Complex Ginzburg-Landau Equations as Perturbations of Nonlinear Schrödinger Equations: Quasi-periodic Solutions

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Abstract

As an application of theory developed in three previous papers, we study the persistence of finite genus solutions of the cubic Schrödinger equation under a cubic complex Ginzburg-Landau perturbation. We use the integrable structure of the cubic Schrödinger equation to derive necesary conditions on the branch points of the Riemann surface asociated to the finite genus solution for their persistence under the perturbation.

1 Introduction

This is the fourth paper in a program concerned with the long time dynamics of a class of infinite dimensional Hamiltonian systems when the symmetries of these systems are broken by a dissipative and driven perturbation. In this paper we study a particular case in which the unperturbed Hamiltonian system is completely integrable by the Inverse Spectral Transform.

The cubic NLS

$$\partial_t A = i\partial_{xx} A + 2i\sigma|A|^2 A\,, \tag{1}$$

is the compatibility condition for the two Zakharov-Shabat linear systems of

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differential equations [?]:

$$\partial_x \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -i\lambda & A \\ -B & i\lambda \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{2}$$

$$\partial_t \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} i(AB - 2\lambda^2) & i\partial_x A + 2\lambda A \\ i\partial_x B - 2\lambda B & i(2\lambda^2 - AB) \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{3}$$

$$B = \sigma A^* . \tag{4}$$

That is, $\partial_t \partial_x \vec{\psi} = \partial_x \partial_t \vec{\psi}$ if and only if A satisfies the cubic NLS equation (1).

Using the Floquet theory of the Zakharov-Shabat system one can find a large class of solutions to the NLS equation periodic in x and quasiperiodic in t, the so called finite genus solutions. In this paper we study the persistence of this kind of solutions when the NLS equation is perturbed to a cubic Ginzburg-Landau equation

$$\partial_t A = \epsilon r A + (\epsilon + i)\partial_{xx} A - 2(\epsilon q - i)|A|^2 A . \tag{5}$$

The particular scaling of the CGL equation (5) was set to emphasize the influence of weak driving, dissipation and nonlinear damping. In the limit when $\epsilon \to 0$, equation (5) reduces to the cubic NLS equation (1). Having explicite knowledge of a particular solution of the NLS equation allows us to investigate its persistence under the CGL perturbation. The difficulty is that the integrable structure of the phase space of the NLS equation is not topologically robust. In general, even under weak dissipative perturbations most of this phase space structures will be destroy. Usually, the best one can expect is the persistence of stationary points. However, for the CGL equation (5), it is observed numerically, see [?]; that a class of nontrivial dynamic objects closely related to some of the NLS finite genus solutions are also preserved.

It is known that using the Zakharov-Shabat system one can find an infinite number of conserved quantities for the NLS equation. In particular, it is possible to write down explicitly the isospectral manifold, which is the collection of all the finite genus solutions with the same Zakharov-Shabat spectrum. This isospectral manifold is characterized by $2g+2$ real parameters, where g is the genus of the Riemann surface asociated to the family of quasiperiodic solutions. We will use the conserved quantities and the selection criteria developed in three previos papers to impose $g+1$ conditions on the $2g+2$ parameters of the persistent solutions. This conditions, complemented with the $g+1$ equations from the boundary conditions, result in a $(2g+2) \times (2g+2)$ system of equations for the $2g+2$ parameters. Finally, we write down explicitly these equations for the parameters for the genus two case with even symmetry.

The paper is organized as follows. In the second section we summarize the previously known material we need on the integrability and finite genus solutions of the NLS equation. For this second section we follow closely the work of Forest and Lee [?]. In the third section we use the method developed in [?] and the integrability of the cubic NLS equation to derive conditions on the parameters of the finite genus solutions for their persistence under the Ginzburg-Landau perturbation. In the fourth section we implement this conditions for the case of genus 2 solutions with even symmetry.

2 Finite Genus Solutions of the NLS Ecuation

The periodic theory for the cubic NLS equation (1) was developed by Ablowitz and Ma [?], Previato [?], and Forest and Lee [?] as an extension of the periodic theory for the Korteweg-de Vries equation created by Lax [?], Novikov [?], McKean, and VanMoerbeke [?]. We will follow closely the work of Forest and Lee [?]. Consider the fundamental matrix solution of (2), $M(x,x_0,\lambda;A)$ normalized to be the identity at $x=x_0$. The transfer matrix $M(x_0+1,x_0,\lambda;A)$ maps solutions of (2) across one period of the potential A. The fundamental object of the periodic spectral theory of (2) is the Floquet discriminant $\Delta(\lambda;A)$, defined as the trace of the transfer matrix,

$$\Delta(\lambda;A) = Tr M(x_0+1,x_0,\lambda;A)\,. \tag{6}$$

Translating any solution $\vec{\psi}$ of (2) n times the period it becomes

$$\vec{\psi}(x+n) = M^n(x_0+1,x_0,\lambda;A)\vec{\psi}(x)\,, \tag{7}$$

therefore $\vec{\psi}$ is bounded if the eigenvalues $\rho^{\pm}$ of $M(x_0+1,x_0,\lambda;A)$ have modulus less or equal to one, where

$$\rho^{\pm} = \frac{\Delta \pm \sqrt{\Delta^2-4}}{2}\,. \tag{8}$$

Hence, the spectrum σ of the Zakharov-Shabat operator (2) over the whole line, is given by

$$\sigma = \{\lambda \in \mathbf{C} \ : \ \Delta(\lambda;A) \in R, -2 < \Delta(\lambda;A) < 2\}\,. \tag{9}$$

The roots λ_j of $\Delta(\lambda; A) = \pm 2$ corresponding to the periodic and antiperiodic eigenvalues.

