 \in H^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, with $R = \max\{\|u_1\|_{H^1\cap L^\infty}, \|u_2\|_{H^1\cap L^\infty}\}$, and let $\psi_1, \psi_2 \in H^2(\mathbb{R}^2)$ be their respective solutions of (3.1b), as in Proposition 2.1. By Corollary 2.1, we see that $0 \leq \psi_1(x), \psi_2(x) < \pi/2 - \theta_0$, for all $x \in \mathbb{R}^2$. We then have

$$\begin{aligned}&(\sin(2(\psi_1+\theta_0)) - \sin(2(\psi_2+\theta_0)))(\psi_1-\psi_2)\\ &\quad\leq 2\cos(2\theta_0)(\psi_1-\psi_2)^2,\end{aligned} \tag{3.10}$$

by the mean value theorem. Also, the difference between two solutions of (3.1a) satisfies

$$\begin{aligned}-\nu\nabla^2(\psi_1-\psi_2) = &\frac{1}{2}E_0^2(\sin(2(\psi_1+\theta_0)) - \sin(2(\psi_2+\theta_0)))\\ &+\frac{1}{2}|u_1|^2(\sin(2(\psi_1+\theta_0)) - \sin(2(\psi_2+\theta_0)))\\ &+\frac{1}{2}\left(|u_1|^2-|u_2|^2\right)\sin(2(\psi_2-\theta_0)),\end{aligned} \tag{3.11}$$

so that multiplying by $\psi_1-\psi_2$ and integrating by parts, we obtain

$$\begin{aligned}\nu\int_{\mathbb{R}^2}|\nabla(\psi_1-\psi_2)|^2\,dx = &\frac{1}{2}E_0^2\int_{\mathbb{R}^2}(\sin(2(\psi_1+\theta_0))\\ &-\sin(2(\psi_2)+\theta_0))(\psi_1-\psi_2)\,dx\\ &+\frac{1}{2}\int_{\mathbb{R}^2}|u_1|^2(\sin(2(\psi_1+\theta_0))\\ &-\sin(2(\psi_2+\theta_0)))(\psi_1-\psi_2)\,dx\\ &+\frac{1}{2}\int_{\mathbb{R}^2}\left(|u_1|^2-|u_2|^2\right)\sin(2(\psi_2+\theta_0))\\ &\times(\psi_1-\psi_2)\,dx.\end{aligned} \tag{3.12}$$

To estimate the right-hand side of (3.12), we use (3.10) and that $\psi_1, \psi_2 \in [0, \pi/2-\theta_0)$ where $\sin(2(\psi+\theta_0))$ is decreasing, to see that

$$\begin{aligned}&\int_{\mathbb{R}^2}(\sin(2(\psi_1+\theta_0)) - \sin(2(\psi_2)+\theta_0))(\psi_1-\psi_2)\,dx\\ &\quad\leq 2\cos(2\theta_0)\|\psi_1-\psi_2\|_{L^2}^2.\end{aligned} \tag{3.13}$$

Also, since $\sin(2(\psi+\theta_0))$ is decreasing in the interval $[0,\pi/2-\theta_0]$, we see that

$$\int_{\mathbb{R}^2}|u_1|^2(\sin(2(\psi_1+\theta_0)) - \sin(2(\psi_2+\theta_0)))(\psi_1-\psi_2)\,dx \leq 0. \tag{3.14}$$

To estimate the third integral in (3.12) we use Hölder's inequality to see that

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(|u_1|^2-|u_2|^2\right)\sin(2(\psi_2+\theta_0))(\psi_1-\psi_2)\,dx\\ &\quad\leq \left(\|u_1\|_{L^4}+\|u_2\|_{L^4}\right)\|u_1-u_2\|_{L^4}\\ &\quad\times\|\psi_1-\psi_2\|_{L^2}.\end{aligned} \tag{3.15}$$

Letting $\alpha = -E_0^2\cos(2\theta_0) > 0$ and using (3.12)–(3.15) and the Gagliardo–Nirenberg inequality (3.5) we have

$$\begin{aligned}&\nu\|\nabla(\psi_1-\psi_2)\|_{L^2}^2 + \frac{\alpha}{2}\|(\psi_1-\psi_2)\|_{L^2}^2\\ &\quad\leq \frac{1}{8\alpha}C\left(\|u_1\|_{H^1}+\|u_2\|_{H^1}\right)^2\|u_1-u_2\|_{L^4}^2,\end{aligned}$$

and therefore

$$\|\nabla(\psi_1-\psi_2)\|_{L^2}^2 \leq \frac{1}{8\alpha\nu}C\left(\|u_1\|_{H^1}+\|u_2\|_{H^1}\right)^2\|u_1-u_2\|_{L^4}^2, \tag{3.16}$$

$$\|(\psi_1-\psi_2)\|_{L^2}^2 \leq \frac{1}{4\alpha^2}C\left(\|u_1\|_{H^1}+\|u_2\|_{H^1}\right)^2\|u_1-u_2\|_{L^4}^2. \tag{3.17}$$

Therefore we obtain

$$\|\psi_1-\psi_2\|_{H^1} \leq C_{\nu,\theta_0,E_0}\left(\|u_1\|_{H^1}+\|u_2\|_{H^1}\right)\|u_1-u_2\|_{L^4}. \tag{3.18}$$

To obtain a Lipschitz estimate for $\|\psi_1-\psi_2\|_{H^2}$ we will use Eq. (3.1b) for the ψ_j, to get

$$\begin{aligned}\nu\left|\nabla^2(\psi_1-\psi_2)\right| \leq &E_0^2|\psi_1-\psi_2| + |u_1|^2|\psi_1-\psi_2|\\ &+\frac{1}{2}(|u_1|+|u_2|)|u_1-u_2|.\end{aligned} \tag{3.19}$$

Using (3.17) it then follows that

$$\begin{aligned}\left\|\nabla^2(\psi_1-\psi_2)\right\|_{L^2} \leq C_{\nu,\theta_0,E_0}&\left(\|u_1\|_{H^1}+\|u_2\|_{H^1}\right)\\ &\times(1+\|u_1\|_{L^\infty}^2+\|u_2\|_{L^\infty}^2)\\ &\times\|u_1-u_2\|_{L^4}.\end{aligned} \tag{3.20}$$

Combining the above inequality, (3.18) and Gagliardo–Nirenberg (3.5), we obtain the estimate (3.9). □

The following two lemmas follow from the Strichartz estimates.

Lemma 3.1. *Let $f \in L^1([0,\zeta], H^1(\mathbb{R}^2))$, and define g by*

$$g(z) = \mathrm{i}\int_0^z W(z-z')f(z')dz'.$$

Then $g \in Y_\zeta$ and satisfies $\|g\|_{Y_\zeta} \leq C_{1,2}\|f\|_{L^1([0,\zeta],H^1)}$.

Proof. Since $W(z)$ is a unitary operator, we have $\|g\|_{C([0,\zeta],H^1)} \leq \|f\|_{L^1([0,\zeta],H^1(\mathbb{R}^2))}$. Using

$$\nabla g(z) = \mathrm{i}\int_0^z W(z-z')\nabla f(z')dz',$$

and the second Strichartz inequality (3.7) with $p = q = 4$, $\gamma = 1$, $r = 2$, we have $\|\nabla g\|_{L^4([0,\zeta],L^4)} \le C_{1,2} \|\nabla f\|_{L^1([0,\zeta],L^2)}$. The statement then follows immediately from the definition of Y_ζ. □

Lemma 3.2. *Let $u_0 \in H^1(\mathbb{R}^2)$, and $h(z) = W(z)u_0$. Then $\|h\|_{Y_\zeta} \le C_4 \|u_0\|_{H^1(\mathbb{R}^2)}$.*

Proof. The statement follows from the first Strichartz estimate (3.7) with $p = q = 4$, and the fact that $z \mapsto h(z) \in C([0, \zeta]; H^1)$. □

To show the existence of local solutions we will use the Picard iteration on $(u, \psi(u))$ in $Y_\zeta \times L^\infty([0, \zeta], H^2(\mathbb{R}^2))$. We first show boundedness and Lipschitz continuity of the nonlinear part of (3.1a) in Y_ζ, Lemmas 3.3, 3.4 respectively.

