On the possibility of wormhole formation due to quantum effects in the gravitational collapse of a small dust shell

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In the present note we outline the main steps towards the analysis of wormhole formation during the quantum collapse of a spherical dust shell. We define the quantum observable θ, corresponding to the classical trace of the expansion tensor, and calculate its expected value in order to obtain information about the geometry of space-time around the shell. We show that the local quantum geometry represents a wormhole.

Keywords: Quantum gravitational collapse; wormholes

En la presente nota mostramos los pasos principales involucrados en el análisis de formación de "agujeros de gusano" durante el colapso cuántico de una cáscara esférica de polvo. Definimos el observable cuántico θ, correspondiente a la traza clásica del tensor de expansión, y calculamos su valor de expectación con objeto de obtener información sobre la geometría del espacio-tiempo en la vecindad de la cáscara. Mostramos que la geometría cuántica local representa un "agujero de gusano".

Descriptores: Colapso gravitacional cuántico; agujeros de gusano

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1. Introduction

In previous work [1] we investigated in detail the quantum mechanics of a collapsing spherical dust shell of a roughly Planck mass. The quantum mechanics was formulated for the radius of the shell in a canonical way [2–4] after dimensional reduction. As a result we obtained a nonlocal Schrödinger equation with vanishing coefficient at the point where the curvature radius R is equal to the classical Schwarzschild radius of the shell, analogous to critical layer absorption in fluid mechanics [6]. The wave function that solves this equation was interpreted as the probability density of finding the shell at radius R. We found that the spectrum is mixed with the continuous spectrum extending from zero to M, the Schwarzschild mass of the classical exterior geometry. The eigenfunctions are localized around $R = 2M$, due to critical layer behavior. However, we derived no information about the surrounding space, and therefore nothing could be said about the quantum evolution of the space-time generated by the shell motion. It is the purpose of this note to take a first step in the analysis of the space around the shell during its quantum collapse.

We derive equations for the local geometry around the shell following [7]. These equations contain information about the contraction and expansion of the local geometry and therefore allow, in a canonical way, to determine an observable depending only on the canonical variables of the shell. Calculating the expectation value of this operator along the trajectory (in Hilbert space) of the quantum collapse, we find, when the bound states do not contribute, that the space-time generated by the shell begins to contract classically towards $R = 0$. The quantum mechanics however does not allow the total collapse, producing as a result an expansion into a different space-time sheet. Thus, at least locally around the shell and on the average, the space-time replicates a wormhole throat.

This note is organized as follows. The second section contains the mathematical formulation of the problem. The third section is devoted to a rather brief sketch (a full account will be the subject of a future publication) of the calculations involved and the main result. The fourth section is devoted to a discussion of the advantages and limitations of the proposed approach.

2. Formulation of the problem

We begin by recalling the essential ideas involved in analyzing the evolution of the space-time in the neighborhood of a spherical collapsing shell. The metric on the shell is given by [3, 7]

$$ds^2 = -d\tau^2 + R^2(\tau)\, d\Omega^2, \tag{1}$$

where τ is the proper time of the shell and $R(\tau)$ determines the proper area of the shell. Outside the shell the solution is the Schwarzschild solution while inside it is taken to be flat Minkowski space. In local Gaussian normal coordinates $(\tau, \eta, \theta, \varphi)$, where η is the proper distance along the geodesic normal to the shell, the metric of the full space in the vicinity of the shell is given by [7]

$$ds^2 = -L^2(\tau,\eta)\, d\tau^2 + d\eta^2 + Q^2(\tau,\eta)\, d\Omega^2. \tag{2}$$

To lowest order in η, $L^2 = 1 + \eta L_1 + \mathcal{O}(\eta^2)$, $Q^2 = R^2(\tau) + \eta Q_1 + \mathcal{O}(\eta^2)$, where the functions L_1 and Q_1 are given functions of the proper time τ.

We are interested in determining whether the local geometry at the shell expands or contracts in terms of the evolution of $R(\tau)$. This is readily evaluated by means of the volume expansion, θ, the trace of the expansion tensor, and is given explicitely by [5]

$$\theta = K = \frac{d}{d\tau} \ln \sqrt{|g|}, \tag{3}$$

where K is the trace of the extrinsic curvature tensor of the hypersurface of the shell and g is the determinant of the hypersurface metric at $\eta = 0$. This gives $\theta = 2\dot{R}(\tau)/R(\tau)$. We thus see the expected classical behavior: as the shell implodes we have $\theta < 0$ and $\theta \to -\infty$ as $R \to 0$. Hence classically the motion ends in a singularity. On the other hand for a classically expanding motion $\theta > 0$.

Using the expression for $\dot{R}$ in terms of P and R derived in Ref. 3 we show that θ, for $R > 2M$ is given by

$$\theta(R,P) = -\frac{2F(R)}{H(R,P)} \sinh P\,, \tag{4}$$

while, for $R < 2M$ we have

$$\theta(R,P) = -2(\operatorname{sgn}\theta) \frac{F(R)}{H(R,P)} \cosh P\,, \tag{5}$$

where $F(R) \equiv \sqrt{1 - 2M/R}$ and the Hamiltonian $H(R,P)$, for $R \geq 2M$ is given by

$$H^2 = 2\big[(R^2 - MR) - R^{3/2}|R - 2M|^{1/2}\cosh P\big], \tag{6}$$

while for $R \leq 2M$

$$H^2 = 2\big[(R^2 - MR) - R^{3/2}|R - 2M|^{1/2}|\sinh P|\big]. \tag{7}$$

Now we need to study the corresponding quantum mechanics. The basic dynamical variables are R and its canonically conjugate momentum P, which operate on the corresponding Hilbert space. We need therefore to realize θ as a selfadjoint operator on the same space. This space was previously determined in Ref. 1. The functions $H(R,P)$ and $\theta(R,P)$ are defined by the spectral calculus. We note that the original canonical Hamiltonian was obtained from an exact dimensional reduction of the full theory [3, 4]. This Hamiltonian is a multivalued function at the phase space and we showed [1] how to quantize this system arriving at the nonlocal expressions for H in (8), given in the Eqs. (6) and (7).