If the function A solves the cubic NLS equation (1), then a direct calculation shows

$$\partial_t M = VM - MV \,, \tag{10}$$

where V is the matrix in the linear system (3). Therefore

$$\partial_t \Delta(\lambda; A) = 0 \,. \tag{11}$$

Thus, in particular, the periodic spectrum of (2) is invariant for solutions $A(x,t)$ the cubic NLS equation. The isospectral classes N_A of all potentials of (2) with the same periodic spectrum as A, arise as the level sets of Δ.

$$N_A = \{B \in F \ : \ \Delta(\lambda; A) = \Delta(\lambda; B) \,\text{for every}\, \lambda \in \mathbf{C}\} \,. \tag{12}$$

The following are properties of spectrum of the Zakharov-Shabat operator (2) on $L^2_{loc}(\mathbf{R}, \mathbf{C}^2)$ for the focusing ($\sigma = 1$) cubic NLS.

- If $\Delta(\lambda; A) = \pm 2$ then $\Delta(\lambda^*; A) = \pm 2$, i.e. the periodic and antiperiodic eigenvalues are either real or occur in complex conjugate pairs.
- $-2 \leq \Delta(\lambda; A) \leq 2$ for all $\lambda \in \mathbf{R}$. The entire real axis is continuous spectrum.
- The simple periodic spectrum,

  $$\sigma^s = \{\lambda_j \ : \ \Delta(\lambda_j) = \pm 2, \ \Delta'(\lambda_j) \neq 0\} \,, \tag{13}$$

  occurs only in complex conjugate pairs off the real axis.
- Given $\vec{\psi}^{(1)} = (\psi_1^{(1)}, \psi_2^{(1)})^T$ and $\vec{\psi}^{(2)} = (\psi_1^{(2)}, \psi_2^{(2)})^T$ two independent solutions of (2)–(3), define the quadratic eigenfunctions

  $$f(x,t;\lambda) = \frac{1}{2}(\psi_1^{(2)}\psi_2^{(1)} + \psi_2^{(2)}\psi_1^{(1)}) \,, \tag{14}$$

  $$g(x,t;\lambda) = \psi_1^{(1)}\psi_1^{(2)} \,, \qquad h(x,t;\lambda) = \psi_2^{(1)}\psi_2^{(2)} \,. \tag{15}$$

  Then $R^2(\lambda) = f^2 - gh$ is independent of x and t. Moreover, these quadratic eigenfunctions provide the fundamental conservation law for the focusing cubic NLS

  $$\partial_t f - \partial_x(4\lambda f + i(Ah + A^* g)) = 0 \,. \tag{16}$$

- Normalize (16) by $R(\lambda)$, then f, g, and h can be expanded in powers of λ^{-1}

$$f = 1 - 2 \sum_1^\infty f_j(x,t) \lambda^{-j-1} \,, \tag{17}$$

$$g = \sum_1^\infty g_j(x,t) \lambda^{-j} \,, \tag{18}$$

$$h = \sum_1^\infty h_j(x,t) \lambda^{-j} \,. \tag{19}$$

Then (16) is an infinite series in λ^{-1}, with conservation laws as coefficients:

$$\sum_1^\infty \lambda_{-j-1} \left(\partial_t f_j - \partial_x (4 f_{j+1} + i(A h_j + A^* g_j)) \right) \,, \tag{20}$$

for example the first two densities are

$$f_1 = |A|^2 \,, \qquad f_2 = \frac{1}{2} (A \partial_x A^* - A^* \partial_x A) \,. \tag{21}$$

- The quadratic eigenfunctions g and h satisfy the linearized NLS equation

$$\partial_t g = i \partial_{xx} g + i(4|A|^2 g - 2A^2 h) \,, \tag{22}$$

$$\partial_t h = -i \partial_{xx} h - i(4|A|^2 h - 2A^{*2} g) \,. \tag{23}$$

- An N-phase solution $A(x,t)$ is a solution of the form $A(x,t) = A(\theta_1, ..., \theta_N)$ of period one in each phase $\theta_j = k_j x - \omega_j t$. If A is a N-phase solution, then $\sigma^s(A)$ has exactly $2N$ elements.
- Decompose $\sigma = \sigma^s \cup \sigma^d$, where $\lambda_d \in \sigma^d$ are called double points, defined by $\Delta(\lambda_d) = \pm 2$, $\Delta'(\lambda_d) = 0$, (Technically, more derivatives can be zero at λ_d). Then, there are at most finitely many double points off the real axis.
- For generic A, $\sigma^d = \emptyset$. I.e., in any open neighborhood of an N-phase solution A, all the eigenvalues are simple.