Lemma 3.3. *Let B be the map defined by*

$$B(u) = u\left(\sin^2(2(\psi(u) + \theta_0)) - \sin^2(2\theta_0)\right),$$

where $\psi(u)$ is a solution of (3.1b). Then B is bounded from Y_ζ to $L^1([0, \zeta], H^1(\mathbb{R}^2))$. Moreover for any $R > 0$ there exists $C > 0$ such that $u \in Y_\zeta$ and $\|u\|_{Y_\zeta} \le R$ imply $\|B(u)\|_{L^1([0,\zeta],H^1(\mathbb{R}^2))} \le C\,\zeta\,\|u\|_{Y_\zeta}$.

Proof. By Proposition 2.1, and the Gagliardo–Nirenberg inequalities (3.4), (3.5), if $u \in Y_\zeta$ then the solution $\psi(u(z))$ of the director equation (3.1b) exists and is in H^2, a.e. in $[0, \zeta]$. The map B is therefore well defined in Y_ζ. The following observations apply to $u = u(z)$, $\psi(u) = \psi(u(z))$ for almost all $z \in [0, \zeta]$. First, by $\left|\sin^2(2(\psi + \theta_0)) - \sin^2(2\theta_0)\right| \le 2$ we have $\|B(u)\|_{L^2} \le 2\|u\|_{L^2}$. Also, by

$$\begin{aligned}\nabla B(u) = &\nabla u\left(\sin^2(2(\psi(u) + \theta_0)) - \sin^2(2\theta_0)\right)\\ &+ 4u \sin(2(\psi(u) + \theta_0)) \cos(2(\psi(u) + \theta_0))\nabla\psi(u),\end{aligned}$$

Lemma 2.3, and Gagliardo–Nirenberg (3.5), we have

$$\begin{aligned}\|\nabla B(u)\|_{L^2} &\le 2\|\nabla u\|_{L^2} + 4\|u\|_{L^4}\|\nabla\psi\|_{L^4}\\ &\le 2\|\nabla u\|_{L^2} + C\|u\|_{L^4}\|\psi\|_{H^2}\\ &\le \tilde{C}(\|\nabla u\|_{L^2} + \|u\|^3_{L^4}) \le \tilde{C}(\|u\|_{H^1} + \|u\|^3_{H^1}).\end{aligned} \tag{3.21}$$

The result follows by integration over $[0, \zeta]$. □

Lemma 3.4. *The map $B : Y_\zeta \to L^1([0, \zeta], H^1(\mathbb{R}^2))$ defined in Lemma 3.3 is locally Lipschitz, i.e. for any $R > 0$ there exists $C > 0$ such that $u_1, u_2 \in Y_\zeta$ and $\|u_1\|_{Y_\zeta}, \|u_2\|_{Y_\zeta} \le R$ imply*

$$\|B(u_1) - B(u_2)\|_{L^1([0,\zeta],H^1(\mathbb{R}^2))} \le C(\zeta + \zeta^{2/3})\|u_1 - u_2\|_{C([0,\zeta],H^1)}. \tag{3.22}$$

Proof. Let $u_1, u_2 \in Y_\zeta$ with $\|u_1\|_{Y_\zeta}, \|u_2\|_{Y_\zeta} \le R$. We use the notation of the previous lemma to establish some pointwise estimates, for almost all z in $[0, \zeta]$.

$$|B(u_1) - B(u_2)| \le 2|u_1 - u_2| + 4|u_2||\psi_1 - \psi_2|,$$

with $\psi_j = \Psi(u_j)$. Therefore

$$\begin{aligned}\|B(u_1) - B(u_2)\|_{L^2} &\le C(\|u_1 - u_2\|_{L^2} + \|u_2\|_{L^4}\|\psi_1 - \psi_2\|_{L^4})\\ &\le C(\|u_1 - u_2\|_{H^1} + \|u_2\|_{H^1}\|\psi_1 - \psi_2\|_{H^1}).\end{aligned} \tag{3.23}$$

On the other hand, it is easy to see that

$$\begin{aligned}|\nabla B(u_1) - \nabla B(u_2)| \le &\,2|\nabla(u_1 - u_2)| + 4|\nabla u_2||\psi_1 - \psi_2|\\ &+ 4|u_1 - u_2||\nabla\psi_1| + 4|u_2||\nabla\psi_2 - \nabla\psi_1|\\ &+ 16|u_2||\psi_2 - \psi_1||\nabla\psi_2|\\ = &\,I_1 + I_2 + I_3 + I_4 + I_5.\end{aligned} \tag{3.24}$$

From the embeddings $H^1(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ and $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, we estimate each term as

$$\begin{aligned}\|I_1\|_{L^2} &\le \|\nabla(u_1 - u_2)\|_{L^2} \le \|u_1 - u_2\|_{H^1},\\ \|I_2\|_{L^2} &\le C\|\nabla u_2\|_{L^2}\|\psi_1 - \psi_2\|_{L^\infty} \le C\|u_2\|_{H^1}\|\psi_1 - \psi_2\|_{H^2},\\ \|I_3\|_{L^2} &\le C\|\nabla\psi_1\|_{L^4}\|u_1 - u_2\|_{L^4} \le C\|\psi_1\|_{H^2}\|u_1 - u_2\|_{H^1},\\ \|I_4\|_{L^2} &\le C\|u_2\|_{L^4}\|\nabla\psi_2 - \nabla\psi_1\|_{L^4} \le C\|u_2\|_{H^1}\|\psi_2 - \psi_1\|_{H^2},\\ \|I_5\|_{L^2} &\le C\|u_2\|_{L^4}\|\nabla\psi_2\|_{L^4}\|\psi_2 - \psi_1\|_{L^\infty}\\ &\le C\|u_2\|_{H^1}\|\psi_2\|_{H^2}\|\psi_2 - \psi_1\|_{H^2}.\end{aligned} \tag{3.25}$$

From Lemma 2.3 we also have $\|\psi_j\|_{H^2} \le C\|u_j\|^2_{H^1}$. Thus, Proposition 3.1, the Gagliardo–Nirenberg inequality (3.4), and (3.23)–(3.25) imply that

$$\|B(u_1) - B(u_2)\|_{H^1} \le C(R)(1 + \|\nabla u_1\|^{4/3}_{L^4} + \|\nabla u_2\|^{4/3}_{L^4})\|u_1 - u_2\|_{H^1}.$$

Integrating over $[0, \zeta]$, we obtain

$$\begin{aligned}\|B(u_1) - B(u_2)\|_{L^1([0,\zeta],H^1)} \le &\,C(R)\|u_1 - u_2\|_{C([0,\zeta],H^1)}\\ &\times \int_0^\zeta (1 + \|\nabla u_1\|^{4/3}_{L^4} + \|\nabla u_2\|^{4/3}_{L^4})\,dz,\end{aligned}$$

and using Hölder's inequality we finally have

$$\|B(u_1) - B(u_2)\|_{L^1([0,\zeta],H^1)} \le C\left(\zeta + \zeta^{2/3}\right)\|u_1 - u_2\|_{C([0,\zeta],H^1)}.$$

for some constant C that depends on R, as stated. □

We now show the existence of local solutions of the evolution equation (3.1).