Now the problem of formation of the wormhole can be formulated as follows. Consider the solution of the nonlocal Schrödinger equation [1]

$$i\frac{\partial\psi}{\partial\tau} = H\psi, \tag{8}$$

with (scattering) condition $\psi \to \psi_{in}$ as $\tau \to -\infty$. Once the solution of (8) with a given initial condition is known, we can calculate the expectation value of the volume expansion

$$\Theta(\tau) = \int_0^\infty \psi^*(x,\tau)\theta(x,P)\psi(x,\tau)\, dx, \tag{9}$$

where we use x instead of R to take into account the change of the relevant measure [1, 2]. It clearly follows from this expression that $\Theta(\tau)$ must be negative as $\tau \to -\infty$, since the initial shell is contracting. Note also that classically $\Theta(\tau)$ will not be defined as $x \to 0$. However when the selfadjointness of H is imposed on the boundary condition at $x = 0$, the result will be a reflected wave (of probability). We thus expect that as $\tau \to \infty$, $\Theta(\tau) > 0$.

The next section contains a sketch of the calculation which shows that this is indeed the case.

3. Asymptotic solution of the problem

The scattering state solution of (8) which matches the incoming wave condition as $\tau \to -\infty$ is given by

$$\psi(x,\tau) = \int_0^M A(E)\varphi(x,E)\, e^{-iE\tau}\, dE, \tag{10}$$

where the function $A(E)$ can be chosen to localize the packet. The asymptotic behavior of $\varphi(x,E)$ for $x \to \infty$ is given in Ref. 1 and is of the form

$$\varphi(x,E) = A_2(x,E)\big[e^{i\theta_2(x,E)} - e^{-i\theta_2(x,E)}\big], \tag{11}$$

where $\theta_2(x,E)$ and $A_2(x,E)$ are defined for $x \geq 2M$, and are determined in the W.K.B. approximation

$$\theta_2(x,E) = \int_{\tilde{R}(E)}^{2M} \ln\big[\sqrt{f^2(\xi,E)+1} + f(\xi,E)\big]\, d\xi + \int_{2M}^{x} \ln\big[\sqrt{f^2(\xi,E)-1} + f(\xi,E)\big]\, d\xi, \tag{12}$$

$$A_2(x,E) = -\frac{i}{2}\big\{x^3(x-2M)[f^2(x,E)-1]\big\}^{-1/4}, \tag{13}$$

where

$$f(\xi, E) = \frac{\xi^2 - M\xi - E^2/2}{\xi^{3/2}|\xi - 2M|^{1/2}}, \tag{14}$$

and

$$\tilde{R}(E) = \frac{M + \sqrt{M^2 + 2E^2}}{2}. \tag{15}$$

We note that $\tilde{R}(E)$ is obtained by solving $f(\tilde{R}(E), E) = 0$ and represents a classical turning point of the phase-space trayectories, for $R \leq 2M$. For $x \to \infty$, $A_2 \sim x^{-1/2}$.

Now, as $\tau \to -\infty$ we evaluate the integral (10) using the stationary phase method. As expected the incoming wave dominates and due to the localized nature of the solution the leading contribution comes for $x \to \infty$. In these limits we can now calculate the leading part of $\Theta(\tau)$, for $\tau \to -\infty$

$$\Theta \approx -\int_{x>x_0} \varphi^*(x) H^{-1}(\sinh\theta_2')\varphi(x)\, dx, \tag{16}$$

here x_0 is large ($x_0 \gg 2M$), $\varphi(x) \equiv \varphi(x, E(\tau))$, $\theta_2'(x,E) \equiv \partial_x\theta_2(x,E)$ and $E = E(\tau)$ is the solution of the equation $\tau = \partial^2\theta_2/\partial E^2$. In the Eq. (16) we consider that $\tau \to -\infty$. Likewise as $\tau \to \infty$ the outgoing wave dominates, and the leading contribution to (10) comes for $x \to \infty$. Then, for $\tau \to \infty$, we have

$$\Theta \approx \int_{x>x_0} \varphi^*(x) H^{-1}(\sinh\theta_2')\, \varphi(x)\, dx. \tag{17}$$

The comparison of the last two equations shows that on the average the space-time contracts and then expands. Thus the average local quantum geometry is indeed the one of a wormhole.

4. Discussion

Using the unambiguous canonical formalism developed in Ref. 3 and 4 we have shown that quantum mechanics (at the level of the Schrödinger equation for a single observable radius of the shell) suggests the possibility of wormhole formation as the result of collapse of a dust shell. We also have shown that the singularity is resolved by the quantum fluctuations via the production of a wormhole. Our approach allows a complete and consistent calculation using the assumption of conservation of probability up to $R = 0$. This is a novel result which should be recovered in calculations based on more realistic assumptions on the behavior of space-time at distances of the order of the Planck length. Unfortunately no calculations of such a nature exist at the present time.

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