2.1 Quasiperiodic Solutions

Whereas the trace of the transfer matrix yields constants of the motion, in the case of quasiperiodic solutions the zeroes of the M_{12} entry provides the "angle variables",

$$M_{12}(x_0 + 1, x_0, \mu_j; A) = 0 \,. \tag{24}$$

The cubic NLS inversion formula for N-phase solutions is:

$$i \partial_x \ln A = \sum_{j=1}^{2N} \lambda_j - 2 \sum_{j=1}^{N-1} \mu_j \,, \tag{25}$$

$$\partial_t \ln A = 2i \left(\sum_{\substack{j>k \\ j,k=1}}^{2N} \lambda_j \lambda_k - \frac{3}{4} \left(\sum_{k=1}^{2N} \lambda_k \right)^2 \right)$$

$$-4i \left(-\frac{1}{2} \left(\sum_{k=1}^{2N} \lambda_k \right) \left(\sum_{j=1}^{N-1} \mu_j \right) + \sum_{\substack{j>k \\ j,k=1}}^{N-1} \mu_j \mu_k \right) , \tag{26}$$

where the μ's evolve in x and t governed by the ODE's

$$\partial_x \mu_k = -2i \frac{R_N(\mu_k)}{\prod_{j \neq k} (\mu_k - \mu_j)} , \tag{27}$$

$$\partial_t \mu_k = \partial_x \mu_k \left(\sum_{j=1}^{2N} \lambda_j - 2 \sum_{\substack{j \neq k \\ j,k=1}}^{N-1} \mu_j \right) , \tag{28}$$

and $R_N(\mu)$ is the hyperelliptic irrationality determined by

$$R_N^2(\mu) = \prod_{k=1}^{2N} (\mu - \lambda_k) . \tag{29}$$

As shown by the μ representation, the entire x, t dependence (except for a plane wave factor) of an N-phase solution is captured by the μ ODE's (27)–(28). It is a remarkable fact that these ODE's linearize via the classical Abel-Jacobi map associated to the Riemann surface $\mathcal{R}$ of genus $g = N - 1$.

$$\mathcal{R} = \left\{ (\lambda, R(\lambda)) \,\middle|\, R^2(\lambda) = \prod_{j=1}^{2N} (\lambda - \lambda_j) \right\} . \tag{30}$$

In fact, the explicit angle variables θ_j in the N-phase description

$$A(x,t) = A_N(\theta_1, ..., \theta_N) , \qquad \theta_j = \kappa_j x - \omega_j t , \tag{31}$$

are the images of μ_j under the Abel-Jacobi map. To be more explicit we require some ingredients from the Riemann surface $\mathcal{R}$.

We choose a canonical basis of closed cycles on $\mathcal{R}$ $\{a_j, b_j\}, j = 1, ..., g$ (see Figure 1). All closed paths on $\mathcal{R}$ are linear combinations of a_j, b_j. Next we choose a normalized basis of holomorphic differentials:

$$\phi_j = \sum_{k=1}^{g} C_{jk} \frac{\lambda_{g-k}}{R(\lambda)} \, d\lambda , \qquad j = 1, ..., g , \tag{32}$$

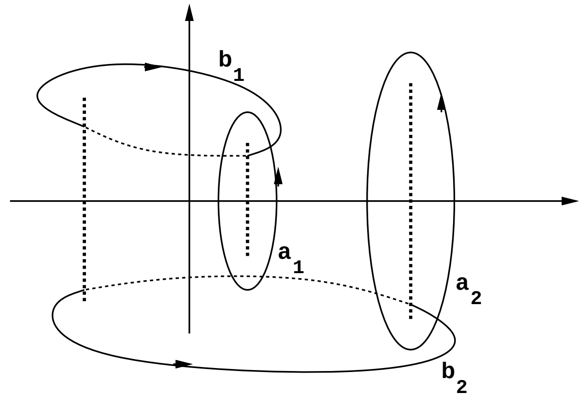

Fig. 1. An example (genus 2) of branch cuts satisfying the focusing cubic NLS restrictions and the corresponding a and b cycles.

where the normalization constants C_{jk} are uniquely specified by the conditions

$$\oint_{a_i} \phi_j = \delta_{ij} \,. \tag{33}$$

The period matrix B of $\mathcal{R}$ is then defined by

$$B_{ij} = \oint_{b_i} \phi_j \,, \qquad i,j = 1, ..., g \,, \tag{34}$$

the matrix B so defined is symmetric and its imaginary part is negative definite. The cubic NLS constraints on $\{\lambda_j\}^{2N}$ (distinct and occurring in complex conjugate pairs), yield the following constraints on the C and B matrices

$$Re(C_{ij}) = 0 \,, \tag{35}$$
$$Re(B_{jj}) = 0 \,, \tag{36}$$
$$Re(B_{ij}) = -\tfrac{1}{2} \;, \qquad i \neq j \,. \tag{37}$$

With these ingredients the Abel-Jacobi map $(\mu_1, ..., \mu_g) \mapsto (l_1, ..., l_g)$ is defined by

$$l_j(\vec{\mu}) = \sum_{k=1}^{g} \int_{\mu_k^0}^{\mu_k} \phi_j \qquad j = 1, ..., g \,, \tag{38}$$

where $(\mu_1^0, ..., \mu_g^0)$ represents a fixed g-tuple of points on $\mathcal{R}$. Geometrically, this map is depicted by

$$\Phi : \vec{\mu} \in (\mathcal{R}\times, ..., \times\mathcal{R}) \to \vec{l} \in \mathbf{C}^g/\Lambda \,, \tag{39}$$

$$\Lambda = (e_1, ..., e_g, B_1, ..., B_g)\,. \tag{40}$$

where e_j are g-vectors with one in the j^{th} coordinate and zero elsewhere.