Theorem 3.1. *Given $u_0 \in H^1(\mathbb{R}^2)$, there exist $\zeta = \zeta\left(\|u_0\|_{H^1}\right) > 0$ and a unique $(u, \psi) \in Y_\zeta \times L^\infty([0, \zeta], H^2(\mathbb{R}^2))$ that satisfies (3.1) and $\psi \in [0, \pi/4]$. Furthermore, the map $u_0 \mapsto u$ is continuous from $H^1(\mathbb{R}^2)$ to Y_ζ.*

Proof. From Lemmas 3.1, 3.2 and 3.3, the map Γ defined in Y_ζ by

$$(\Gamma u)(z) = W(z)\,u_0 + \mathrm{i}\int_0^z W(z - z')B\left(u\left(z'\right)\right)dz', \quad z \in [0, \zeta], \tag{3.26}$$

satisfies $\Gamma u \in Y_\zeta$. (The dependence of Γ on u_0 is not made explicit in this notation.) By Lemma 2.3, we also have that $u \in Y_\zeta$ implies $\psi = \Psi(u) \in L^\infty([0, \zeta], H^2(\mathbb{R}^2))$. Define $h \in Y_\zeta$ by $h(z) = W(z)\,u_0$, $z \in [0, \zeta]$, and consider the closed ball $\overline{\mathcal{B}}_h(R) \subset Y_\zeta$ that is centered at h and has radius $R > 0$. Using Lemmas 3.1, 3.3 we see that if ζ is sufficiently small then

$$\|\Gamma u - h\|_{Y_\zeta} \le C(\|h\|_{Y_\zeta} + R)\zeta \le R. \tag{3.27}$$

Thus Γ maps the closed ball to its interior. To complete the argument we will prove that Γ is a contraction in $\overline{\mathcal{B}}_h(R)$. Then it will have a unique fixed point in $\overline{\mathcal{B}}_h(R)$.

Let $u_1, u_2 \in Y_\zeta$. Then for $0 \le z \le \zeta$

$$(\Gamma u_1)(z) - (\Gamma u_2)(z) = \mathrm{i}\int_0^z W(z-z')\left(B\left(u_1\left(z'\right)\right) - B\left(u_2\left(z'\right)\right)\right) dz', \tag{3.28}$$

and by Lemmas 3.1, 3.4 we have

$$\begin{aligned}\|\Gamma(u_1) - \Gamma(u_2)\|_{Y_\zeta} &\le C\,\|B(u_1) - B(u_2)\|_{L^1([0,\zeta],H^1)} \\ &\le C(R)\,(\zeta + \zeta^{2/3})\,\|u_1 - u_2\|_{Y_\zeta}.\end{aligned} \tag{3.29}$$

Thus taking ζ such that $C(R)(\zeta + \zeta^{2/3}) < 1$ we see that Γ is a contraction in $\overline{\mathcal{B}}_h(R)$.

To see the continuity on the initial conditions, we consider solutions u_j, with respective initial conditions v_j, $j = 1, 2$. We use the notation $\Gamma_{v_j}(u_j)$ for the map Γ of (3.26). By $u_j = \Gamma_{v_j}(u_j)$ we can immediately combine (3.29) and Lemma 3.2 to see that for ζ sufficiently small we have $\|u_1 - u_2\|_{Y_\zeta} \le C\,\|v_1 - v_2\|_{H^1}$, as required. □

The global existence uses the conservation of the L^2-norm of u, and of the Hamiltonian H of the coupled system, given by

$$\begin{aligned}H(u,\psi) = \frac{1}{4}\int_{\mathbb{R}^2} \Big(&|\nabla u|^2 + \nu|\nabla\psi|^2 + \gamma E_0^2\left(\sin^2(\theta_0)\right. \\ &\left. - \sin^2(\theta_0+\psi) + \sin(2\theta_0)\psi\right) \\ &- \gamma\left(\sin^2(\theta_0+\psi) - \sin^2(\theta_0)\right)|u|^2 \Big)\, dx.\end{aligned} \tag{3.30}$$

Define the function

$$h(\psi) = \sin^2(\theta_0) - \sin^2(\theta_0+\psi) + \sin(2\theta_0)\psi, \tag{3.31}$$

with $\pi/4 < \theta_0 < \pi/2$. We check that if $\psi \ge 0$, then

$$\begin{aligned}h(\psi) &\ge \sin^2(\theta_0) - \sin^2(\theta_0+\psi) + \frac{1}{2}\sin(2\theta_0)\sin(2\psi) \\ &= 2\sin\left(\theta_0 - \frac{\pi}{4}\right)\sin\left(\theta_0 + \frac{\pi}{4}\right)\sin^2(\psi) \ge 0.\end{aligned} \tag{3.32}$$

Then by (3.30) we have

$$\begin{aligned}\frac{1}{4}\int_{\mathbb{R}^2} |\nabla u|^2 dx &\le H + \frac{1}{4}\gamma\int_{\mathbb{R}^2}\left(\sin^2(\theta_0+\psi) - \sin^2(\theta_0)\right)|u|^2 dx \\ &\le H + \frac{1}{4}\gamma\|u\|_{L^2}^2\end{aligned}$$

and therefore $\|u\|_{H^1}$ should remain bounded for all time.

The above result of local existence, and the conservation of energy (3.30), leads to the following global existence statement.

Theorem 3.2 (*Global Existence*). *Given $u_0 \in H^1(\mathbb{R}^2)$, there exists a unique $(u,\psi) \in C(\mathbb{R}, H^1(\mathbb{R}^2)) \times L^\infty(\mathbb{R}, H^2(\mathbb{R}^2))$ such that $\psi \in [0,\pi/4]$ and $\nabla u \in L^4_{loc}(\mathbb{R}, L^4(\mathbb{R}^2))$ solution of (1.1).*

Proof. To obtain global existence we use smoother solutions (u,ψ) and the continuous dependence. Given $u \in H^2(\mathbb{R}^2)$, we use the fact that $H^2(\mathbb{R}^2)$ is a Banach algebra, and the argument of Proposition 3.1, to prove that $\Psi(u) \in H^4(\mathbb{R}^2)$. Moreover, we use the argument of Theorem 3.1 to see that for $u_0 \in H^2(\mathbb{R}^2)$ we have the local solution $(u,\psi) \in C([0,\zeta], H^2(\mathbb{R}^2)) \times L^\infty([0,\zeta], H^4(\mathbb{R}))$, with $u \in C^1([0,\zeta], L^2(\mathbb{R}^2))$. Considering such (u,ψ), we use the explicit form of the energy in (3.30), to see that

$$H(u,\psi) \ge \frac{1}{4}\|\nabla u\|_{L^2}^2 - \frac{\gamma}{4}\|u\|_{L^2}^2,$$

and therefore

$$\|u\|_{H^1}^2 \le 4H(u,\psi) + (1+\gamma)\|u\|_{L^2}^2, \tag{3.33}$$

where the right side is a constant depending on $\|u_0\|_{H^1}$. From continuous dependence on initial data, we obtain an a priori bound for $\|u\|_{H^1}^2$. Now, by an usual prolongation argument we can assert that u is defined on $\mathbb{R}$. □

We conclude this section showing that if $\|u_0\|_{L^2}$ is sufficiently small, then the solution of (3.1) satisfies $u \in L^4(\mathbb{R}, L^4(\mathbb{R}^2))$. As a consequence the soliton solutions considered in the next section cannot have arbitrarily small L^2-norm.

Proposition 3.2. *There exists $C > 0$ such that if $(u,\psi) \in Y_\zeta \times L^\infty([0,\zeta], H^2)$ is the solution of (3.1), then*

$$\|u\|_{L^4([0,\zeta],L^4)} \le C\,\|u_0\|_{L^2} + C\,\|u\|_{L^4([0,\zeta],L^4)}^3. \tag{3.34}$$

Proof. The solution u of (3.1a) satisfies

$$\begin{aligned}u(z) = W(z)u_0 + \mathrm{i}\gamma\int_0^z W(z-z')u(z')(&\sin^2(\theta_0+\psi(z')) \\ &- \sin^2(\theta_0))dz' = h(z) + g(z).\end{aligned}$$

We have $\sin^2(\theta_0+\psi) - \sin^2(\theta_0) = \sin 2(\theta_0+\chi)\psi \le \psi$, where $\chi \in [0,\psi]$, since $\psi \in [0,\pi/4)$. Using the Strichartz estimates (3.7) we then obtain

$$\begin{aligned}\|h\|_{L^4([0,\zeta],L^4)} &\le C_4\,\|u_0\|_{L^2}, \\ \|g\|_{L^4([0,\zeta],L^4)} &\le \gamma C_{4,4/3}\,\|u\psi\|_{L^{4/3}([0,\zeta],L^{4/3})}.\end{aligned} \tag{3.35}$$

Then Hölder's inequality, and $\|\psi\|_{L^2} \le \tilde{C}\,\|u\|_{L^4}^2$ from Lemma 2.3 yield

$$\begin{aligned}\|u\psi\|_{L^{4/3}([0,\zeta],L^{4/3})}^{4/3} &\le C_{4,4/3}^{4/3}\int_0^\zeta \|u(z)\|_{L^4}^{4/3}\,\|\psi(z)\|_{L^2}^{4/3}\,dz \\ &\le \tilde{C}^{4/3}C_{4,4/3}^{4/3}\int_0^\zeta \|u(z)\|_{L^4}^4\,dz \le C\,\|u\|_{L^4([0,\zeta],L^4)}^4.\end{aligned} \tag{3.36}$$

The statement follows immediately from (3.35), (3.36), with C depending on $C_{4,2}$, $C_{4,4/3}$, and $\tilde{C}$. □

Bound (3.34) for arbitrary ζ implies the decay statement, see also [4]:

Proposition 3.3. *There exists $a_0 > 0$ such that if $u_0 \in H^1(\mathbb{R}^2)$ satisfies $\|u_0\|_{L^2} < a_0$, then the solution of (1.1) satisfies $\|u\|_{L^4(\mathbb{R},L^4)} < \infty$.*

4. Existence of ground states

In this section we show the existence of solutions (u,ψ) of the stationary problem associated to the system (1.1). Using the soliton ansatz $u(x,z) = e^{\mathrm{i}\sigma z}v(x)$ with $\sigma \in \mathbb{R}$ and $\psi(x,z) = \phi(x)$, Eqs. (1.1) become

$$\begin{aligned}0 &= \nabla^2 v - 2\sigma v + 2\gamma(\sin^2(\theta_0+\phi) - \sin^2(\theta_0)), \\ 0 &= \nabla^2\phi + \frac{1}{2\nu}E_0^2(\sin(2(\theta_0+\phi)) - \sin(2\theta_0)) \\ &\quad + \frac{1}{2\nu}|v|^2\sin(2(\theta_0+\phi)).\end{aligned} \tag{4.1}$$

The idea is to look for solutions of (4.1) by minimizing the Hamiltonian $H(v,\phi)$ of (3.30) over configurations $(v,\phi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ with $\|v\|_{L^2}^2$ fixed. These lowest energy configurations may be termed ground state solitons.

Before stating the main result, we show however that H is not bounded below. This fact motivates a modification of the minimization problem. We formulate this precisely after the lemma.

In what follows we will use the fact that for $\theta_0 \in (\pi/4, \pi/2)$, the function $h(\psi)$ defined in (3.31) is decreasing on the interval $(\pi/4 - \theta_0, 0]$, and we can therefore extend the inequality (3.32) to $(\pi/4 - \theta_0, +\infty)$.

Lemma 4.1. *Let H be the Hamiltonian defined in (3.30) and let $u \in H^1(\mathbb{R}^2)$. Then*

$$\inf_{\psi \in H^1} H(u, \psi) = -\infty.$$

Proof. Consider $u \in H^1(\mathbb{R}^2)$ fixed, and the following function in $H^1(\mathbb{R}^2)$

$$\phi(x) = \begin{cases} 0, & \text{if } \|x\| \ge 2, \\ 2 - \|x\|, & \text{if } 1 < \|x\| < 2, \\ 1, & \text{if } \|x\| \le 1. \end{cases} \tag{4.2}$$

We define the sequence of functions $\psi_n(x) = -\psi^* \phi(r_n x)$, where ψ^* will be chosen large enough and so that $h(-\psi^*) < 0$, and $\{r_n\}_{n\in\mathbb{N}}$ such that $r_n \to 0$. Using that $\psi_n(x) = 0$ for $\|x\| \ge 2/r_n$, and $h(0) = 0$, we deduce that

$$\begin{aligned} H(u, \psi_n) \le \frac{1}{4}\|\nabla u\|^2_{L^2} + \frac{\gamma}{2}\|u\|^2_{L^2} + \frac{\nu}{4}\int_{B(0,2/r_n)} \|\nabla \psi_n(x)\|^2 dx \\ + \frac{\gamma E_0^2}{4}\int_{B(0,2/r_n)} h(\psi_n(x)) dx. \end{aligned} \tag{4.3}$$

We compute the third term of the right hand side as

$$\begin{aligned} \int_{B(0,2/r_n)} \|\nabla \psi_n(x)\|^2 dx &= \int_{B(0,2/r_n)-B(0,1/r_n)} \|\nabla \psi_n(x)\|^2 dx \\ &= 3\pi(\psi^*)^2. \end{aligned}$$

Let $-\delta$ be such that if $x < -\delta$, then $h(x) < 0$, and let $M = \max_{x\in[-\delta,0]} h(x)$. Then we estimate the fourth term on the right hand side of (4.9) as

$$\begin{aligned} \int_{B(0,2/r_n)} h(\psi_n(x)) dx &= \int_{B(0,1/r_n)} h(-\psi^*) dx \\ &+ \int_{B(0,2/r_n)\setminus B(0,1/r_n)} h(-\psi^*\phi(r_n x)) dx \\ &= \frac{\pi h(-\psi^*)}{r_n^2} dx \\ &+ \int_{B(0,(2-\delta/\psi^*)/r_n)\setminus B(0,1/r_n)} h(-\psi^*\phi(r_n x)) dx \\ &+ \int_{B(0,2/r_n)\setminus B(0,(2-\delta/\psi^*)/r_n)} h(-\psi^*\phi(r_n x)) dx \\ &\le \frac{\pi h(-\psi^*)}{r_n^2} + M\pi \frac{4-(2-\delta/\psi^*)^2}{r_n^2} \\ &< \pi \frac{h(-\psi^*) + 3M}{r_n^2}. \end{aligned}$$

Then choosing ψ^* large enough, we have that $h(-\psi^*) + 3M < 0$ and therefore $H(u, \psi_n) \to -\infty$. □

We see that the lack of a lower bound for H is due to the fact that function h of (3.31) is unbounded below. On the other hand, we observe that h can be bounded below by suitably restricting the range of ψ.