It follows by a straightforward computation that the μ-ODE's (27)–(28) are transformed under the Abel-Jacobi transformation into

$$\partial_x l_j(\vec{\mu}) = -2iC_{j1}\,, \tag{41}$$

$$\partial_t l_j(\vec{\mu}) = -2i\left(\left(\sum_{k=1}^{2N}\lambda_k\right)C_{j1} + 2C_{j2}\right)\,. \tag{42}$$

Thus, the $l_j(\vec{\mu})$ are indeed phases (linear sums in x and t).

$$l_j(x,t) = -2i\left(C_{j1}x + \left(\left(\sum_{k=1}^{2N}\lambda_k\right)C_{j1} + 2C_{j2}\right)t\right) + l_j(0,0)\,. \tag{43}$$

The constraints (35)–(37) for the focusing cubic NLS translate into the following constraints in the phases l_j

$$Im\Big(l_j(x,t) - l_j(0,0)\Big) = 0\,. \tag{44}$$

Next we want to give the precise form of the phases $\theta_j = \kappa_j x + \omega_j t$ with explicit formulas for the wave numbers κ_j and frequencies ω_j, and allowable values of the integration constants $l_j(0,0)$. These facts require a delicate analysis of the Abel-Jacobi map. For our purposes we simple state that to write an explicit formula for the cubic NLS solution $A(x,t)$, (26)–(26) says one needs explicit formulae for $\sum_1^{N-1}\mu_j$ and $\sum_{j>k}\mu_j\mu_k$. This is the classical problem of the inversion of the Abel-Jacobi map, its solution is given by ratios of Riemann theta functions asociated to the Riemann Surface (see [**?**]). The quasiperiodic solutions of the cubic NLS equation can be given explicitelly in terms of these Riemann theta functions. We will skip these details and give only the relevant phase information:

$$l_j(x,t) = \theta_j(x,t) + l_j(0,0)\,, \tag{45}$$

$$\theta_j(x,t) = \kappa_j x + \omega_j t\,, \tag{46}$$

$$\kappa_j = -2\pi i C_{j1}\,, \tag{47}$$

$$\omega_j = -4\pi i\left(\left(\sum_{k=1}^{2N}\lambda_k\right)C_{j1} + 2C_{j2}\right)\,. \tag{48}$$

Finally, $l_j(0,0)$ must satisfy technical constraints (see [**?**]), which we omit, but which are fundamental to produce the explicit quasiperiodic theta function solution of cubic NLS. The conditions for periodicity in space are given by (see (47))

$$\frac{1}{2\pi}\kappa_j = n_j\,, \qquad n_j \in Z\,. \tag{49}$$

Because of this last constraint, these solutions, up to arbitrary phase shifts, are parametrized by N discrete parameters n_j and the remaining N real parameters. In the next section we will study the conditions for the persistence of this class of solutions.

3 Persistence Criteria for Finite Genus Solutions

In this section we study the persistence of the isospectral manifold

$$N_A = \{B \in F\,:\, \Delta(\lambda; A) = \Delta(\lambda; B) \quad \text{for all} \quad \lambda \in C\}\,, \tag{50}$$

of quasiperiodic solutions for $0 < \epsilon << 1$ in the complex Ginzburg-Landau perturbation

$$\partial_t A = \epsilon r A + (\epsilon + i)\partial_{xx} A - 2(\epsilon q + i)|A|^2 A\,. \tag{51}$$

Notice that the isospectral manifold is a *nonresonant* N-torus of NLS solutions

$$A(x,t) = F(\vec{k}\,x + \vec{\omega}\,t + \vec{z}\,)\,, \tag{52}$$

as defined in [?] for a Floquet spectrum with $2N$ periodic eigenvalues. We enphasize that our definition of *nonresonant*, i.e.

$$\{\vec{l}\,t_0 + \vec{k}\,t_1 + \vec{\omega}\,t_2\,:\,(t_0,t_1,t_2) \in \mathcal{R}^3\} \quad \text{dense in } \mathcal{T}^n. \tag{53}$$

is weaker than assuming the components of $\vec{\omega}$ rationally independent. This is due to the subtorus swept out by the terms $\vec{l}\,t_0 + \vec{k}\,t_1$.

We now rewrite from [?] our definition 3.1 of persistence of nonresonant N-torus of solutions of the NLS equation.

Definition 3.0.1 *A nonresonant torus of NLS solutions of the form (52) is said to persist under the CGL perturbation if and only if there exists an* ϵ

dependent family of smooth tori of solutions of the CGL equation (51) of the form

$$A_\epsilon(x,t) = F_\epsilon(\vec{k}\,x + \vec{\omega}_\epsilon\,t + \vec{z}\,)\,, \tag{54}$$

where the frequency vectors $\vec{\omega}_\epsilon$ converge to $\vec{\omega}$ in $\mathcal{R}^n$ and the functions F_ϵ converge to F in $C^\infty(\mathcal{T}^n)$ as ϵ tends to zero.

Notice that equations (1) and (5) can be written as

$$\partial_t A = -i\frac{\delta \mathcal{H}}{\delta A^*}\,, \tag{55}$$

$$\partial_t A = -i\frac{\delta \mathcal{H}}{\delta A^*} - \epsilon\frac{\delta \mathcal{G}}{\delta A^*}\,, \tag{56}$$

where the functionals $\mathcal{H}$ and $\mathcal{G}$ are given by

$$\mathcal{H} = \int\limits_0^1 \left(|\partial_x A|^2 - |A|^4\right)\,dx\,, \tag{57}$$

$$\mathcal{G} = \int\limits_0^1 \left(|\partial_x A|^2 + q|A|^4 - r|A|^2\right)\,dx\,. \tag{58}$$

We now rewrite proposition 3.1 of [**?**] on the persistence of nonresonant N-torus of solutions of the NLS equation.

Proposition 3.0.1 *A necessary condition for the persistence of a nonresonant NLS torus in the sense of Definition 3.0.1 is the following. For each conserved functional $\mathcal{F}$, the function F that characterizes the GNLS torus through (52) satisfies the 'Melnikov' selection criterion*

$$0 = \mathcal{M}^{\mathcal{F}}(F) \equiv \int\limits_{\mathcal{T}^n} \left(\frac{\delta \mathcal{F}}{\delta A^*}\frac{\delta \mathcal{G}}{\delta A} + \frac{\delta \mathcal{F}}{\delta A}\frac{\delta \mathcal{G}}{\delta A^*}\right)(F(\vec{z}))\,d\overset{n}{z}\,, \tag{59}$$

or equivalently, any solution A on a persisting GNLS torus must satisfy

$$0 = \mathcal{M}^{\mathcal{F}}(A) = \lim_{T\to\infty}\frac{1}{2T}\int\limits_{-T}^{T} SS^{\mathcal{F}}(A(t))\,dt\,, \tag{60}$$

where

$$SS^{\mathcal{F}}(A) \equiv \int_0^1 \left(\frac{\delta \mathcal{F}}{\delta A^*} \frac{\delta \mathcal{G}}{\delta A} + \frac{\delta \mathcal{F}}{\delta A} \frac{\delta \mathcal{G}}{\delta A^*} \right)(A)\, dx \,. \tag{61}$$

We use the above condition and one collection of conserved quantities of the cubic Schrödinger equation to write down explicitly necessary conditions for the persistence of this isospectral manifold. There are several equivalent families of conserved quantities. We will see that for this calculation it is convenient to use the simple critical points of the spectrum (defined below).

Because $\mathcal{G}$ is a linear combination of the mass the energy and the L^4 norm of A

$$\mathcal{G} = \mathcal{H} - r\mathcal{M} + (q+1) \int_0^1 |A|^4\, dx \,, \tag{62}$$

both terms in the persistence condition (**??**) are equal. Therefore (**??**) can be written if the following way

$$J_{\mathcal{F}} = 2 \lim_{T \to \infty} \frac{1}{T} \int_0^T \int_0^1 \frac{\delta \mathcal{F}}{\delta A^*} \frac{\delta \mathcal{G}}{\delta A}(A_0)\, dx\, dt = 0 \,. \tag{63}$$

We will use the following family of conserved quantities called the simple critical points of the spectrum. These simple critical points of Δ are defined by

$$\left\{ \lambda_c \;\middle|\; \frac{d\Delta}{d\lambda}(A, \lambda_c) = 0 \;\text{ and }\; \Delta(A, \lambda) \neq \pm 2 \right\}. \tag{64}$$

For a quasiperiodic potential A_N with N quasiperiods, there are exactly N of these simple double points. Like the periodic spectrum, the simple double points are either real or appear in complex conjugate pairs. In the following lemma we compute the derivative of λ_c with respect to A^*

LEMMA 1. The derivative of λ_c with respect to A^* is given by:

$$\frac{\delta \lambda_c}{\delta A^*} = -\frac{i}{F(\lambda_c)} g(x,t;\lambda_c) \,, \qquad F(\lambda_c) = \int_0^1 \psi_1 \phi_2 + \psi_2 \phi_1\, dx \,, \tag{65}$$

where $g(x,t;\lambda)$ is the quadratic eigenfunction (15) and $F(\lambda)$ is the integral of the quadratic eigenfunction (14), therefore $F(\lambda)$ is independent of t.

PROOF. Let $M = (\vec{\psi}, \vec{\phi})$ be a solution of the Zakharov-Shabat system

$$\partial_x M = UM\,, \qquad M(0,\lambda_c) = I\,, \tag{66}$$

where

$$U = \begin{pmatrix} -i\lambda_c & \delta A \\ -\delta A^* & i\lambda_c \end{pmatrix}\,. \tag{67}$$

Taking the variation of (66), the following equation is obtained

$$\delta\partial_x M = \delta U M + U\delta M\,. \tag{68}$$

Multiplying both sides of this equation by M^{-1} and integrating yields

$$\int_0^1 M^{-1}\delta\partial_x M\,dx = \int_0^1 M^{-1}(\delta U M + U\delta M)\,dx\,. \tag{69}$$

If the variation and the derivative in x can be interchanged then integrating by parts the left side

$$\int_0^1 M^{-1}\partial_x\delta M\,dx = -\int_0^1 \partial_x M^{-1}\delta M\,dx = \int_0^1 M^{-1}U\delta M\,dx\,. \tag{70}$$

where the last integral is obtained using the fact that if $\partial_x M = UM$ then $\partial_x M^{-1} = -M^{-1}U$. The last integral is equal to the second term in (69). Therefore

$$\int_0^1 M^{-1}\delta U M\,dx = 0\,. \tag{71}$$

Taking the variation with respect to A^* and A of this last equation we obtain

$$i\frac{\delta\lambda_c}{\delta A^*}\int_0^1 \psi_1\phi_2 + \psi_2\phi_1\,dx = \phi_1\psi_1\,, \tag{72}$$

from which the lemma follows.