Motivated by the above we define

$$\begin{aligned} S_a &= \{(v, \phi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2) : \|v\|^2_{L^2} \\ &= a,\ \phi > \pi/4 - \theta_0 \text{ a.e.}\}, \end{aligned} \tag{4.4}$$

where $H(v, \phi)$ is given in (3.30), and the variational problem

$$J_a = \inf_{(v,\phi)\in S_a} H(v, \phi).$$

The main result of this section, Proposition 4.3, is the existence of an element (v, ϕ) of S_a attaining the infimum J_a, provided $a \ge b$ for some $b > 0$. We also show that the minimizers can be chosen to be radially symmetric, decreasing and to satisfy $v \ge 0$ and $\phi \in [0, \pi/2 - \theta_0]$ everywhere. By Corollary 4.2, an element (v, ϕ) of S_a attaining J_a, $a \ge b$, is a smooth solution of (4.1) for some real σ.

We first observe that for $(v, \phi) \in S_a$, $H(v, \phi) \ge -\frac{\gamma}{4} a$ using (3.32), and therefore $J_a > -\infty$. Note that by $\theta_0 \in (\pi/4, \pi/2)$, $\pi/4 - \theta_0$ is strictly negative. Also, we have $H(0, \phi), H(v, 0) \ge 0$.

Let P be the function defined as follows:

$$P(\phi) = \begin{cases} 0, & \text{if } \phi \le 0, \\ \phi, & \text{if } 0 < \phi < \frac{\pi}{2} - \theta_0, \\ \frac{\pi}{2} - \theta_0, & \text{if } \phi \ge \frac{\pi}{2} - \theta_0. \end{cases}$$

Lemma 4.2. *Let $(v, \phi) \in S_a$. Then $(|v|, P(\phi)) \in S_a$, and*

$$H(|v|, P(\phi)) \le H(v, \phi).$$

Proof. First, $|\nabla P(\phi)| = |P'(\phi)||\nabla\phi| \le |\nabla\phi|$ implies $\|\nabla P(\phi)\|_{L^2} \le \|\nabla\phi\|_{L^2}$. Also, $|\nabla|v|| \le |\nabla v|$ implies $\|\nabla|v|\|_{L^2} \le \|\nabla v\|_{L^2}$.

For $\phi \in (\pi/4 - \theta_0, 0]$, the inequality $H(v, \phi) \ge H(v, 0)$ follows from the fact that $\sin(\theta_0 + \phi) \le \sin(\theta_0)$ and using that $h(\phi) \ge 0$, where h is the function defined in (3.31). For $\phi \in [0, \pi/2 - \theta_0]$ the inequality is immediate. Finally, for $\phi > \pi/2 - \theta_0$, the inequality $H(u, \pi/2 - \theta_0) \le H(u, \phi)$ follows from

$$\begin{aligned} \sin^2(\theta_0) - 1 \le \sin^2(\theta_0) - \sin^2(\theta_0 + \phi), \\ \sin^2(\theta_0) + \sin(2\theta_0)(\pi/2 - \theta_0) - 1 \le \sin^2(\theta_0) + \sin(2\theta_0)\phi \\ - \sin^2(\theta_0 + \phi). \quad \square \end{aligned}$$

By Lemma 4.2 we can restrict our attention to functions (v, ϕ) such that $v \ge 0$ and $0 \le \phi \le \pi/2 - \theta_0$ almost everywhere in $\mathbb{R}^2$.

Let $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ be a measurable function such that $|\{x \in \mathbb{R}^n : \varphi(x) > t\}| < \infty$ for any $t > 0$, and let φ^* denote the symmetric decreasing rearrangement of φ. We recall the following lemma, see [15].

Lemma 4.3. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be an increasing continuous function such that $f(0) = 0$, then for all $\varphi : \mathbb{R}^n \to \mathbb{R}_+$ measurable, $(f \circ \varphi)^* = f \circ \varphi^*$*

Proposition 4.1. *Let $(v, \phi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ with $v \ge 0$ and $0 \le \phi \le \pi/2 - \theta_0$, a.e. in $\mathbb{R}^2$. Then $H(v^*, \phi^*) \le H(v, \phi)$, where v^* and ϕ^* are the symmetric decreasing rearrangements of v and ϕ respectively.*

Proof. Applying the Pólya-Szegö inequality, we have that

$$\frac{1}{4}\left\|\nabla v^*\right\|^2_{L^2} + \frac{\nu}{4}\left\|\nabla\phi^*\right\|^2_{L^2} \le \frac{1}{4}\left\|\nabla v\right\|^2_{L^2} + \frac{\nu}{4}\left\|\nabla\phi\right\|^2_{L^2}. \tag{4.5}$$

The functions $h(\phi)$, defined in (3.31), and $\sin^2(\theta_0 + \phi) - \sin^2(\theta_0)$ are increasing and continuous on $[0, \pi/2 - \theta_0]$, and vanish at the origin. Lemma 4.3 then implies $(h(\phi))^* = h(\phi^*)$ and $(\sin^2(\theta_0 + \phi) - \sin^2(\theta_0))^* = \sin^2(\theta_0 + \phi^*) - \sin^2(\theta_0)$. The first equality yields

$$\begin{aligned} &\int_{\mathbb{R}^2} \sin^2(\theta_0) - \sin^2(\theta_0 + \phi) + \sin(2\theta_0)\phi dx \\ &= \int_{\mathbb{R}^2} \sin^2(\theta_0) - \sin^2(\theta_0 + \phi^*) + \sin(2\theta_0)\phi^* dx, \end{aligned} \tag{4.6}$$

while the second equality, and the product rearrangement inequality of [15], ch.3.4, imply

$$\begin{aligned} \int_{\mathbb{R}^2} v^2(\sin^2(\theta_0 + \phi) - \sin^2(\theta_0)) dx \le \int_{\mathbb{R}^2} (v^2)^*(\sin^2(\theta_0 + \phi) \\ - \sin^2(\theta_0))^* dx \end{aligned}$$

$$= \int_{\mathbb{R}^2} (v^*)^2 (\sin^2(\theta_0 + \phi^*) - \sin^2(\theta_0))dx. \quad (4.7)$$

The conclusion follows immediately from (4.5)–(4.7). □

Proposition 4.2. *There exists $\tilde{a} > 0$ such that if $0 < a \le \tilde{a}$, then $J_a = 0$. Also, there exists $b > \tilde{a} > 0$ such that $J_a < 0$ for all $a \ge b$.*

Proof. We look for a lower bound of the Hamiltonian (3.30). By $\pi/4 < \theta_0 < \pi/2$ and $0 \le \phi \le \pi/2 - \theta_0 < \pi/4$ we have that

$$h(\phi) = \sin^2(\theta_0) - \sin^2(\theta_0 + \phi) + \sin(2\theta_0)\phi \ge -\cos(2\theta_0)\phi^2 > 0, \quad (4.8)$$

therefore

$$\int_{\mathbb{R}^2} \frac{\gamma E_0^2}{4} \left(\sin^2(\theta_0) - \sin^2(\theta_0 + \phi) - \sin(2\theta_0)\phi\right) dx \ge \frac{|\cos(2\theta_0)|\gamma E_0^2}{4} \|\phi\|_{L^2}^2 > 0$$