Using these simple critical points λ_c as conserved quantities. The persistence condition (63) can be written as

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^1 g(x,t;\lambda_c) \left(-\partial_{xx} A^* + 2q|A|^2 A^* - rA^*\right) \, dx \, dt = 0 \, , \tag{73}$$

for all the simple critical points λ_c.

For an $(N-1)$-genus potential A_N, the corresponding quadratic eigenfunctions f, g, and h are polynomial in λ. In particular, because of (24) the μ variables are the zeros of g

$$g(x,t;\lambda) = -iA \prod_{j=1}^{N-1} (\lambda - \mu_j) \, . \tag{74}$$

Because $g(x,t;\lambda)$ is a polynomial of degree N–1, instead of requiring condition (73) for each of the N simple critical points λ_c, it is equivalent to impose the same condition for all λ,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^1 \prod_{j=1}^{N-1} (\lambda - \mu_j) A \left(-\partial_{xx} A^* + 2q|A|^2 A^* - rA^*\right) \, dx \, dt = 0 \, . \tag{75}$$

In order to express the integrand in terms of only the μ's and not their conjugates (this requirement will be important below, where we integrate in the μ representation). It is necessary to integrate by parts the first term, so the conditions become

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \int_0^1 \Bigg[\prod_{j=1}^{N-1} (\lambda - \mu_j) \left(-A^* \partial_{xx} A + 2q|A|^4 - r|A|^2\right) \\ -\partial_x \left(\prod_{j=1}^{N-1} (\lambda - \mu_j) \right) \left(A^* \partial_x A - A \partial_x A^*\right) \Bigg] \, dx \, dt = 0 \, . \tag{76}$$

When all the periodic double points are real, the isospectral manifold N_A is a N-torus (see [?]). In this case, if the frequencies ω_j (48) are not rationally related (which is the generic case), the solution A_N fills ergodically the torus

N_A. Hence for any function $F[A(x,t)]$, the average

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\int_0^1 F[A(x,t)]\,dx\,dt\,, \tag{77}$$

is equal to the average of F over the torus Prop 3.0.1

$$\int_0^1 \dots \int_0^1 F[A(\theta_1,\dots\theta_{N-1})]\,d\theta_1\dots d\theta_{N-1}\,. \tag{78}$$

The integral in (78) involves only N–1 phases θ_j, because the plane wave factor introduced when integrating the trace formulas, can be factorized. Moreover, because the integral in (77) is the average in space and time, the integral (77) is equal to the averages over the torus except when the frequencies and wave numbers are commensurable simultaneously, i.e. when there is a vector $\vec{\xi}=(\xi_1,\dots,\xi_{N-1})$ with integer coordinates such that

$$(\vec{\xi},\vec{k})=(\vec{\xi},\vec{\omega})=0\,. \tag{79}$$

The second term in (76) can be written like a perfect derivative in x using the fact the $A^*\partial_x A - A\partial_x A^*$ is polynomial in the μ variables. If the solution A_N fills ergodically the torus N_A then the average in time is independent of the initial condition and therefore the second term in (76) is zero. Hence condition (76) in the $\vec{\theta}$ representation becomes

$$\int_0^1 \dots \int_0^1 \left[\prod_{j=1}^{N-1}(\lambda-\mu_j)\left(-A^*\partial_{xx}A+2q|A|^4-r|A|^2\right)\right]d\theta_1\dots d\theta_{N-1}=0\,. \tag{80}$$

The integrand in (80) can be more easily written in terms of the μ variables than in terms of the phases θ_j. Therefore it is better to change variables in the integral (80) from the $\vec{\theta}$ representation to the $\vec{\mu}$ representation.

$$\oint_{\mu_1}\dots\oint_{\mu_{N-1}}\left[\prod_{j=1}^{N-1}(\lambda-\mu_j)\left(-A^*\partial_{xx}A+2q|A|^4-r|A|^2\right)\right]\frac{\partial\vec{\theta}}{\partial\vec{\mu}}\,d\mu_1\dots d\mu_{N-1}=0\,, \tag{81}$$

where the integrals are taken over the μ-cycles. The transformation $\vec{\mu}\to\vec{\theta}$ is given by the Abel-Jacobi map (38). The Jacobean of this transformation can be computed directly from its definition,

$$\frac{\partial\vec{\theta}}{\partial\vec{\mu}}=\det\left(\frac{\mu_j^{N-1-k}}{R(\mu_j)}\right)=\frac{\prod_{j>k}(\mu_j-\mu_k)}{\prod_{j=1}^{N-1}R(\mu_j)}\,. \tag{82}$$

For the A_N potential the generating function for the densities is polynomial in λ

$$f = \sum_{j=0}^{N} f_j \lambda^j \,. \tag{83}$$

In this expansion, f_0 and f_1 are constants. Using f_2 the densities for the mass can be express in terms of the μ variables.

$$|A|^2 = \left(2 \sum_1^{N-1} \mu_j - \Lambda_1 \right) \sum_1^{N-1} \mu_j - 2 \sum_{j>k} \mu_j \mu_k + \Lambda_2 - \frac{1}{4} \Lambda_1 \,, \tag{84}$$

where

$$\Lambda_1 = \sum_{j=1}^{2N} \lambda_j \qquad \Lambda_2 = \sum_{j>k} \lambda_j \lambda_k \,. \tag{85}$$

Using the trace formula (26)

$$\frac{\partial_{xx} A}{A} = 4\Lambda_1 \sum_1^{N-1} \mu_j - 4 \left(\sum_1^{N-1} \mu_j \right)^2 + \Lambda_1^2 \,. \tag{86}$$

Substituting all these expressions into (81), it becomes

$$\oint_{\mu_1} \ldots \oint_{\mu_{N-1}} \frac{\prod_{j>k} (\mu_j - \mu_k)}{\prod_{j=1}^{N-1} R(\mu_j)} \prod_{j=1}^{N-1} (\lambda - \mu_j) \Gamma_1 \left(\Gamma_2^2 - 2\Lambda_1^2 + 2q\Gamma_1 - r \right) d\mu_1 \ldots d\mu_{N-1} = 0 \,. \tag{87}$$

where

$$\Gamma_1 = \Gamma_2 \sum_1^{N-1} \mu_j - 2 \sum_{j>k} \mu_j \mu_k + \Lambda_2 - \frac{1}{4} \Lambda_1 \,, \qquad \Gamma_2 = 2 \sum_1^{N-1} \mu_j - \Lambda_1 \,. \tag{88}$$

The integration in the new μ variables needs to be done over the μ-cycles. When the corresponding Zakharov-Shabat operator is self adjoint, like for the defocusing cubic Schrödinger equation, the μ-cycles are fairly simple. In our case, because the Zakharov-Shabat operator (2) is not self adjoint, the μ-cycles are very complicated. However, Ercolani and Forest [?] have shown that the Abel-Jacobi map (38) restricted to the cubic NLS flow, imposes topological constraints on the closed μ-cycles:

$$\mu_j \sim a_j \,, \tag{89}$$

where $\sim$ denotes "is homologous to", or equivalent as a path of integration over the Riemann surface $\mathcal{R}$ (30). Therefore because the integrand in (87) is analytic, it is possible to change the path of integration in (87) to the a-cycles.

$$\oint_{a_1} \cdots \oint_{a_{N-1}} \frac{\prod_{j>k}(\mu_j - \mu_k)}{\prod_{j=1}^{N-1} R(\mu_j)} \prod_{j=1}^{N-1} (\lambda - \mu_j)\Gamma_1 \left(\Gamma_2^2 - 2\Lambda_1^2 + 2q\Gamma_1 - r \right) d\mu_1 ... d\mu_{N-1} = 0\,. \quad (90)$$

Notice that condition (92) is a polynomial in λ of degree $N-1$. Therefore it impose N conditions over the periodic spectrum $\{\lambda_j\}_{j=1}^{2N}$. These conditions together with the N conditions for periodicity (49) uniquely determine the periodic spectrum $\{\lambda_j\}_{j=1}^{2N}$. Summarizing, we have the following proposition.

PROPOSITION 6.1. A necessary condition for the persistence of a N-phase solution of the NLS equation (1) with periodic spectrum $\{\lambda_j\}_{j=1}^{2N}$ under the perturbation (51) is that the periodic eigenvalues satisfy the following $2N$ by $2N$ system of equations

$$\frac{1}{2\pi}\kappa_j = n_j\,, \qquad n_j \in Z\,, \quad (91)$$

and

$$\oint_{a_1} \cdots \oint_{a_{N-1}} \frac{\prod_{j>k}(\mu_j - \mu_k)}{\prod_{j=1}^{N-1} R(\mu_j)} \prod_{j=1}^{N-1} (\lambda - \mu_j)\Gamma_1 \left(\Gamma_2^2 - 2\Lambda_1^2 + 2q\Gamma_1 - r \right) d\mu_1 ... d\mu_{N-1} = 0\,, \quad (92)$$

where Λ_1 and Λ_2 are given by (88) and the wave numbers κ are given by (47).

REMARK. For the genus one case ($N = 2$), writing conditions (92) in the θ representation, they reduce to conditions (**??**)–(**??**) of [**?**]. In [**?**] we proved that this conditions are sufficient, provided the Jacobian of the conditions is nonsingular.

4 Genus 2 Case

In this section we analyze the persistence of even genus two solutions of the cubic Schrödinger equation (**??**) under the complex Ginzburg-Landau perturbation (51). We will show numerical simulations that strongly indicate the existence of a family of solutions to the CGL equation that smoothly deform to a even genus two solution of the CNLS equation as ϵ goes to zero.

Expanding in powers of λ conditions (92) in the genus two case (N=3), one gets the following three necessary conditions for persistence

$$\oint_{a_1}\oint_{a_2} \frac{\mu_2-\mu_1}{R(\mu_1)R(\mu_2)}\Gamma_1\left(\Gamma_2^2-2\Lambda_1^2+2q\Gamma_1-r\right)d\mu_1\,d\mu_2=0\,, \tag{93}$$

$$\oint_{a_1}\oint_{a_2} \frac{\mu_1^2-\mu_2^2}{R(\mu_1)R(\mu_2)}\Gamma_1\left(\Gamma_2^2-2\Gamma_1^2+2q\Gamma_1-r\right)d\mu_1\,d\mu_2=0\,, \tag{94}$$

$$\oint_{a_1}\oint_{a_2} \frac{\mu_1-\mu_2}{R(\mu_1)R(\mu_2)}\mu_1\mu_2\Gamma_1\left(\Gamma_2^2-2\Gamma_1^2+2q\Gamma_1-r\right)d\mu_1\,d\mu_2=0\,, \tag{95}$$

where

$$\Gamma_1=\Gamma_2(\mu_1+\mu_2)-2\mu_1\mu_2+\Lambda_2-\frac{1}{4}\Lambda_1\,,\qquad \Gamma_2=2(\mu_1+\mu_2)-\Lambda_1\,. \tag{96}$$