From inequality (3.32), and $0 \le \phi \le \pi/2 - \theta_0$, we have $\sin^2(\theta_0 + \phi) - \sin^2(\theta_0) \le 2\sin(2\theta_0)\phi$, therefore

$$\int_{\mathbb{R}^2} v^2(\sin^2(\theta_0 + \phi) - \sin^2(\theta_0))dx \le \int_{\mathbb{R}^2} v^2 \sin(2\theta_0)\phi dx \le \|v\|_{L^4}^2 \|\phi\|_{L^2} \le \frac{1}{2\varepsilon} \|v\|_{L^4}^4 + \frac{\varepsilon}{2} \|\phi\|_{L^2}^2 .$$

Thus, taking $\varepsilon = 2E_0^2|\cos(2\theta_0)|$,

$$H(v, \phi) \ge \frac{1}{4} \|\nabla v\|_{L^2}^2 + \frac{\nu}{4} \|\nabla \phi\|_{L^2}^2 - \left(\frac{\gamma}{16E_0^2|\cos(2\theta_0)|}\right) \|v\|_{L^4}^4 ,$$

so that by the Gagliardo–Nirenberg inequality (3.5), $\|v\|_{L^4}^4 \le C\, a \|\nabla v\|_{L^2}^2$, we have $J_a \ge 0$ if $a \le \tilde{a} = \frac{4E_0^2|\cos(2\theta_0)|}{\gamma C}$.

Consider now some $v \in H^1(\mathbb{R}^2)$ with $\|v\|_{L^2}^2 = a$, and let $u_\lambda(x) = \lambda v(\lambda x)$. We have $\|u_\lambda\|_{L^2}^2 = a$ and $\|\nabla u_\lambda\|_{L^2}^2 = \lambda^2 \|\nabla v_\lambda\|_{L^2}^2$, therefore $H(u_\lambda, 0) = \lambda^2 H(v, 0) \to 0$ as $\lambda \to 0$. We conclude that $J_a = 0$ for $a \le \tilde{a}$, proving the first statement.

To show the second statement, we use the fact that if $\phi \in [0, \pi/2 - \theta_0]$ then $h(\phi) \le 2\phi^2$, and $\sin^2(\theta_0 + \phi) - \sin^2(\theta_0) \ge \frac{\cos^2(\theta_0)}{\pi/2 - \theta_0}\phi$. Letting $v = \alpha\phi$, $\alpha \in \mathbb{R}$, we then have

$$\begin{aligned} H(\alpha\phi, \phi) &\le \frac{\alpha^2 + \nu}{4} \|\nabla\phi\|_{L^2}^2 + \frac{\gamma E_0^2}{2} \|\phi\|_{L^2}^2 - \frac{\gamma\alpha^2\cos^2(\theta_0)}{4(\pi/2 - \theta_0)} \|\phi\|_{L^3}^3 \\ &= \frac{\alpha^2 + \nu}{4} \left(\|\nabla\phi\|_{L^2}^2 + \frac{2\gamma E_0^2}{\alpha^2 + \nu} \|\phi\|_{L^2}^2 - \frac{\gamma\alpha^2\cos^2(\theta_0)}{(\alpha^2 + \nu)(\pi/2 - \theta_0)} \|\phi\|_{L^3}^3 \right). \end{aligned} \quad (4.9)$$

Consider some $\phi_1 \in H^1(\mathbb{R}^2)$ satisfying $\phi_1 \not\equiv 0$, and $0 \le \phi_1 \le \pi/2 - \theta_0$ everywhere, and let $\phi_\lambda(x) = \phi_1(\lambda x)$. We have $\|\phi_\lambda\|_{L^2}^2 = \lambda^{-2} \|\phi_1\|_{L^2}^2$, $\|\phi_\lambda\|_{L^3}^3 = \lambda^{-2} \|\phi_1\|_{L^3}^3$ and $\|\nabla\phi_\lambda\|_{L^2}^2 = \|\nabla\phi_1\|_{L^2}^2$. Then, for all $\alpha \in \mathbb{R}$, (4.9) implies

$$H(\alpha\phi_\lambda, \phi_\lambda) \le \frac{\alpha^2 + \nu}{4} \left(\|\nabla\phi_1\|_{L^2}^2 + \frac{2\gamma E_0^2 \lambda^{-2}}{\alpha^2 + \nu} \|\phi_1\|_{L^2}^2 - \frac{\gamma\alpha^2\cos^2(\theta_0)\lambda^{-2}}{(\alpha^2 + \nu)(\pi/2 - \theta_0)} \|\phi_1\|_{L^3}^3 \right). \quad (4.10)$$

Fixing $\alpha > \left(\frac{2E_0^2(\pi/2 - \theta_0)\|\phi_1\|_{L^2}^2}{\cos^2(\theta_0)\|\phi_1\|_{L^3}^3}\right)^{1/2}$, we see from (4.10) that there exists $\lambda_0 > 0$ (depending on α, $\|\nabla\phi_1\|_{L^2}$) such that $0 < \lambda < \lambda_0$ implies $H(\alpha\phi_\lambda, \phi_\lambda) < 0$. On the other hand, $\|v\|^2 = \frac{\alpha^2}{\lambda^2}\|\phi_1\|_{L^2}^2$, therefore $a \ge b = \frac{\alpha^2}{\lambda_0^2}\|\phi_1\|_{L^2}^2$ implies $J_a < 0$. □

We now use Proposition 4.2 to prove the existence of minimizers of the Hamiltonian (3.30) in S_a, assuming a is such that $J_a < 0$. By Proposition 4.1 it is enough to look for these minimizers in $H^1_{rad}(\mathbb{R}^2) \times H^1_{rad}(\mathbb{R}^2)$.

Proposition 4.3. *Let $a \ge b > 0$ with b as in 4.2. Then there exists $(v, \phi) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ that satisfies $(v, \phi) \in S_a$, and $H(v, \phi) = J_a$. In addition, we may assume that $(v, \phi) \in H^1_{rad}(\mathbb{R}^2) \times H^1_{rad}(\mathbb{R}^2)$, are decreasing, and satisfy $v \ge 0$ and $\phi \in [0, \pi/2 - \theta_0]$ everywhere.*

Proof. Let $\mathcal{A} = \{(v_n, \phi_n)\}_{n\in\mathbb{Z}^+} \subset S_a$ be a minimizing sequence for H. By Lemma 4.2, and Proposition 4.1 we may assume that the minimizing sequence $\mathcal{A}$ also belongs to $H^1_{rad}(\mathbb{R}^2) \times H^1_{rad}(\mathbb{R}^2)$, and that its elements (v_n, ϕ_n) satisfy $v_n \ge 0$, $\phi_n \in [0, \pi/2 - \theta_0]$ a.e. in $\mathbb{R}^2$, for all $n \in \mathbb{Z}^+$.