If the potential A is a spatially even function, then the periodic spectrum $\{\lambda_j\}_{j=1}^{2N}$ satisfy one more symmetry condition. If λ_j is a periodic (or antiperiodic) eigenvalue, then $-\lambda_j$ is also a periodic (or antiperiodic) eigenvalue. In figure (2) a typical configuration of the periodic eigenvalues for these even potentials (represented by the ends of the branch cuts) and a choice of the a and b cycles. Using this a cycles it can be proven that the external wave number κ_3 is zero, and the remaining two are given by

$$\kappa_1=\kappa_2=-2\pi i\left(\oint_{a_1}\frac{\lambda\,d\lambda}{R(\lambda)}\right)^{-1}\,. \tag{97}$$

The corresponding frequencies are

$$\omega_1=-\omega_2=-4\pi i\left(\oint_{a_1}\frac{d\lambda}{R(\lambda)}\right)^{-1}\,. \tag{98}$$

The phase velocities are equal in magnitude but of opposite sign, therefore, these solutions represent standing waves.

Using the symmetry imposed on the spectrum by evenness of the potential, it follows that $\Lambda_1=\sum_1^{2N}\lambda_j=0$. In this case the momentum $\mathcal{J}=i/2(A^*\partial_x A-A\partial_x A^*)$ is identically zero. Writing

$$g=\sum_1^{N-1} g_j\lambda^j\,, \tag{99}$$

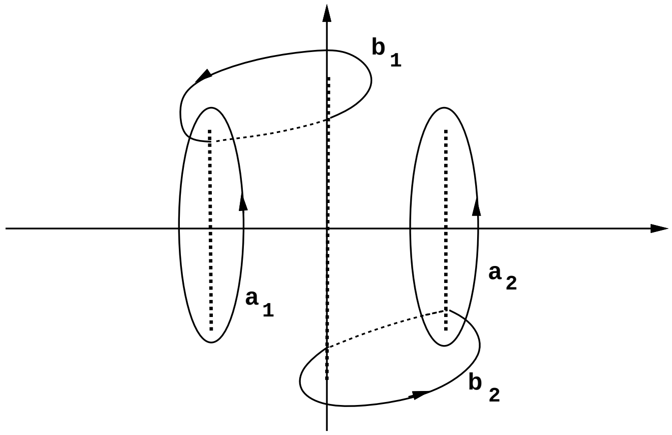

Fig. 2. An example (genus 2) of branch cuts satisfying the focusing cubic NLS restrictions and the even symmetry. It shows also one choice of a and b cycles.

then

$$g_{N-2} = i\frac{\delta \mathcal{J}}{\delta A^*} + \frac{i}{2}\Lambda_1 \frac{\delta \mathcal{M}}{\delta A^*}\,. \tag{100}$$

Therefore for an even potential of genus two, condition (94) is identically satisfied. For this kind of potentials the conditions for persistence (93)–(95) transform into the following conditions

$$\oint_{a_1}\oint_{a_2} \frac{\mu_2-\mu_1}{R(\mu_1)R(\mu_2)}\Gamma_1\left(4(\mu_1+\mu_2)^2+2q\Gamma_1-r\right)d\mu_1\,d\mu_2 = 0\,, \tag{101}$$

$$\oint_{a_1}\oint_{a_2} \frac{\mu_2-\mu_1}{R(\mu_1)R(\mu_2)}\mu_1\mu_2\Gamma_1\left(4(\mu_1+\mu_2)^2+2q\Gamma_1-r\right)d\mu_1\,d\mu_2 = 0\,, \tag{102}$$

where

$$\Gamma_1 = 2(\mu_1+\mu_2)^2 - 2\mu_1\mu_2 + \Lambda_2\,. \tag{103}$$

Therefore we have the following proposition.

PROPOSITION 6.2. A necessary condition for the persistence of an even genus two solution of the NLS equation (1) with periodic eigenvalues $\{\lambda_j\}_{j=1}^{6}$, is that the periodic eigenvalues satisfy the corresponding symmetries and the following 3 by 3 system of equations

$$i\oint_{a_1}\frac{d\lambda}{R(\lambda)} = \frac{1}{n}\,, \tag{104}$$

$$\oint_{a_1} \oint_{a_2} \frac{\mu_2 - \mu_1}{R(\mu_1)R(\mu_2)} \Gamma_1 \left(4(\mu_1 + \mu_2)^2 + 2q\Gamma_1 - r\right) d\mu_1 \, d\mu_2 = 0 \,, \qquad (105)$$

$$\oint_{a_1} \oint_{a_2} \frac{\mu_2 - \mu_1}{R(\mu_1)R(\mu_2)} \mu_1\mu_2\Gamma_1 \left(4(\mu_1 + \mu_2)^2 + 2q\Gamma_1 - r\right) d\mu_1 \, d\mu_2 = 0 \,, \qquad (106)$$

where n is a positive integer and Γ_1 is given by (103).