From (3.33), we obtain the following upper bound of the norm H^1 of v_n :

$$\|v_n\|_{H^1}^2 \le \sup_n 4H(v_n, \phi_n) + (1 + \gamma)a. \quad (4.11)$$

Also, using the lower bound for $h(\phi)$ in (4.8), and $\sin^2(\theta_0 + \phi_n) - \sin^2(\theta_0) \le 1$ for $\phi \in [0, \pi/2 - \theta_0]$, we have that

$$\begin{aligned} H(v_n, \phi_n) &\ge \frac{\nu}{4}\|\nabla\phi_n\|_{L^2}^2 - \frac{\gamma}{4}\|v_n\|_{L^2}^2 + \frac{(-\cos(2\theta_0))\gamma E_0^2}{4}\|\phi_n\|_{L^2}^2 \\ &\ge -\frac{\gamma}{4}a + \frac{1}{4}\min\{\nu, -\cos(2\theta_0)\gamma E_0^2\}\|\phi_n\|_{H^1}^2, \end{aligned}$$

therefore, we obtain the following upper bound for the norm H^1 of ϕ_n :

$$\|\phi_n\|_{H^1}^2 \le C_{\gamma,E_0,\nu,\theta_0}\left(\sup_n 4H(v_n, \phi_n) + \gamma a\right). \quad (4.12)$$

Then there exists a subsequence of $\mathcal{A}$ that is weakly convergent to (v, ϕ) in $H^1_{rad}(\mathbb{R}^2) \times H^1_{rad}(\mathbb{R}^2)$. We denote this subsequence also by $\mathcal{A} = \{(u_n, \phi_n)\}_{n\in\mathbb{Z}^+}$. Since $H^1_{rad}(\mathbb{R}^2)$ is compactly embedded in $L^p(\mathbb{R}^2)$, for any $2 < p < \infty$, see [9], the subsequence converges strongly to $(v, \phi) \in L^3(\mathbb{R}^2) \times L^3(\mathbb{R}^2)$. This implies that $v_n \to v$ and $\phi_n \to \phi$ a.e., therefore we may assume $v \ge 0$, $0 \le \phi \le \pi/2 - \theta_0$ a.e. in $\mathbb{R}^2$.

To see that the limit is the minimizer, we use the weak semi-continuity of the L^2 norm to obtain

$$\|v\|_{L^2}^2 \le \liminf_n \|v_n\|_{L^2}^2 = a, \quad (4.13)$$

$$\frac{1}{4}\|\nabla v\|_{L^2}^2 + \frac{\nu}{4}\|\nabla\phi\|_{L^2}^2 \le \liminf_n \left(\frac{1}{4}\|\nabla v_n\|_{L^2}^2 + \frac{\nu}{4}\|\nabla\phi_n\|_{L^2}^2\right). \quad (4.14)$$

Since the function $h(\phi)$ given in (3.31) is non-negative for $\phi_n \in [0, \pi/2 - \theta_0]$ we can apply Fatou's lemma to the sequence $h(\phi_n)$ to obtain

$$\int_{\mathbb{R}^2} h(\phi)dx \le \liminf_n \int_{\mathbb{R}^2} h(\phi_n)dx. \quad (4.15)$$

Also, $v_n \to v$ in $L^3(\mathbb{R}^2)$ implies $v_n^2 \to v^2$ in $L^{3/2}(\mathbb{R}^2)$ and $\left|\sin^2(\theta_0 + \phi_n) - \sin^2(\theta_0 + \phi)\right| \le |\phi_n - \phi|$ implies $\sin^2(\theta_0 + \phi_n)$

$\to \sin^2(\theta_0 + \phi)$ in $L^3(\mathbb{R}^2)$. Therefore

$$\lim_n \int_{\mathbb{R}^2} v_n^2 \left(\sin^2(\theta_0+\phi_n) - \sin^2(\theta_0)\right) dx = \int_{\mathbb{R}^2} v^2 \left(\sin^2(\theta_0+\phi) - \sin^2(\theta_0)\right) dx. \tag{4.16}$$

Collecting (4.14)–(4.16), we therefore have

$$H(v, \phi) \le \liminf_n H(u_n, \phi_n) = J_a < 0. \tag{4.17}$$

Using the fact that $H(v, 0)$, $H(0, \phi) \ge 0$, we conclude that $v \not\equiv 0$ and $\phi \not\equiv 0$, and moreover by (4.13) we have $0 < \|v\|_{L^2}^2 \le a$. Let $\lambda = \sqrt{a}/\|v\|_{L^2} \ge 1$, we check that $(\lambda v, \phi) \in S_a$, and that

$$J_a \le H(\lambda v, \phi) \le \lambda^2 H(v, \phi) \le \lambda^2 J_a \le J_a < 0. \tag{4.18}$$

It follows that $\lambda = 1$. Therefore $(v, \phi) \in S_a$, and $H(v, \phi) = J_a$. □

The following statement is shown as in [4].

Corollary 4.1. *There exists $a_0 > 0$ such that $J_a = 0$ for $0 < a \le a_0$, and $J_a < 0$ for $a > a_0$. Moreover, the map $a \mapsto J_a$ is decreasing in (a_0, ∞).*

We finally show that the minimizers of the proposition above are smooth solutions of (4.1). The arguments are standard and we only sketch the proof.

Corollary 4.2. *Let $(v^*, \phi^*) \in H^1_{\text{rad}}(\mathbb{R}^2) \times H^1_{\text{rad}}(\mathbb{R}^2)$, $v^* \ge 0$ and $\phi^* \in [0, \pi/2 - \theta_0]$ everywhere, be minimizer of H in S_a, a as in Proposition 4.3. Then there exists $\sigma \in \mathbb{R}$ such that v^*, ϕ^* satisfy (4.1) in $H^{-1}_{\text{rad}}(\mathbb{R}^2) \times H^{-1}_{\text{rad}}(\mathbb{R}^2)$. Moreover, v^*, ϕ^* are C^2.*

Proof. We easily see that H is differentiable in $H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. Let a as in Proposition 4.3. Considering variations in S_a of the first component around (v^*, ϕ^*) we have the first equation (4.1) in the dual space $H^{-1}(\mathbb{R}^2)$ for some real σ. We also consider variations in S_a of the second component around (v^*, ϕ^*), for instance $\phi^* + \tilde{\phi}$. Since the condition for $(v^*, \phi^* + \tilde{\phi}) \in S_a$ requires that $\phi^* + \tilde{\phi} > \pi/4 - \theta_0$ a.e., we consider $\tilde{\phi} \in H^2(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2)$ with $\|\tilde{\phi}\|_{H^2}$ sufficiently small. The derivative along the second component then vanishes for any $\tilde{\phi} \in H^2(\mathbb{R}^2)$ and we obtain the second equation (4.1) in the dual space $H^{-1}(\mathbb{R}^2)$ by density and the existence of the derivative.

To show that (v^*, ϕ^*) is smooth we write (4.1) as $(-\nabla^2 + 1)v^* = f_1(v^*, \phi^*)$, $(-\nabla^2 + 1)\phi^* = f_2(v^*, \phi^*)$, and check that $(v^*, \phi^*) \in S_a$ implies that $f_1(v^*, \phi^*)$, $f_2(v^*, \phi^*) \in L^2(\mathbb{R}^2)$ using again Gagliardo–Nirenberg. System (4.1) then implies that v^*, $\phi^* \in H^2(\mathbb{R}^2)$ and we can iterate the argument to increase the regularity: from v^*, $\phi^* \in H^2(\mathbb{R}^2)$ we check that $f_1(v^*, \phi^*)$, $f_2(v^*, \phi^*) \in H^2(\mathbb{R}^2)$, therefore v^*, $\phi^* \in H^4(\mathbb{R}^2) \subset C^2(\mathbb{R}^2)$ using (4.1) and Sobolev. □

5. Discussion

In this paper we have extended the theory of optical solitons in liquid crystals to a model that allows large angle deviations of the liquid crystal orientation. Our results give new information on the saturation of the nonlinear effects in the system. The saturation effect we show appears to be optimal and is quite intuitive given the geometry of the electric field–director angle interaction, but has received less attention in the literature as most studies have considered small angle models. Saturation provides an additional regularizing mechanism, seen in the way the conservation of the Hamiltonian was used to show long-time existence.

We expect that our results can be extended to cover models that take into account two more recent experimental methods to adjust the pre-tilt angle θ_0 that do not use an external electric field E_0, either by suitably "anchoring" the angle θ_0 at the boundary, see [16], or by using external magnetic fields [17]. Also, while the condition $\theta_0 > \pi/4$ appears naturally in the derivation of models considered in this paper and in the literature, see [4], it does not seem to be necessary in the experimental set-up, and appears to be a technical condition that facilitates the analysis of the small scale models on the plane. This point, and the more general question of justifying the NLS models should motivate further study of finite domain effects.

CRediT authorship contribution statement

Juan Pablo Borgna: Conceptualization, Methodology, Formal analysis. **Panayotis Panayotaros:** Conceptualization, Methodology, Formal analysis. **Diego Rial:** Conceptualization, Methodology, Formal analysis. **Constanza Sánchez de la Vega:** Conceptualization, Methodology, Formal analysis.

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Appendix

We present a heuristic derivation of system (1.1). We start by describing the physics and geometry of the experiment modeled by the system.

The experimental setup is a rectangular box containing a nematic liquid crystal. Following the notation in the experimental literature, the horizontal axes are y and z, while x denotes the vertical axis. We are interested in the propagation of a laser beam along the z-axis (the "optical axis") in this box. We assume that the electric field of the laser beam has only one component, along the x-axis. The complex scalar $u(x, y, z)$ represents the amplitude of this electric field. (The laser electric field has the form of a slowly modulated plane wave, see [4].) Furthermore, there is a constant electric field of magnitude E_0 applied to the sample, also along the x-axis. (This field can be produced by placing the liquid crystal sample between two capacitor plates that are parallel to the y, z plane.)

The nematic liquid crystal we consider can be thought of as consisting of cylindrically symmetric molecules. We describe the macroscopic state of the material by a field of unit vectors (the "director field") that represent the macroscopic molecular orientation at each point and model the material using the Oseen–Frank equations for the director field [18,19]. In the experiment considered we assume that this unit vector is everywhere on the $x-z$ plane and can be thus described by the angle $\theta(x, y, z)$ between the molecular axis unit vector and the z-axis.

The experimental system is described by a coupled Maxwell–Oseen–Frank system of equations for the external and laser electric fields and director field respectively, see [4]. The interaction between electric fields and the director field has a simple geometrical interpretation, namely electric fields create dipoles in the nematic liquid crystal molecules and the electric and director fields tend to align. We can then see that the assumptions that all electric fields are along the x-axis, and that the molecular orientation vector is on the z, $x-$plane are consistent with the Maxwell–Oseen–Frank description.

If the laser beam is absent, the total electric field is $E_0\hat{x}$, with E_0 a real constant, $\hat{x}$ the unit vector along the x-axis. The director field produced by this constant electric field is denoted by θ_0. This angle field is referred to as "pretilt angle". The Maxwell–Oseen–Frank equations for the pretilt angle θ_0 produced by a constant electric field are complicated by the boundary conditions for the director field at the boundaries of the domain occupied by the liquid crystal. A simplification can be obtained if we ignore lateral boundaries along the optical axis, i.e. assume that the domain is only bounded by parallel planes at $x = \pm d/2$, $y = \pm c/2$ for some $d, c > 0$. We may also assume that θ_0 is independent of z. The Maxwell–Oseen–Frank equations then reduce to

$$\nu \nabla^2 \theta_0 = -\frac{1}{2} E_0^2 \sin 2\theta_0, \tag{A.1}$$

with Dirichlet boundary conditions for θ_0 at $x = \pm d/2$, $y = \pm c/2$. Here $\nabla^2 = \partial_x^2 + \partial_y^2$ the Laplacian in the transverse directions.

The director field in the presence of both the laser electric field (described by u) and the constant field $E_0\hat{x}$ is written as $\theta(x, y, z) = \theta_0(x, y, z) + \psi(x, y, z)$, i.e. ψ is the "additional angle" produced by the laser beam. Note that the time-dependence of the laser field is assumed known and is averaged, i.e. the Maxwell–Oseen–Frank system is approximated by a time-independent Helmholtz–Oseen–Frank equation. The system is further simplified assuming slow variation of u and θ in along the optical axis, see [4], and we obtain the intermediate model of [1,3]

$$\partial_z u = \frac{1}{2} \mathrm{i} \nabla^2 u + \mathrm{i}\gamma (\sin^2 \theta - \sin^2 \theta_0) u, \tag{A.2a}$$

$$\nu \nabla^2 \theta = -\frac{1}{2} (E_0^2 + |u|^2) \sin 2\theta, \tag{A.2b}$$

with $\theta = \theta_0 + \psi$. Using the relation (A.1) between E_0 and θ_0 we then obtain

$$\partial_z u = \frac{1}{2} \mathrm{i} \nabla^2 u + \mathrm{i}\gamma (\sin^2 \theta - \sin^2 \theta_0) u, \tag{A.3a}$$

$$\nu \nabla^2 \psi = \frac{1}{2} E_0^2 \sin 2\theta_0 - \frac{1}{2} (E_0^2 + |u|^2) \sin 2(\theta_0 + \psi). \tag{A.3b}$$

Note that θ_0 in (A.3) is still not determined since we have not solved (A.1) with its boundary conditions. Instead we use a function that is a reasonable approximation for θ_0 given E_0. The simplest choice is a constant that is close to the experimental value of θ_0 in the region where the laser beam varies rapidly. We thus obtain (1.1), with θ_0 a constant. The constant θ_0 approximation is justified by the observation that the beam has a width measured in micrometers (10^{-6} m) while the box has dimensions of centimeters (10^{-2} m) and θ_0 varies over the longer scale of the box [3]. Related experiments consider non-constant external field E_0, e.g. the case $E_0 = E_0(y)$, $\theta_0 = \theta_0(x, y)$, both periodic in y, see [20,21].

The main physical advantage of the new model over system (1.2) is that we have not made any assumptions on the size of ψ. We also see that the coupling between the laser field u and θ in both (A.2) and (A.3) vanishes at $\theta = \pi/2$, i.e. precisely when the director angle is aligned with the laser electric field.

Note that the physical assumptions behind (A.2), and (1.1) are the same. In both cases we may use a reasonable approximation of θ_0 as a parameter of the system. The system we consider here is closer to the systems (1.2), (1.3). Their derivation from (A.2) goes through (A.3), and where E_0 appears explicitly.

The description of the molecular orientation by the director field and the geometry of the interaction between electric fields and the molecular orientation imply certain symmetries that may be broken by specifying the director field in some parts of the problem, e.g. the boundaries. Thus we start by assuming that in the absence of the laser electric field, θ_0 takes values in $[-\pi/2, \pi/2)$, and that the experimentally observed θ_0 takes values in $[0, \pi/2)$. Adding the laser field produces an additional angle ψ. The range of ψ is deduced from Eqs. (A.3b) in Section 2.

Note that the assumption $\theta_0 > \pi/4$ in (1.1) is a technical condition that was also used in the studies of (1.2), (1.3). Its role becomes clear in the proofs of Sections 2–4. Also the existence of a nontrivial solution θ_0 of (A.1) requires $E_0 > \mathcal{E}_0$ for some $\mathcal{E}_0 > 0$ that depends on the boundary conditions, see [4]. Our results on (1.1) are valid for any $E_0 > 0$, i.e. this threshold does not affect the analysis of (1.1), it is rather related to choosing E_0, θ_0 that are consistent with (A.1) and boundary conditions.

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